On the spectral analysis of the Schrodinger operator with a periodic PT-symmetric potential

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I am going to give a talk about the one dimensional Schrödinger operator $L(q)$ generated in $L_2(-\infty, \infty)$ by the differential expression

$$-y''(x) + q(x)y(x), \tag{1}$$

where q is 1-periodic $q(x+1) = q(x)$, complex-valued and PT symmetric $\overline{q(-x)} = q(x)$ potential. It is well-known [Rofe-Beketov (1963), McGarvey (1965)] that the spectrum $\sigma(L(q))$ of the operator $L(q)$ is the union of the spectra $\sigma(L_t(q))$ of the operators $L_t(q)$ for $t \in (-\pi, \pi]$ generated in $L_2[0,1]$ by (1) and the boundary conditions

$$y(1) = e^{it}y(0), \;\; y'(1) = e^{it}y'(0). \tag{2}$$

The spectrum of L_t consists of the eigenvalues $\lambda_1(t)$, $\lambda_2(t), \ldots..$ such that $\lambda_n(-t) = \lambda_n(t)$ and $\lambda_n(t) \to \infty$ as $n \to \infty$. Moreover they can be numbered so that λ_n is a continuous function on $[0, \pi]$. Thus

$$\Gamma_n = \{\lambda_n(t) : t \in [0, \pi]\}$$

is a continuous curve in the complex plane and the spectrum of $L(q)$ is the union of the curves Γ_n for $n = 1, 2, \ldots$:

$$\sigma(L(q)) = \bigcup_{t\in[0,\pi]} \sigma(L_t(q)) = \bigcup_{n\in\mathbb{N}} \Gamma_n.$$

In the first papers [Bender et al (1999)] about the PT-symmetric periodic potential, the disappearance of real energy bands for some complex-valued PT-symmetric periodic potentials have been reported. Shin (2004) showed that the disappearance of such real energy bands implies the existence of nonreal band spectra. He involved some condition on the Hill discriminant to show the existence of nonreal curves in the spectrum. Caliceti and Graffi (2009) found explicit condition on the Fourier coefficient of the potential providing the nonreal spectra for small potentials.

I prove that the main part of the spectrum of $L(q)$ is real and contains the large part of $[0, \infty)$. However, in general, the spectrum contains also infinitely many nonreal acts. The necessary and sufficient condition on the potential for finiteness of the number of the nonreal arcs is determined. Moreover, I find necessary and sufficient conditions for the equality of the spectrum of $L(q)$ to the half line $[c, \infty)$. Then, I consider the connections between spectrality of $L(q)$ and the reality of its spectrum. Finally, I consider in detail the optical potential

$$4\cos^2 x + 4iV\sin 2x \tag{3}$$

The steps of my investigations are the followings:
1. General Properties of the spectrum of $L(q)$ with PT-symmetric potential q.
2. General necessary and sufficient conditions for finiteness of the number of the nonreal arcs in the spectrum $\sigma(L(q))$ and for $\sigma(L(q)) = [c, \infty)$
3. Necessary and sufficient condition on the potential.
4. The connections between reality of $\sigma(L(q))$ and spectrality of $L(q)$.
5. Detail investigations of the optical potential

General Properties of the spectrum of $L(q)$.

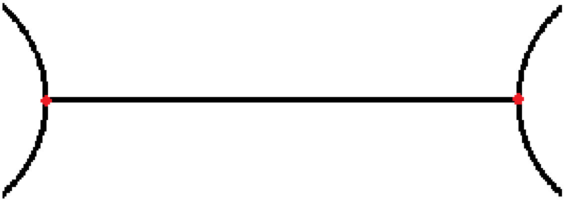
Theorem

(a) If $\lambda_n(t_1)$ and $\lambda_n(t_2)$ are real numbers, where $0 \leq t_1 < t_2 \leq \pi$ then $\gamma := \{\lambda_n(t) : t \in [t_1, t_2]\}$ is an interval of the real line.
(b) If the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi)$ are real numbers then $\lambda_n(t)$ are real eigenvalues of $L_t(q)$ for all $t \in (0, \pi)$, that is, Γ_n is either $[\lambda_n(0), \lambda_n(\pi)]$ or $[\lambda_n(\pi), \lambda_n(0)]$.
(c) The spectrum of $L(q)$ is completely real if and only if all eigenvalues of $L_0(q)$ and $L_\pi(q)$ are real.

