Resonances for Open Quantum Maps

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joint work with Semyon Dyatlov (MIT/Clay Institute)

Resonances: Geometric Scattering and Dynamics

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Open quantum map: overview

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- Many quantum open chaotic system can be reduced to open quantum maps: Nonnenmacher-Sjöstrand-Zworski '11.
- Applications going as far as computer networks: Ermann-Frahm-Shepelyansky '15.

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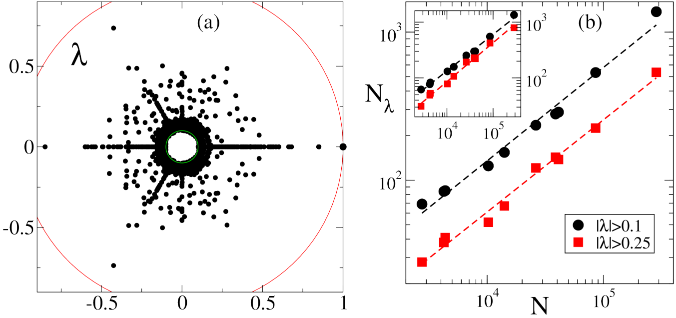


Figure: Eigenvalues for the Google Matrix of the Linux kernel and Weyl asymptotics, Ermann-Frahm-Shepelyansky 15.

Open baker's maps

Open baker's maps $\varkappa = \varkappa_{M,\mathcal{A}}$ are determined by

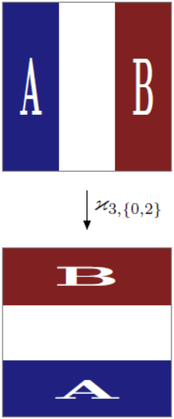
- an integer $M \geq 3$, the base
- a set $\mathcal{A} \subset \{0, \dots, M-1\}$, the alphabet
- we always assume $1 < |\mathcal{A}| < M$

$\varkappa$ is a canonical relation on $(0,1)_x \times (0,1)_\xi$:

$$\varkappa : (x, \xi) \mapsto \left(Mx - a, \frac{\xi + a}{M} \right)$$

$$\text{if} \quad x \in \left(\frac{a}{M}, \frac{a+1}{M} \right), \quad a \in \mathcal{A}$$

Basic model for a hyperbolic transformation with 'holes' through which one can escape



Discrete Cantor sets

For $k \in \mathbb{N}$, the domain and range of $\varkappa^k$ are

$$\Gamma_k^- := \mathrm{Domain}(\varkappa^k) = \big\{(x,\xi) \colon \lfloor M^k \cdot x \rfloor \in \mathcal{C}_k\big\}$$
$$\Gamma_k^+ := \mathrm{Range}(\varkappa^k) = \big\{(x,\xi) \colon \lfloor M^k \cdot \xi \rfloor \in \mathcal{C}_k\big\}$$

where $\mathcal{C}_k \subset \{0, \dots, M^k - 1\}$ is a discrete Cantor set:

$$\mathcal{C}_k = \mathcal{C}_k(M, \mathcal{A}) = \Big\{ \sum\nolimits_{r=0}^{k-1} a_r M^r \colon a_0, \dots, a_{k-1} \in \mathcal{A} \Big\}$$

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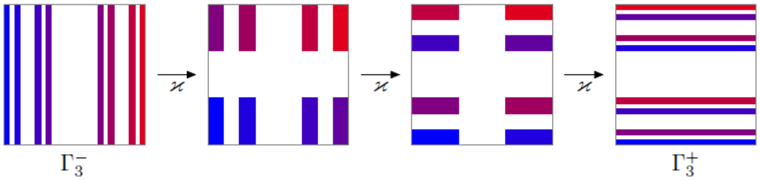
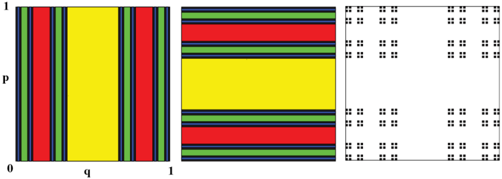


Limiting Cantor set and trapped set

The trapped set in the dynamic of $\varkappa$ is defined as $K = \Gamma^+ \cap \Gamma^-$ where $\Gamma^\pm = \bigcap_k \Gamma_k^\pm$ are the incoming/outgoing tails



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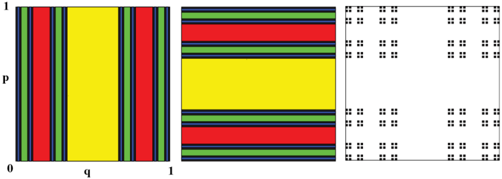

It is given by $\mathcal{C}_\infty \times \mathcal{C}_\infty$ where $\mathcal{C}_\infty$ is the limiting Cantor set:

$$\mathcal{C}_\infty := \bigcap_k \bigcup_{c \in \mathcal{C}_k} \left[\frac{c}{M^k}, \frac{c+1}{M^k} \right] \subset [0,1].$$

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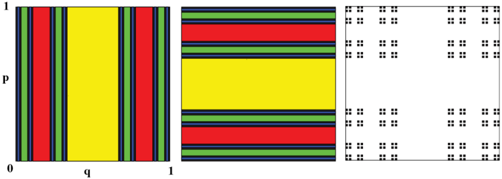

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$$\delta := \frac{\log |\mathcal{A}|}{\log M} \in (0,1)$$

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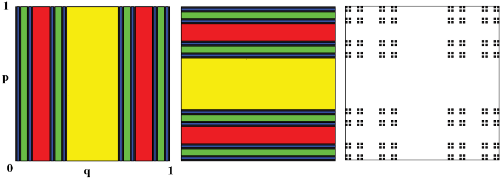

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The topological pressure is given by $P(s) = \delta - s$, $s \in \mathbb{R}$.

