The Hamilton flow and the Schrödinger evolution for degree-2 complex-valued Hamiltonians

Joe Viola

Laboratoire de Mathématiques Jean Leray
Université de Nantes
Joseph.Viola@univ-nantes.fr

5 June 2017

The objects discussed

Let P be the Weyl quantization of a degree-2 complex-valued polynomial $p(x,\xi)$ and let Q be the Weyl quantization of a quadratic form $q(x,\xi)$.
Our goal is to describe how to get information about the Schrödinger evolution

$$U(t,x) = \exp(-\mathrm{i}tP)u(x)$$

from elementary symplectic linear algebra. Specifically, we'll get under suitable conditions the $L^2(\mathbb{R}^n)$ operator norm using the Hamilton flow

$$\mathbf{K} = \exp(tH_q)$$

of the quadratic part. Recall that $H_q = (\partial_\xi q, -\partial_x q)$, which we will regard as a matrix.

Application: the NSAHO

For $\theta \in (-\pi/2, \pi/2)$, let

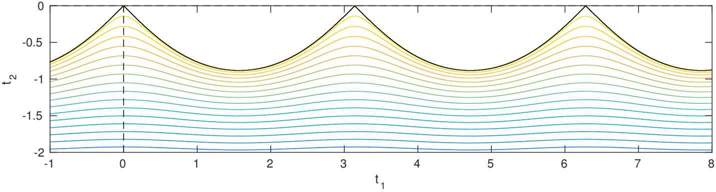
$$Q_\theta = \frac{1}{2}\left(\mathrm{e}^{-\mathrm{i}\theta}D^2 + \mathrm{e}^{\mathrm{i}\theta}x^2\right).$$

For $t \in \mathbb{C}$, let

$$a = |\cos t|^2 + \cos(2\theta)|\sin t|^2.$$

Then [V. 2016, 2017] when $\Im t \leq 0$ and $a \geq 1$,

$$\|\mathrm{e}^{-\mathrm{i}tQ_\theta}\|_{\mathcal{L}(L^2(\mathbb{R}))} = \left(a - \sqrt{a^2-1}\right)^{1/4}$$



Application: the shifted harmonic oscillator

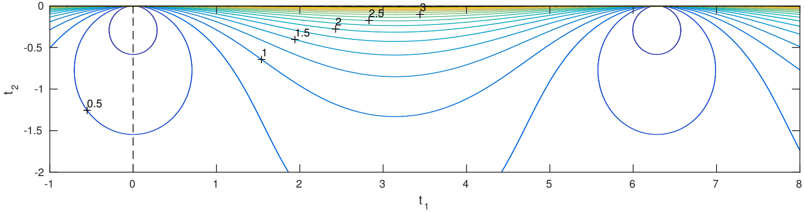
Let

$$P = \frac{1}{2}(D^2 + (x - \mathrm{i})^2)$$

be obtained by shifting the harmonic oscillator Q_0 by $\mathbf{v} = (\mathrm{i}, 0)$. Then

$$G = \frac{\|\mathrm{e}^{-\mathrm{i}tP}\|}{\|\mathrm{e}^{-\mathrm{i}tQ}\|} = \exp\left(\frac{\cos t_1 - \cosh t_2}{\sinh t_2}\right).$$

(The image is $\log\log G$, so “3” represents 1.9×10^8.)



The NSAHO

The non-self-adjoint harmonic oscillator

$$Q_\theta = \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\theta}D^2 + \mathrm{e}^{\mathrm{i}\theta}x^2), \quad \theta \in (-\pi/2, \pi/2)$$

has symbol

$$q_\theta(x,\xi) = \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\theta}\xi^2 + \mathrm{e}^{\mathrm{i}\theta}x^2).$$