Now to consider, the reality of the spectrum in detail, we investigate the points in which the spectrum ceases to be real. These points are crucial and can be defined as follows.

Definition

A real number $\lambda \in \sigma(L)$ is said to be a left (right) complexation point of the spectrum if there exists $\varepsilon > 0$ such that $[\lambda, \lambda + \varepsilon] \subset \sigma(L)$ ($[\lambda - \varepsilon, \lambda] \subset \sigma(L)$) and $\sigma(L)$ contains a nonreal number in any neighborhood of λ. Both left and right complexation points are called complexation points.



Pic. 1. Complexation points

Theorem

If $\lambda_n(t)$ is a complexation point, then it is a multiple eigenvalue of $L_t(q)$ and spectral singularities of $L(q)$. Moreover, the multiplicity of $\lambda_n(t)$ is greater than 2 if $t = 0, \pi$. In any case if $\lambda_n(t)$ is a complexation point then the eigenspace corresponding to $\lambda_n(t)$ contains an associated function.

The spectral singularities was defined as follows.

Definition

We say that $\lambda \in \sigma(L(q))$ is a spectral singularity of $L(q)$ if for all $\varepsilon > 0$ there exists a sequence $\{\gamma_n\}$ of the arcs $\gamma_n \subset \{z \in \mathbb{C} :| z - \lambda |< \varepsilon\}$ such that $\gamma_n \subset \sigma(L)$ and

$$\lim_{n\to\infty} \| P(\gamma_n) \| = \infty. \tag{4}$$

Theorem

Suppose that n is a large number.
(a) There may exists at most one number ε_n and $\pi - \delta_n$ in the neighborhoods of 0 and π respectively, such that $\lambda_n(\varepsilon_n)$ and $\lambda_n(\pi - \delta_n)$ are double eigenvalues.
(b) The eigenvalues $\lambda_n(t)$ for $t \neq [0, \pi] \setminus \{\varepsilon_n, \pi - \delta_n\}$ are simple and are not the complexation points. Moreover $\varepsilon_n \to 0$ and $\delta_n \to 0$ as $n \to \infty$.
(c) The double eigenvalues $\lambda_n(\varepsilon_n)$ and $\lambda_n(\pi - \delta_n)$ are the complexation points if and only if $\varepsilon_n \neq 0$ and $\delta_n \neq 0$ respectively.
(d) The main part $\{\lambda_n(t) : t \in [\varepsilon_n, \pi - \delta_n]\}$ of Γ_n is real. The other parts $\{\lambda_n(t) : t \in [0, \varepsilon_n)\}$ and $\{\lambda_n(t) : t \in (\pi - \delta_n, \pi]\}$ (if exist) are the pure nonreal parts of Γ_n and are called the tails of Γ_n.

Pic. 2, Γ_n for large n.

General necessary and sufficient condition

Theorem

If n is a large number, then the component Γ_n has left (right) nonreal tail, that is, it contains a left (right) complexation points point if and only if $\lambda_n(0)$ $(\lambda_n(\pi))$ is a nonreal number.

Theorem

If n is a large number, then the followings are equivalent
(a) Component Γ_n is the real interval.
(b) The eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi)$ are the real numbers.
(c) $\lambda_n(t)$ for $t \in (0, \pi)$ are simple eigenvalues of $L_t(q)$.
(d) The eigenvalues $\lambda_n(t)$ for $t \in (0, \pi)$ are not the complexation point.