Quantization on the torus: Discrete microlocal analysis

Quantization of observable on the torus $\mathbb{T}^2 = \mathbb{S}^1_x \times \mathbb{S}^1_\xi$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$:

$$a \in C^\infty(\mathbb{T}^2) \mapsto \mathrm{Op}_N(a) : \ell^2_N \to \ell^2_N.$$

Here the Hilbert space $\ell^2_N := \ell^2(\mathbb{Z}_N)$ has dimension $N \gg 1$.
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($N \sim h^{-1}$.) Discrete Fourier transform $\mathcal{F}_N : \ell^2_N \to \ell^2_N$

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_\ell e^{2\pi i j\ell/N} u(\ell).$$

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Properties of quantization

- $a = a(x) \Rightarrow \mathrm{Op}_N(a) = a_N, a_N(j) = a(j/N)$;
- $a = a(\xi) \Rightarrow \mathrm{Op}_N(a) = \mathcal{F}_N^* a_N \mathcal{F}_N$;
- $[\mathrm{Op}_N(a), \mathrm{Op}_N(b)] = -\frac{i}{2\pi N} \mathrm{Op}_N(\{a, b\}) + O(N^{-2})_{\ell^2_N \to \ell^2_N}$.

Open quantum baker's maps

Example: $M = 3$, $\mathcal{A} = \{0, 2\}$. We put $N := M^k$ and

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} \end{pmatrix} : \ell^2_N \to \ell^2_N$$

where we fix $\chi \in C_0^\infty((0,1);[0,1])$, $\chi_N(j) = \chi(j/N)$

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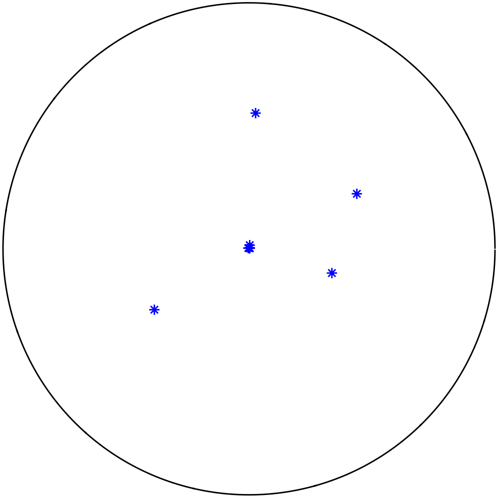
- B_N is a quantization of $\varkappa_{M,\mathcal{A}}$: Egorov's theorem

$$B_N \operatorname{Op}_N(a) = \operatorname{Op}_N(b) B_N + \mathcal{O}(N^{-1})_{\ell_N^2 \to \ell_N^2}$$

  if $\quad a(x,\xi) = b(y,\eta) \quad$ when $\varkappa_{M,\mathcal{A}}(x,\xi) = (y,\eta)$, $\xi, y \in \operatorname{supp} \chi$

- Resonances are eigenvalues of B_N. They are in the unit disk $\{\lambda \in \mathbb{C} : |\lambda| \le 1\}$.
- Similar construction for any base M and alphabet $\mathcal{A}$.

Numerical example: $M = 5$, $\mathcal{A} = \{1, 3\}$



$\mathrm{Spec}(B_N)$ for $k = 2$, $N = M^k$

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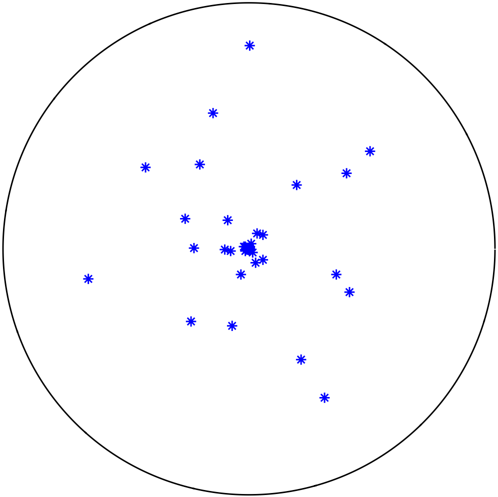

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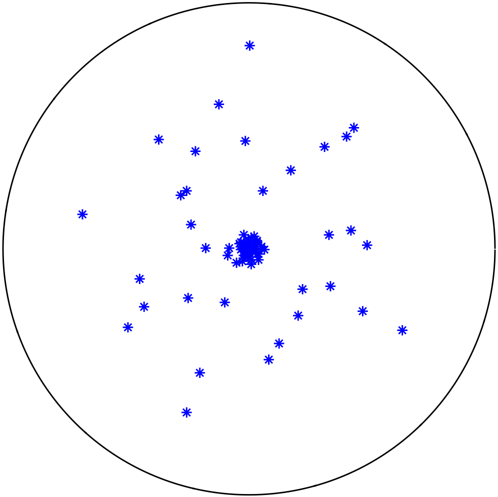