These operators are obtained from the self-adjoint harmonic oscillator Q_0 via a formal change of variables: if

$$\mathcal{V}_\mu u(x) = \mu^{1/2}u(\mu x), \quad \mu = \mathrm{e}^{\mathrm{i}\theta/2},$$

then

$$Q_\theta = \mathcal{V}_\mu Q_0 \mathcal{V}_\mu^{-1}.$$

The Hamilton flow of the NSAHO

Any change of variables is associated with a formal Egorov theorem

$$\mathcal{V}_\mu a^w \mathcal{V}_\mu^{-1} = (a \circ \mathbf{V}_\mu^{-1})^w, \quad \mathbf{V}_\mu = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu^\top \end{pmatrix}.$$

This allows us to decompose the Hamilton vector field

$$H_{q_\theta} = \begin{pmatrix} 0 & \mu^{-2} \\ -\mu^2 & 0 \end{pmatrix} = \mathbf{V}_\mu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{V}_\mu^{-1}$$

and the Hamilton flow

$$\exp(tH_{q_\theta}) = \mathbf{V}_\mu \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mathbf{V}_\mu^{-1}.$$

Hamilton-Schrödinger connection: Egorov

We have (at least formally) the Egorov relation

$$\exp(-\mathrm{i}Q)a^w = (a \circ \mathbf{K}^{-1})^w \exp(-\mathrm{i}Q).$$

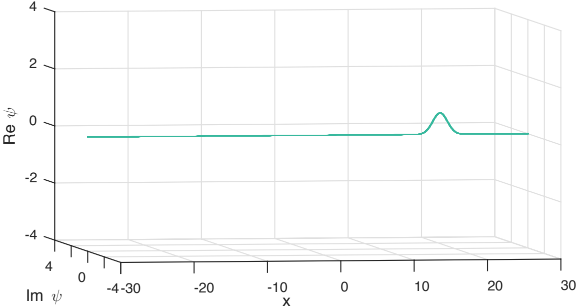
If we write a coherent state[1] as the kernel of an annihilation operator, we can track the center using the Egorov relation.

[1] Taken to mean $f(x) = \mathrm{e}^{\varphi(x)} \in L^2$ where $\varphi(x)$ is a degree 2 polynomial

Example: the quantum harmonic oscillator

If $f(x) = \mathrm{e}^{-(x-20)^2/2}$, then $(D - \mathrm{i}(x-20))f(x) = 0$. Setting $a(x,\xi) = \xi - \mathrm{i}(x-20)$, we see that

$$(a \circ \exp(tH_{q_0})^{-1})(x,\xi) = \mathrm{e}^{\mathrm{i}t}((\xi + 20\sin t) - \mathrm{i}(x - 20\cos t))$$



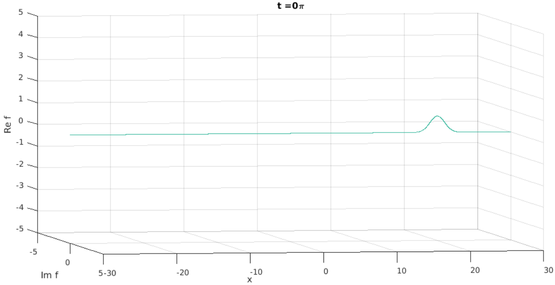
Example: the quantum harmonic oscillator

If $f(x) = \mathrm{e}^{-(x-20)^2/2}$, then $(D - \mathrm{i}(x-20))f(x) = 0$. Setting $a(x,\xi) = \xi - \mathrm{i}(x-20)$, we see that

$$(a \circ \exp(tH_{q_0})^{-1})(x,\xi) = \mathrm{e}^{\mathrm{i}t}((\xi + 20\sin t) - \mathrm{i}(x - 20\cos t))$$

Example: blowup for the NSAHO

When we try to do the same thing for the non-self-adjoint harmonic oscillator, the norm doesn't stay constant (here $\theta = \pi/180$).