Theorem

The spectrum of $L(q)$ is a half line if and only if the followings hold: (i) one eigenvalue of $L_0(q)$ is simple and the all others are double, (ii) all eigenvalues of $L_\pi(q)$ are double.

Necessary and sufficient condition on the potential for the reality of Γ_n for large n.

Let S_p be the set of 1 periodic PT-symmetric functions $q \in W_1^p[0,1]$ such that

$$q(1) = q(0),\ q'(1) = q'(0), \ldots,\ q^{(s-1)}(1) = q^{(s-1)}(0) \tag{5}$$

for some $s \leq p$ and there exist positive constants c_1, c_2, c_3 and N satisfying

$$\mid q_n \mid > c_1 n^{-s-1} \ \& \ c_2 \ |q_n| \leq |q_{-n}| \leq c_3 |q_n|, \ \forall n > N,$$

where $q_n = \int_0^1 q(x) e^{-i2\pi x} dx$. In particular, S_0 is the set of 1 periodic PT-symmetric functions $q \in L_1[0,1]$ satisfying

$$\mid q_n \mid > c_1 n^{-1} \ \& \ c_2 \ |q_n| \leq |q_{-n}| \leq c_3 |q_n|, \ \forall n > N.$$

Theorem

Suppose that $q \in S_p$ and n is a large number. Then $\Gamma_n \subset \mathbb{R}$ if and only if

$$q_n q_{-n} > 0. \tag{6}$$

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Let Q_n and S_n be the Fourier coefficients of the function Q and S defined by

$$Q(x) = \int_0^x q(t)\,dt, \quad S(x) = Q^2(x).$$

Suppose that equality (5) and the inequality

$$|P_n| > cn^{-2s-2}, \tag{7}$$

holds, where $P_n = q_n q_{-n} - q_n\left(S_{-n} - 2Q_0 Q_{-n}\right) - q_{-n}(S_n - 2Q_0 Q_n)$.

Theorem

If n is a large number and $P_n < 0$ then Γ_n has the nonreal tails.

The connections between reality of $\sigma(L(q))$ and spectrality of $L(q)$.

Theorem

(a) Let n be a large number and $t \in (0, \pi)$. Then the followings are equivalent.
1) $\lambda_n(t)$ is a spectral singularity of $L(q)$.
2) $\lambda_n(t)$ is a complexation points point of $\sigma(L(q))$.
3) $\lambda_n(t)$ is a multiple eigenvalue of $L_t(q)$.
4) If $t \in (0, h]$, where h is a small number, then $\{\lambda_n(s) : s \in [0, t)\}$; if $t \in [\pi - h, \pi)$ then $\{\lambda_n(s) : s \in (t, \pi]\}$ is a nonreal tail of Γ_n.
(b) Let $q \in S_p$ for some $p = 0, 1, ...$ Then the followings are equivalent.
1) $L(q)$ is an asymptotically spectral operator.
2) There exists a large number m such that (6) holds for all $|n| > m$.
3) There exist a large number m such that Γ_n for $n > m$ are real pairwise disjoint intervals.

Detail investigations of the optical potential

$$4\cos^2 x + 4iV\sin 2x = 2 + (1+2V)e^{i2x} + (1-2V)e^{-i2x}, \;\; V \geq 0. \quad (3)$$

For the first time, the explanation of the nonreality of the spectrum of $L(q)$, redenoted by $L(V)$, for $V > 0.5$ was done by Makris et al (2008). For $V = 0,85$ they sketch the real and imaginary parts of the first two bands by using the numerical methods. Midya et al (2010) reduce the operator $L(q)$ with potential (3) to the Mathier operator and using the tabular values establish that there is second critical point $V_2 \sim 0.888437$ after which no part of the first and second bands remains real.
Note that the corresponding Mathier operator in the case $0 \leq V < 1/2$ is self-adjoint and hence the spectrum consists of the real intervals. The case $V = 1/2$ for the first time is considered by Gasimov (1980) and it was proved that the spectrum is $[0, \infty)$.