Spec(B_N) for $k = 4$, $N = M^k$

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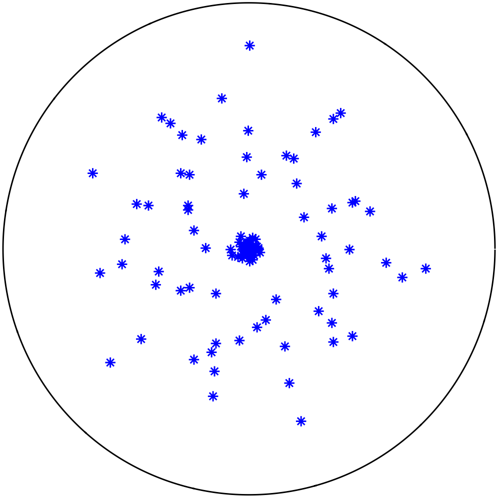

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Previous results: Walsh quantized baker's map

A different quantization using Walsh Fourier transform W_N (the discrete Fourier transform on the group $(\mathbb{Z}_M)^k$) instead of the standard discrete Fourier transform $\mathcal{F}_N$ (the discrete Fourier transform on the group $\mathbb{Z}_N$, $N = M^k$) has been studied by Nonnenmacher-Zworski '07.

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- Positive spectral gap for $M = 3$, $\mathcal{A} = \{0, 2\}$, but no gap for $M = 4$, $\mathcal{A} = \{0, 2\}$.
- Fractal Weyl law and uniform angular distribution.

Results: spectral gap

Let R_N be the spectral radius of B_N:

$$R_N := \max\{|\lambda| : \lambda \in \mathrm{Spec}(B_N)\}.$$

Theorem 1 [Dyatlov-J '16]

There exists (explicitly computable!)

$$\beta = \beta(M, \mathcal{A}) > \max\left(0, \frac{1}{2} - \delta\right)$$

such that B_N has an asymptotic spectral gap of size β:

$$\limsup_{N\to\infty} R_N \le M^{-\beta} < 1 \tag{1}$$

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Remark: The pressure gap is given by $\beta = -P(1/2) = \frac{1}{2} - \delta$, valid under the pressure condition $\delta < 1/2$.

Numerical example: $M = 5$, $\mathcal{A} = \{1, 3\}$, $N = M^5$

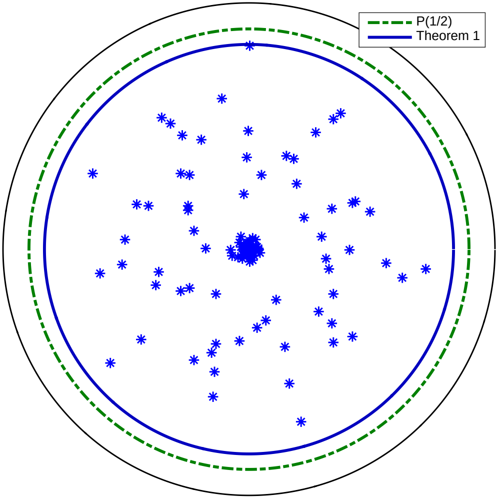

Figure: For some cases the gap of Theorem 1 approximates the spectral radius well.

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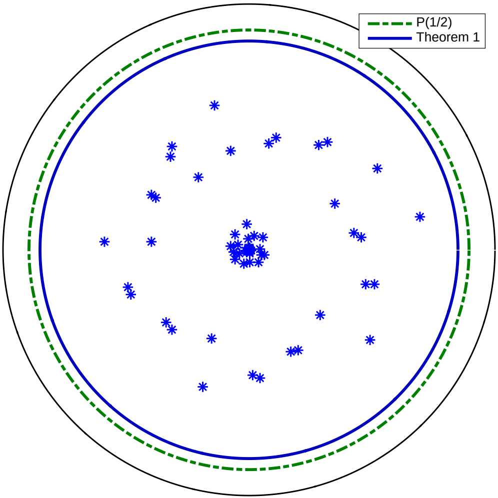

Figure: and for some cases, this upper bound is far from sharp.

(Essential) Spectral gaps in open quantum chaos

- Pressure Gap: $\beta = -P(1/2)$ if $P(1/2) < 0$. Patterson '76, Sullivan '79, Ikawa '88, Gaspard-Rice '89, Nonnenmacher-Zworski '09.

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- Pressure Gap: $\beta = -P(1/2)$ if $P(1/2) < 0$. Patterson '76, Sullivan '79, Ikawa '88, Gaspard-Rice '89, Nonnenmacher-Zworski '09.
- Improved Gap $\beta = -P(1/2) + \epsilon$ for some systems with $P(1/2) \leq 0$ where $\epsilon > 0$ depends on the system in an unspecified way. Naud '05, Petkov-Stoyanov '10, Stoyanov '11, '12, Bourgain-Gamburd-Sarnak '11, Oh-Winter '16, Magee-Oh-Winter '14. The ideas originate from Dolgopyat '98 on spectral radius of transfer operator for Anosov flow.

(Essential) Spectral gaps for convex co-compact hyperbolic surfaces

For convex co-compact hyperbolic surfaces, using Fractal uncertainty principle, improvement over both the pressure gap $\beta = -P(1/2) = \frac{1}{2} - \delta$ and the trivial gap $\beta = 0$ has been obtained recently.