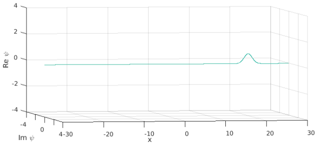


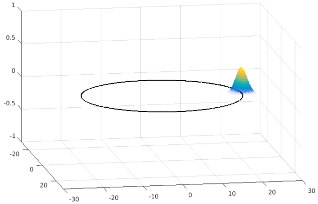
Example: blowup for the NSAHO

When we try to do the same thing for the non-self-adjoint harmonic oscillator, the norm doesn't stay constant (here $\theta = \pi/180$.

Rotation on the Bargmann side

The Bargmann transform lets us represent a coherent state as function of two variables.



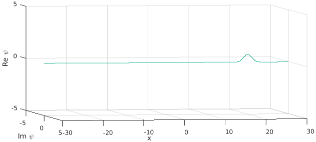


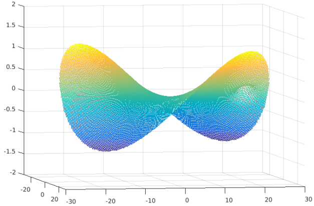
Rotation on the Bargmann side

The Bargmann transform lets us represent a coherent state as function of two variables.

Bent phase space for the NSAHO

The complex scaling for the NSAHO induces a bend in phase space, which allows us to heuristically understand the non-unitary evolution.





Bent phase space for the NSAHO

The complex scaling for the NSAHO induces a bend in phase space, which allows us to heuristically understand the non-unitary evolution.

Egorov and boundedness

Notice that

$$(\mathrm{e}^{-\mathrm{i}Q})^* \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}, \quad (\mathrm{e}^{-\mathrm{i}Q})^*\mathrm{e}^{-\mathrm{i}Q} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

In order for bounded Gaussians to stay bounded, we need to assume that

$$\mathrm{i}\sigma(\overline{\mathbf{z}}, \overline{\mathbf{K}}^{-1}\mathbf{K}\mathbf{z}) \geq \mathrm{i}\sigma(\overline{\mathbf{z}}, \mathbf{z}), \quad \forall \mathbf{z} = (x, \xi) \in \mathbb{C}^{2n},$$

where $\sigma((x,\xi),(y,\eta)) = \xi \cdot y - \eta \cdot x$. When this holds strictly on $\mathbf{z} \neq 0$, we say that $\mathbf{K}$ is strictly positive.

Classical-quantum correspondence

Egorov and annihilation operators don’t tell us the norm because we ignore a constant multiple.

Classical-quantum correspondence

Egorov and annihilation operators don't tell us the norm because we ignore a constant multiple.

Let $Q_j = q_j^w$ and $\mathbf{K}_j = \exp(H_{q_j})$ for $j = 1, 2, 3$. Then from Egorov it's obvious that

$$\mathrm{e}^{-\mathrm{i}Q_1}\mathrm{e}^{-\mathrm{i}Q_2} = C\mathrm{e}^{-\mathrm{i}Q_3} \implies \mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_3.$$

Using the Mehler formula, we can find the remarkable inverse

$$\mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_3 \implies \mathrm{e}^{-\mathrm{i}Q_1}\mathrm{e}^{-\mathrm{i}Q_2} = \pm\mathrm{e}^{-\mathrm{i}Q_3}$$

(See Hörmander, 1995; he calls this the "metaplectic semigroup.")

Proof that $\|e^{-iQ}\| = \prod\{\mu_j^{1/4} \ : \ \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.

With this miracle in hand,

$$(e^{-iQ})^* e^{-iQ} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Proof that $\|\mathrm{e}^{-\mathrm{i}Q}\| = \prod\{\mu_j^{1/4} \ : \ \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.

With this miracle in hand,

$$(\mathrm{e}^{-\mathrm{i}Q})^* \mathrm{e}^{-\mathrm{i}Q} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Defining q_1 so that $\exp H_{q_1} = \overline{\mathbf{K}}^{-1}\mathbf{K}$, we get

$$(\mathrm{e}^{-\mathrm{i}Q})^* \mathrm{e}^{-\mathrm{i}Q} = \pm \mathrm{e}^{-\mathrm{i}Q_1}.$$

Proof that $\|e^{-iQ}\| = \prod\{\mu_j^{1/4} \ : \ \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.