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- Dyatlov-Zahl '16: Improved gap $\beta > 0$ for hyperbolic surfaces with $P(1/2) = 0$ and nearby surfaces, some with $P(1/2) > 0$; β is given explicitly in terms of the Ahlfors-David regularity constant C_R and the Hausdorff dimension δ of the limit set. (Additive energy, Freĭman theorem)

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- Dyatlov-J '17: Improved spectral gap $\beta > \frac{1}{2} - \delta$ with explicit β in terms of C_R and δ. (A quantitative version of Naud '05, combining Dolgopyat's idea with the fractal structure)

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- Bourgain-Dyatlov '16: Improved spectral gap $\beta > 0$ with β unspecified, but only depending on C_R and δ. (Beurling-Mallivan multiplier theorem, harmonic measures)

The proof: Reduction to fractal uncertainty principle

Let $(B_N - \lambda)u = 0$, $\|u\|_{\ell^2_N} = 1$ and $|\lambda| \geq c > 0$.

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- $a \equiv 1$, $b = 1_{\Gamma_k^+} \Rightarrow u = \operatorname{Op}_N(1_{\Gamma_k^+})u + O(N^{-\infty})$;
- $b \equiv 1$, $a = 1_{\Gamma_k^-} \Rightarrow \|\operatorname{Op}_N(1_{\Gamma_k^-})u\| \geq |\lambda|^k - O(N^{-\infty})$;
- Contradiction if $|\lambda| \geq M^{-\beta}$ and the fractal uncertainty principle holds with exponent β:

$$\|\operatorname{Op}_N(1_{\Gamma_k^-})\operatorname{Op}_N(1_{\Gamma_k^+})\|_{\ell^2_N \to \ell^2_N} \leq CN^{-\beta}.$$

Fractal uncertainty principle

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$$\| \operatorname{Op}_N(1_{\Gamma_k^-}) \operatorname{Op}_N(1_{\Gamma_k^+}) \|_{\ell_N^2 \to \ell_N^2} \le C N^{-\beta}$$

can be rewritten as

$$\| 1_{\mathcal{C}_k} \mathcal{F}_N 1_{\mathcal{C}_k} \|_{\ell_N^2 \to \ell_N^2} \le C N^{-\beta}.$$

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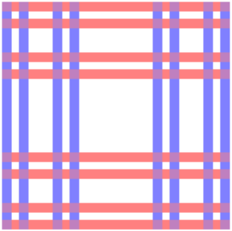

Figure: Functions cannot be localized on $\mathcal{C}_k$ both in position and in frequency.

Recovering the pressure gap

In the fractal uncertainty principle

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We can improve both of the trivial gap $\beta = 0$ and the pressure gap $\beta = \frac{1}{2} - \delta$:

Theorem 2 [Dyatlov-J '16]

The fractal uncertainty principle holds for some

$$\beta = \beta(M, \mathcal{A}) > \max\left(0, \frac{1}{2} - \delta\right).$$

Proof of the fractal uncertainty principle

Observation: For $N = M^k$, $N_1 = M^{k_1}$, $N_2 = M^{k_2}$, $k = k_1 + k_2$, the Walsh quantization satisfies the tensor product formula:

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$$r_k := \|1_{\mathcal{C}_k} \mathcal{F}_N 1_{\mathcal{C}_k}\|_{\ell^2_N \to \ell^2_N}$$

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Therefore it is enough to show that for some k,

$$r_k < \min(1, N^{\delta - 1/2}).$$

Proof of FUP: improve the trivial gap

First, we show $r_k < 1$: If not, then we can find u such that

$$\|u\|_{\ell^2_N} = 1, \quad u = 1_{\mathcal{C}_k} u, \quad \mathcal{F}_N u = 0 \text{ on } \mathbb{Z}_N \setminus \mathcal{C}_k.$$

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- Contradiction for large k.

Proof of FUP: improve the pressure gap

Now we show that $r_k < N^{\delta-1/2} = |\mathcal{C}_k|/\sqrt{N}$: If not, then

$$\|1_{\mathcal{C}_k}\mathcal{F}_N 1_{\mathcal{C}_k}\|_{\ell^2_N \to \ell^2_N} = \frac{|\mathcal{C}_k|}{\sqrt{N}} = \|1_{\mathcal{C}_k}\mathcal{F}_N 1_{\mathcal{C}_k}\|_{\mathrm{HS}}.$$

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- So all 2×2 minors are zero.
- Contradiction when $|\mathcal{A}| > 1$, $k \geq 2$.

More on fractal uncertainty exponents

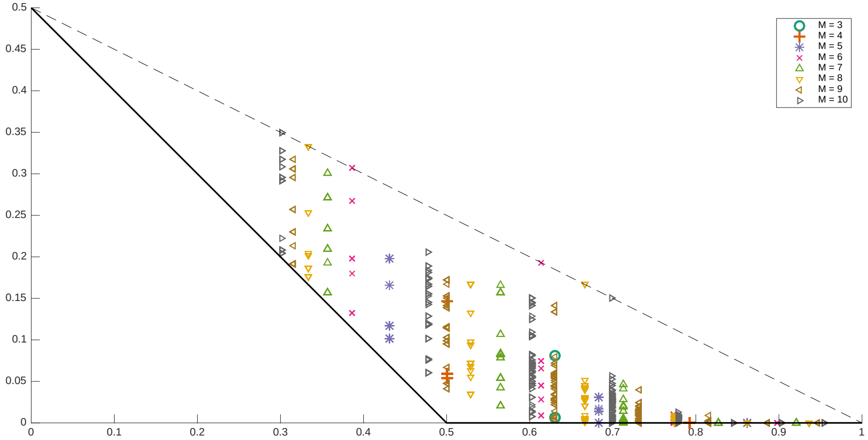

Figure: X axis: δ; Y axis: FUP exponent β (numerics); all alphabets with $M \leq 10$. Solid line: $\beta = \max(0, \frac{1}{2} - \delta)$ (trivial/pressure gap), dashed line: $\beta = -\frac{P(1)}{2} = 1 - \frac{\delta}{2}$.

More on fractal uncertainty exponents

Bounds on β as $M \to \infty$:

$\delta \le 1/2$:

$$\beta - \left(\tfrac{1}{2} - \delta\right) \gtrsim \frac{1}{M^8 \log M}$$

$\delta \approx 1/2$: using additive energy,

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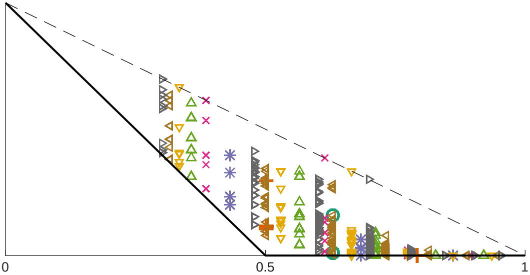
$\delta \ge 1/2$:

$$\beta \gtrsim \exp\left(- M^{\frac{\delta}{1-\delta} + o(1)} \right)$$



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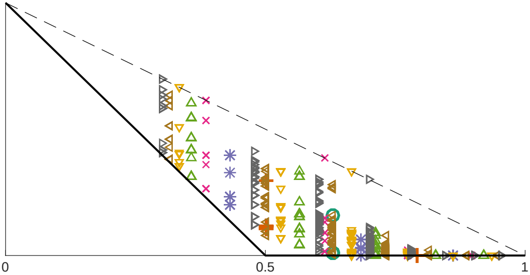

Solid: $\beta = \max(0, \frac{1}{2} - \delta)$, dashed: $\beta = \frac{1-\delta}{2}$

- Examples of alphabets (arithmetic progressions) with $\delta \le 1/2$ and

$$\beta - \left(\tfrac{1}{2} - \delta\right) \lesssim \frac{M^{2\delta - 1}}{\log M}$$

- Examples of special alphabets with $\beta = \frac{1-\delta}{2}$

Special alphabets with maximal β

We call $\mathcal{A}$ a special alphabet, if

$$\text{for all } \ j,\ell \in \mathcal{A},\ j \neq \ell, \quad \text{we have } \ \mathcal{F}_M(1_{\mathcal{A}})(j-\ell) = 0 \qquad (2)$$

Such $\mathcal{A}$ have $\beta = \frac{1-\delta}{2} = -\frac{P(1)}{2}$, which is the largest possible value of β and all nonzero singular values of $1_{\mathcal{C}_k}\mathcal{F}_N 1_{\mathcal{C}_k}$ are equal to $N^{-\beta}$

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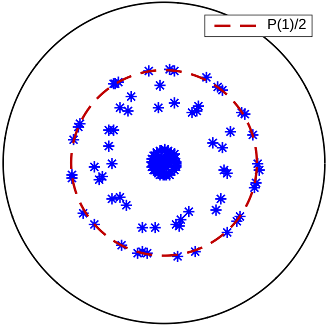

Example: $M = 6$, $\mathcal{A} = \{1, 4\}$, $N = M^5$

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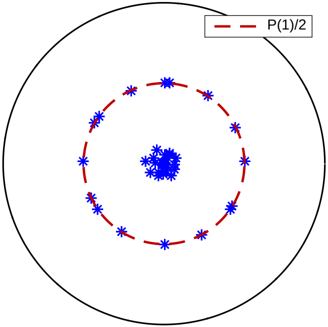

Example: $M = 8$, $\mathcal{A} = \{2, 4\}$, $N = M^4$

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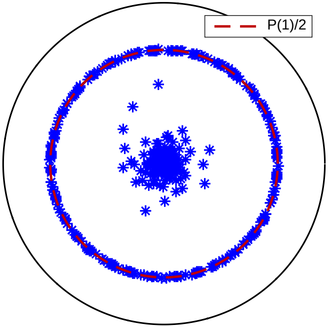

Example: $M = 8$, $\mathcal{A} = \{1, 2, 5, 6\}$, $N = M^4$

Conjecture on band structure for special alphabets

Conjecture 1 (band structure)

Assume that $\mathcal{A}$ is a special alphabet. Then there exists $\mu > \frac{1-\delta}{2}$ such that:

- For any $\varepsilon > 0$ and N large, there is a second gap

$$\mathrm{Spec}(B_N) \cap \{M^{-\mu} \leq |\lambda| \leq M^{-\frac{1-\delta}{2}-\varepsilon}\} = \emptyset$$

- Eigenvalues in the first band satisfy exact fractal Weyl law:

$$\big|\,\mathrm{Spec}(B_N) \cap \{|\lambda| \geq M^{-\mu}\}\big| = |\mathcal{A}|^k = N^\delta$$

Open quantum baker's map with general N

In the definition of open quantum baker's map B_N (again, say $M = 3$, $\mathcal{A} = \{0, 2\}$)

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} \end{pmatrix} : \ell_N^2 \to \ell_N^2$$

we can take N to be any multiple of M. The spectral gap still follows from fractal uncertainty principle:

$$\|1_{\mathcal{C}_k(N)} \mathcal{F}_N 1_{\mathcal{C}_k(N)}\|_{\ell_N^2 \to \ell_N^2} \leq C N^{-\beta}.$$

Here k is chosen so that $M^k \leq N < M^{k+1}$ and $\mathcal{C}_k(N) \subset \mathbb{Z}_N$ is a set of size $|\mathcal{A}|^k$ that looks like $\mathcal{C}_k \subset \mathbb{Z}_{M^k}$. $\mathcal{C}_k(N)$ do not have good "tensor" structures like $\mathcal{C}_k$, but they are still Ahlfors-David regular!