With this miracle in hand,

$$(e^{-iQ})^* e^{-iQ} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Defining q_1 so that $\exp H_{q_1} = \overline{\mathbf{K}}^{-1}\mathbf{K}$, we get

$$(e^{-iQ})^* e^{-iQ} = \pm e^{-iQ_1}.$$

Since this operator is real positive definite, we can take iQ_1 of harmonic oscillator type (iq_1 pos. def.)

Proof that $\|e^{-iQ}\| = \prod\{\mu_j^{1/4} : \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.

With this miracle in hand,

$$(e^{-iQ})^* e^{-iQ} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Defining q_1 so that $\exp H_{q_1} = \overline{\mathbf{K}}^{-1}\mathbf{K}$, we get

$$(e^{-iQ})^* e^{-iQ} = \pm e^{-iQ_1}.$$

Since this operator is real positive definite, we can take iQ_1 of harmonic oscillator type (iq_1 pos. def.)

We find $\|e^{-iQ}\|$ from $\|e^{-iQ_1}\|$

Proof that $\|\mathrm{e}^{-\mathrm{i}Q}\| = \prod\{\mu_j^{1/4} \,:\, \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.

With this miracle in hand,

$$(\mathrm{e}^{-\mathrm{i}Q})^* \mathrm{e}^{-\mathrm{i}Q} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Defining q_1 so that $\exp H_{q_1} = \overline{\mathbf{K}}^{-1}\mathbf{K}$, we get

$$(\mathrm{e}^{-\mathrm{i}Q})^* \mathrm{e}^{-\mathrm{i}Q} = \pm \mathrm{e}^{-\mathrm{i}Q_1}.$$

Since this operator is real positive definite, we can take $\mathrm{i}Q_1$ of harmonic oscillator type ($\mathrm{i}q_1$ pos. def.)

We find $\|\mathrm{e}^{-\mathrm{i}Q}\|$ from $\|\mathrm{e}^{-\mathrm{i}Q_1}\|$from $\mathrm{Spec}\, Q_1$

Proof that $\|e^{-iQ}\| = \prod\{\mu_j^{1/4} \ : \ \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.
With this miracle in hand,

$$(e^{-iQ})^* e^{-iQ} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Defining q_1 so that $\exp H_{q_1} = \overline{\mathbf{K}}^{-1}\mathbf{K}$, we get

$$(e^{-iQ})^* e^{-iQ} = \pm e^{-iQ_1}.$$

Since this operator is real positive definite, we can take iQ_1 of harmonic oscillator type (iq_1 pos. def.)
We find $\|e^{-iQ}\|$ from $\|e^{-iQ_1}\|$from Spec $Q_1$from Spec H_{q_1}

Proof that $\|\mathrm{e}^{-\mathrm{i}Q}\| = \prod\{\mu_j^{1/4} \ : \ \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.

With this miracle in hand,

$$(\mathrm{e}^{-\mathrm{i}Q})^* \mathrm{e}^{-\mathrm{i}Q} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Defining q_1 so that $\exp H_{q_1} = \overline{\mathbf{K}}^{-1}\mathbf{K}$, we get

$$(\mathrm{e}^{-\mathrm{i}Q})^* \mathrm{e}^{-\mathrm{i}Q} = \pm \mathrm{e}^{-\mathrm{i}Q_1}.$$

Since this operator is real positive definite, we can take $\mathrm{i}Q_1$ of harmonic oscillator type ($\mathrm{i}q_1$ pos. def.)

We find $\|\mathrm{e}^{-\mathrm{i}Q}\|$ from $\|\mathrm{e}^{-\mathrm{i}Q_1}\|$from Spec $Q_1$from Spec $H_{q_1}$from Spec $\exp H_{q_1} = \mathrm{Spec}\,\overline{\mathbf{K}}^{-1}\mathbf{K}$.