Ahlfors-David regular sets

Let $X \subset \mathbb{R}$ be a non-empty compact set, $\delta \in (0,1)$, $C_R \geq 1$, and $0 \leq \alpha_0 \leq \alpha_1 \leq \infty$, we say that X is δ-regular with constant C_R on scales from α_0 to α_1 if there exists a Borel measure μ_X on $\mathbb{R}$ such that

- μ_X is supported on X: $\mu_X(\mathbb{R} \setminus X) = 0$;
- for any interval I of size $[\alpha_0, \alpha_1]$, we have $\mu_X(I) \leq C_R|I|^\delta$;
- if in additionally I is centered at a point in X, then $\mu_X(I) \geq C_R^{-1}|I|^\delta$.

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Examples:

- The limit Cantor set is δ-regular with constant $M^{2\delta}$ from scales 0 to 1 where $\delta = \log |\mathcal{A}| / \log M$.
- $N^{-1}\mathcal{C}_k(N)$ is δ-regular with constant $8M^{3\delta}$ from scales N^{-1} to 1 where $\delta = \log |\mathcal{A}| / \log M$.

FUP for 1-dimensional regular sets I: General form

Theorem (Dyatlov-J. '17)

Assume that (X, μ_X) is δ-regular, and (Y, μ_Y) is δ'-regular, from scales h to 1 with constant C_R, where $0 < \delta, \delta' < 1$, and $X \subset I_0$, $Y \subset J_0$ for some intervals I_0, J_0. Consider an operator $\mathcal{B}_h : L^1(Y, \mu_Y) \to L^\infty(X, \mu_X)$ of the form

$$\mathcal{B}_h f(x) = \int_Y \exp\left(\frac{i\Phi(x,y)}{h}\right) G(x,y) f(y)\, d\mu_Y(y)$$

where $\Phi(x,y) \in C^2(I_0 \times J_0; \mathbb{R})$ satisfies $\partial^2_{xy}\Phi \neq 0$ and $G(x,y) \in C^1(I_0 \times J_0; \mathbb{C})$.
Then there exist constants $C, \varepsilon_0 > 0$ such that

$$\|\mathcal{B}_h\|_{L^2(Y,\mu_Y) \to L^2(X,\mu_X)} \leq C h^{\varepsilon_0}.$$

Here ε_0 depends only on δ, δ', C_R as follows

$$\varepsilon_0 = (5C_R)^{-80\left(\frac{1}{\delta(1-\delta)} + \frac{1}{\delta'(1-\delta')}\right)}$$

FUP for 1-dimensional regular sets I: Fourier transform

Consider the semiclassical Fourier transform

$$\mathcal{F}_h u(\xi) = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi/h} u(x) dx$$

If Λ is δ-regular from scales h to 1, and $X = \Lambda(h) = \Lambda + [-h, h]$, then

$$\|1_{\Lambda(h)} \mathcal{F}_h 1_{\Lambda(h)}\|_{L^2 \to L^2} \leq C h^{\frac{1}{2} - \delta + \epsilon_0}.$$

Note that X with $h^{\delta - 1}$ times the Lebesgue measure is δ-regular from scale h to 1. The volume bound $|\Lambda(h)| \leq Ch^{1-\delta}$ and $L^1 \to L^\infty$ bound for $\mathcal{F}_h$ gives $O(h^{\frac{1}{2} - \delta})$.

FUP for 1-dimensional regular sets II

Theorem (Bourgain-Dyatlov '16)

Let $B = B(h) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *be defined as*

$$Bf(x) = h^{-1/2} \int e^{i\Phi(x,y)/h} b(x,y) f(y) dy$$

where $\Phi \in C^\infty(U;\mathbb{R})$, $b \in C_0^\infty(U)$ *on some open set* $U \subset \mathbb{R}^2$ *satisfy* $\partial^2_{xy}\Phi \neq 0$ *on* U*. Let* $\delta \in (0,1)$ *and* $C_R > 0$*. If* X, Y *are* δ*-regular sets from scales 0 to 1 with constant* C_R*, then there exists* $\beta > 0$, $\rho \in (0,1)$ *depending only on* δ, C_R *and* $C > 0$ *depending on* δ, C_R, b, Φ *such that for all* $h \in (0,1)$

$$\|1_{X(h^\rho)} B 1_{Y(h^\rho)}\|_{L^2 \to L^2} \leq C h^\beta.$$

FUP for 1-dimensional regular sets II: Fourier transform

Again, consider the semiclassical Fourier transform

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If Λ is δ-regular from scales h to 1, and $X = \Lambda(h) = \Lambda + [-h, h]$, then

$$\|1_{\Lambda(h)} \mathcal{F}_h 1_{\Lambda(h)}\|_{L^2 \to L^2} \le Ch^\beta.$$

Note that the $L^2 \to L^2$ bound for $\mathcal{F}_h$ gives bound $O(1)$.
Both results can be translated to discrete Fourier transform to get FUP for $1_{C_k(N)} \mathcal{F}_N 1_{C_k(N)}$.