Proof that $\|e^{-iQ}\| = \prod\{\mu_j^{1/4} \;:\; \mu_j \in \mathrm{Spec}(\overline{\mathbf{K}}^{-1}\mathbf{K}) \cap (0,1)\}$

Proof.
With this miracle in hand,

$$(e^{-iQ})^* e^{-iQ} \overset{\text{Egorov}}{\sim} \overline{\mathbf{K}}^{-1}\mathbf{K}.$$

Defining q_1 so that $\exp H_{q_1} = \overline{\mathbf{K}}^{-1}\mathbf{K}$, we get

$$(e^{-iQ})^* e^{-iQ} = \pm e^{-iQ_1}.$$

Since this operator is real positive definite, we can take iQ_1 of harmonic oscillator type (iq_1 pos. def.)
We find $\|e^{-iQ}\|$ from $\|e^{-iQ_1}\|$from $\mathrm{Spec}\, Q_1$from $\mathrm{Spec}\, H_{q_1}$from $\mathrm{Spec} \exp H_{q_1} = \mathrm{Spec}\, \overline{\mathbf{K}}^{-1}\mathbf{K}$. $\square$

Polynomials of degree 2

When $\mathbf{K}$ is strictly positive, it's enough to consider

$$p(x,\xi) = q((x,\xi) - \mathbf{v}), \quad \mathbf{v} \in \mathbb{C}^{2n}$$

to describe all lower-degree perturbations. With

$$\mathcal{S}_{(v_x, v_\xi)} u(x) = \mathrm{e}^{-\frac{\mathrm{i}}{2} v_x \cdot v_\xi + \mathrm{i} v_\xi \cdot x} u(x - v_x),$$

Egorov gives that

$$P = \mathcal{S}_{\mathbf{v}} Q \mathcal{S}_{\mathbf{v}}^{-1}.$$

Example:

$$\begin{aligned} P &= \frac{1}{2}(D^2 + (x - \mathrm{i})^2) \\ &= \mathcal{S}_{(\mathrm{i},0)} Q_0 \mathcal{S}_{(\mathrm{i},0)}^{-1}. \end{aligned}$$

The norm

Imitating the singular-value decomposition (but with symplectic linear algebra), we can find $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^{2n}$ such that

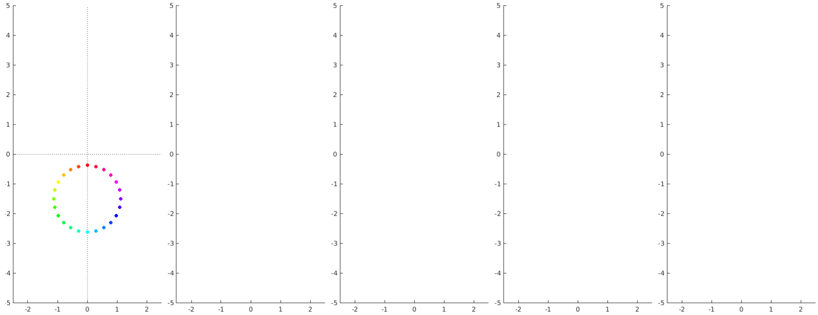
$$\mathrm{e}^{-\mathrm{i}P} = \mathcal{S}_{\mathbf{v}} \mathrm{e}^{-\mathrm{i}Q} \mathcal{S}_{\mathbf{v}}^{-1} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} \mathrm{e}^{-\mathrm{i}Q} \mathcal{S}_{\mathbf{a}_1}^{*}$$

Applying some half-Egorov theorems to Mehler formulas, we get equality up to a geometric correction factor:

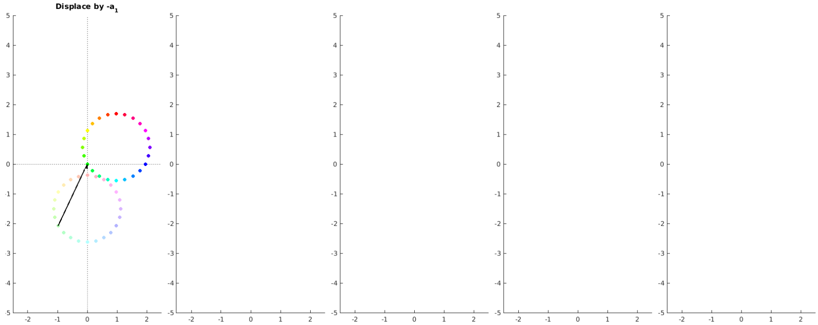
$$\mathrm{e}^{-\mathrm{i}P} = \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(\mathbf{a}_2 - \mathbf{a}_1, \mathbf{v})} \mathcal{S}_{\mathbf{a}_2} \mathrm{e}^{-\mathrm{i}Q} \mathcal{S}_{\mathbf{a}_1}^{*}.$$

Since the real shifts are unitary and we know $\|\mathrm{e}^{-\mathrm{i}Q}\|$, we get the norm.

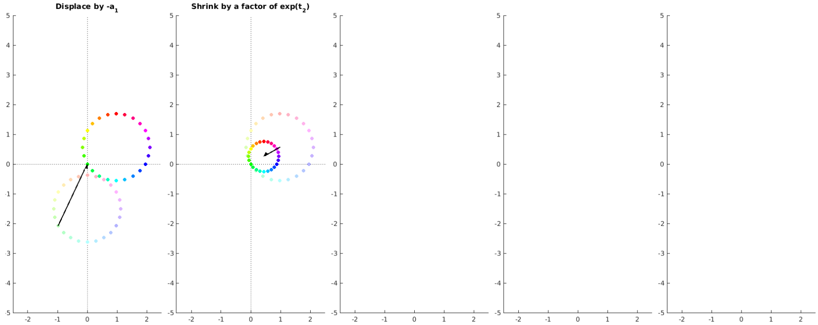
$$e^{-itP} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} e^{-it_1 Q_0} e^{-0.8 Q_0} \mathcal{S}_{\mathbf{a}_1}^*$$



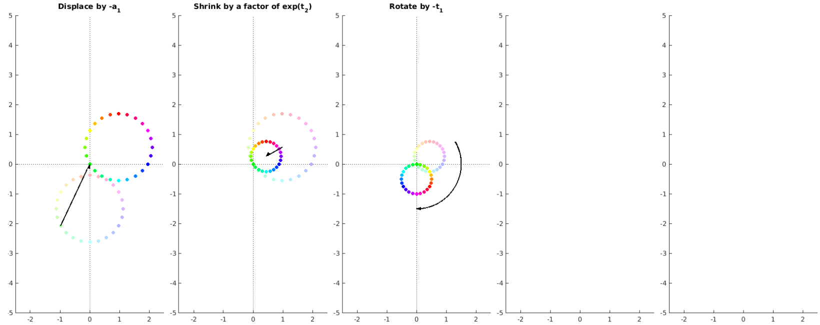
$$e^{-itP} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} e^{-it_1 Q_0} e^{-0.8 Q_0} \mathcal{S}^*_{\mathbf{a}_1}$$



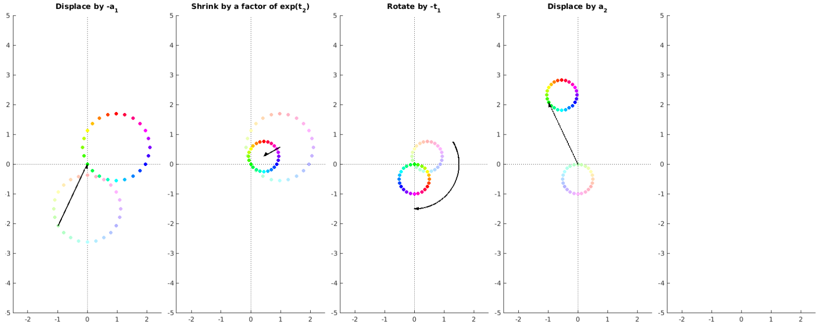
$$e^{-itP} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} e^{-it_1 Q_0} e^{-0.8 Q_0} \mathcal{S}^*_{\mathbf{a}_1}$$



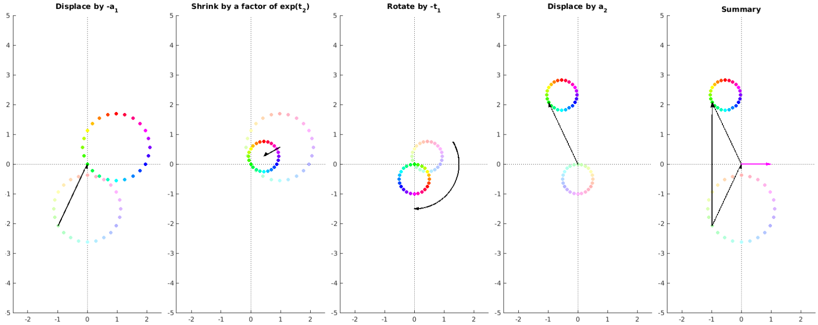
$$e^{-itP} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} e^{-it_1 Q_0} e^{-0.8 Q_0} \mathcal{S}^*_{\mathbf{a}_1}$$



$$e^{-itP} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} e^{-it_1 Q_0} e^{-0.8 Q_0} \mathcal{S}^*_{\mathbf{a}_1}$$



$$\mathrm{e}^{-\mathrm{i}tP} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} \mathrm{e}^{-\mathrm{i}t_1 Q_0} \mathrm{e}^{-0.8 Q_0} \mathcal{S}_{\mathbf{a}_1}^*$$

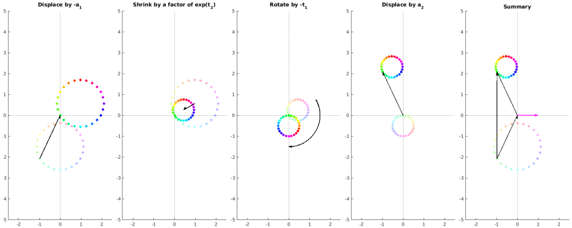


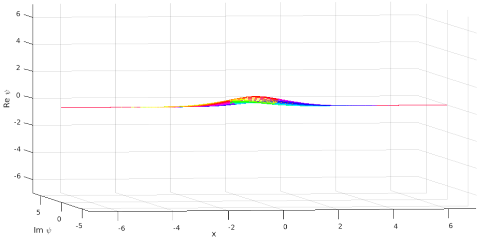
$$\mathrm{e}^{-\mathrm{i}tP} \overset{\text{Egorov}}{\sim} \mathcal{S}_{\mathbf{a}_2} \mathrm{e}^{-\mathrm{i}t_1 Q_0} \mathrm{e}^{-0.8 Q_0} \mathcal{S}_{\mathbf{a}_1}^*$$

Correction

Knowing the geometry gives a simple correction:

$$\mathrm{e}^{-\mathrm{i}P} = \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(\mathbf{v},\mathbf{a}_2-\mathbf{a}_1)}\mathcal{S}_{\mathbf{a}_2}\mathrm{e}^{-\mathrm{i}Q}\mathcal{S}^*_{\mathbf{a}_1}$$





Correction

Knowing the geometry gives a simple correction:

$$\mathrm{e}^{-\mathrm{i}P} = \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(\mathbf{v},\mathbf{a}_2-\mathbf{a}_1)}\mathcal{S}_{\mathbf{a}_2}\mathrm{e}^{-\mathrm{i}Q}\mathcal{S}^*_{\mathbf{a}_1}$$