Open quantum baker's map with general N

Theorem [Dyatlov-J '17]

There exists

$$\beta = \beta(M, \mathcal{A}) > \max\left(0, \frac{1}{2} - \delta\right)$$

such that B_N has an asymptotic spectral gap of size β:

$$\limsup_{N\to\infty} R_N \le M^{-\beta} < 1$$

But we only get explicit constant β for $\delta \le 1/2$:

$$\beta \ge \frac{1}{2} - \delta + (40M^{3\delta})^{-\frac{160}{\delta(1-\delta)}}.$$

(Still polynomially in M!)

Results: resonance counting

We count eigenvalues of B_N in annuli:

$$\#(N,\nu) = \left|\operatorname{Spec}(B_N) \cap \{|\lambda| \geq M^{-\nu}\}\right|$$

Theorem 3 [Dyatlov-J. '16]

For each $\varepsilon > 0$ and $\nu > 0$ we have the fractal Weyl upper bound

$$\#(N,\nu) \leq C_{\nu,\varepsilon} N^{m(\delta,\nu)+\varepsilon}, \quad m(\delta,\nu) = \min(\delta, 2\nu + 2\delta - 1)$$

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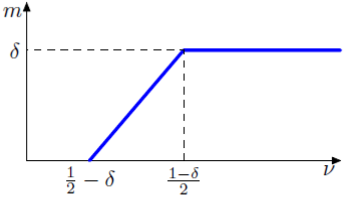
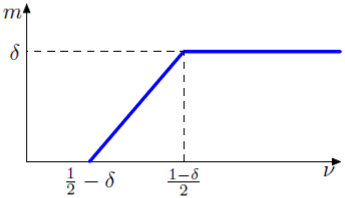
- $m = \delta$ for $\nu \geq \frac{1-\delta}{2} = -\frac{1}{2}P(1)$;
- $m < 0$ for $\nu < \frac{1}{2} - \delta = -P(\frac{1}{2})$.



Conjecture: Fractal Weyl Law



Conjecture 2 (fractal Weyl law)

For each $\nu > -\frac{1}{2}P(1) = \frac{1-\delta}{2}$, we have

$$\#(N, \nu) \geq c_\nu N^\delta > 0.$$

Note: For convex co-compact hyperbolic surfaces, Jakobson-Naud conjectured the gap to be of size $-\frac{1}{2}P(1)$.

Fractal Weyl law in open quantum chaos

- Upper bound on general hyperbolic situations: $\mathcal{N}(R,\nu) \leq C(\nu)R^{\delta}$. Sjöstrand '90, Guillopé-Lin-Zworski '04, Sjöstrand-Zworski '07, Nonnenmacher-Sjöstrand-Zworski '11, '14, Datchev-Dyatlov '13.

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- Naud '14, Jakobson-Naud '14: $\mathcal{N}(R,\nu) \leq C(\nu)R^{m(\nu)}$ for some $m(\nu) < \delta$ when $\nu < \frac{1}{2} - \delta$ for convex co-compact hyperbolic surfaces.

Fractal Weyl law in open quantum chaos

- Upper bound on general hyperbolic situations: $\mathcal{N}(R,\nu) \leq C(\nu)R^{\delta}$. Sjöstrand '90, Guillopé-Lin-Zworski '04, Sjöstrand-Zworski '07, Nonnenmacher-Sjöstrand-Zworski '11, '14, Datchev-Dyatlov '13.
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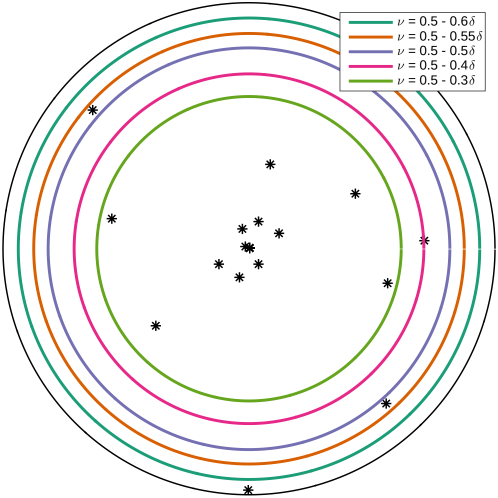
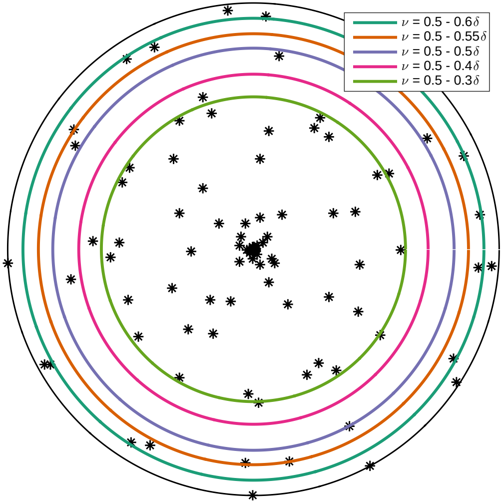
- Dyatlov '15: $\mathcal{N}(R,\nu) \leq C(\nu)R^{m(\delta,\nu)+0}$ where $m(\delta,\nu) = \min(\delta, 2\nu + 2\delta - 1)$ for convex co-compact hyperbolic surfaces.

Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$



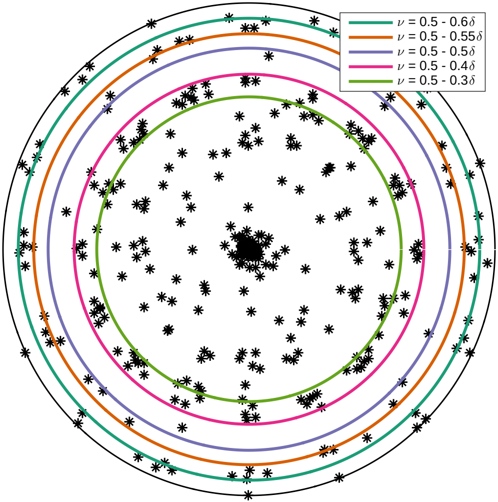
$k = 2$

Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$



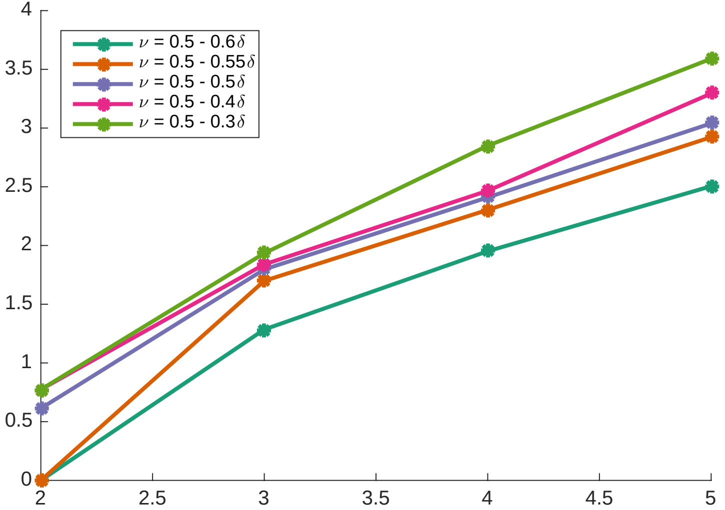
$k = 3$

Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$



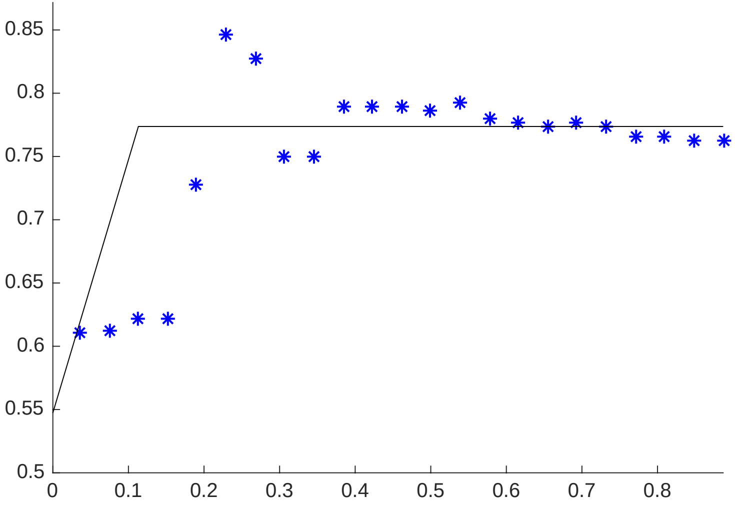
$k = 4$

Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$



Plot of $\log \#(M^k, \nu)/\log M$ as a function of k

Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$



Linear fits for the growth exponent of $\#(N, \nu)$ and the bound of Theorem 3

Results: dependence on cutoffs

Recall that the defininition of $B_N = B_{N,\chi}$ involved a cutoff function

$$\chi \in C_0^\infty((0,1);[0,1])$$

e.g. for $M = 3$, $\mathcal{A} = \{0,2\}$

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} \end{pmatrix}$$

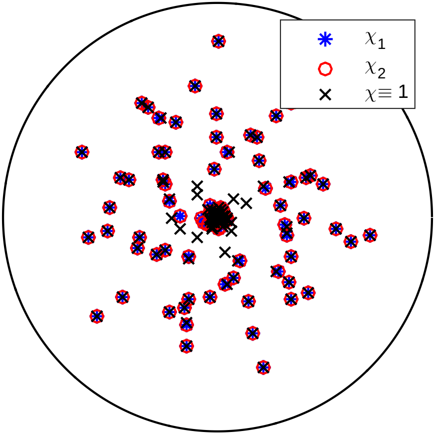
Theorem 4 [Dyatlov-J '16]

Assume that $\chi_1, \chi_2 \in C_0^\infty((0,1);[0,1])$ and $\chi_1 = \chi_2$ near the Cantor set $\mathcal{C}_\infty \subset [0,1]$. Then for each ν, eigenvalues of B_{N,χ_1} in $\{|\lambda| \geq M^{-\nu}\}$ are $\mathcal{O}(N^{-\infty})$ quasimodes of B_{N,χ_2}.

Dependence on cutoff

If $0, M-1 \notin \mathcal{A}$ it is natural to take $\chi = 1$ near $\mathcal{C}_\infty$.

However we cannot take $\chi \equiv 1$:



$M = 5$, $\mathcal{A} = \{1, 3\}$, $N = M^5$, $\chi_1 = \chi_2 = 1$ near $\mathcal{C}_\infty$

Summary

- We obtain results on spectral gap which lie well beyond what is known for more general systems
- We use fractal uncertainty principle, the fine structure of the associated Cantor sets, and simple tools from harmonic analysis, algebra, combinatorics, and number theory
- We also show a fractal Weyl upper bound
- We discover that the studied systems form a rich class with a variety of different types of behavior

Thanks for your attention!