Thus the shapes of the spectrum of $L(V)$ for $0 \leq V \leq 1/2$ are well-known

Pic. 3. $0 \leq V < 1/2$. As in the self-adjoint Mathier operator

Pic.4. $V = 1/2$. All periodic (except first) and antiperiodic eigenvalues are double and spectral singularities of $L(q)$.

Note that the reality of the spectrum for $0 \leq V \leq 1/2$ also follows from the following. I proved that if $ab = cd$, then

$$\sigma(L(ae^{-i2x} + be^{i2x})) = \sigma(L(ce^{-i2x} + de^{i2x})$$

Therefore

$$\sigma(L((1+2V)e^{i2x} + (1-2V)e^{-i2x}) = \sigma(L(2c\cos 2x)), \tag{8}$$

where $c = \sqrt{1-4V^2}$

Theorem

The PT-symmetric operator $L(ae^{-i2x} + be^{i2x})$ is a spectral operator if and only if $a = b$, that is, $q(x)$ is the real potential $2a\cos 2x$. The optical operator with potential (3) is spectral if and only if $V = 0$.

I give a complete description, provided with a mathematical proof, of the shape of the spectrum of the Hill operator $L(V)$ with potential (3), when V changes from $1/2$ to $\sqrt{5}/2$, that is, $c = ir$ and $r \in (0, 2)$.

For this first we prove the followings

Theorem

If $1/2 < V < \sqrt{5}/2$, then all antiperiodic eigenvalues are nonreal and simple.

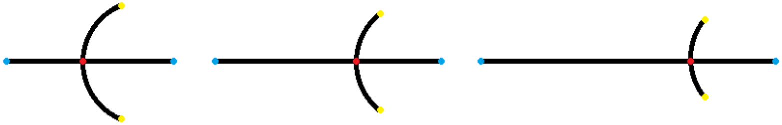
Theorem

If $1/2 < V < \sqrt{5}/2$, then all periodic eigenvalues (except first and second) are real and simple.

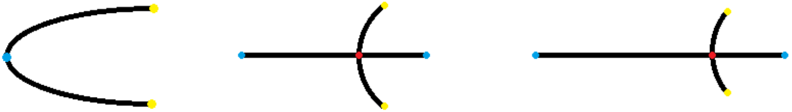
Theorem

There exists unique number V_2 from $(1/2, \sqrt{5}/2)$ such that
(a) If $1/2 < V < V_2$ then the first and second eigenvalues are real and simple.
(b) If $V = V_2$, then $\lambda_1(0) = \lambda_2(0) \in \mathbb{R}$
(c) If $V_2 < V < \sqrt{5}/2$ then the first and second periodic eigenvalues $\lambda_1(0)$ and $\lambda_2(0)$ are simple and nonreal and $\lambda_2(0) = \overline{\lambda_1(0)}$.

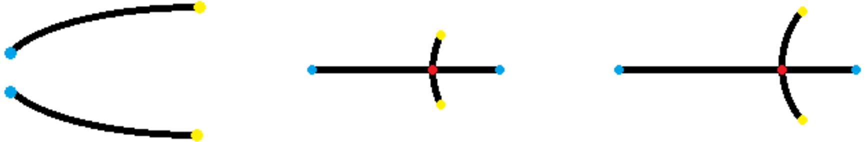
I prove that V_2 is the second critical point (after which no part of the first and second bands remains real) and is a number between 0.8884370025 and 0.8884370117. Moreover, the last theorem shows that V_2 is the unique degeneration point for the first periodic eigenvalue, in the sense that the first periodic eigenvalue of the potential (3) is simple for all $V \in [0, \sqrt{5}/2) \setminus \{V_2\}$ and is double if $V = V_2$. My approach give the possibility to find the arbitrary close values of the k-th critical point V_k and prove that no part of the $(2k-3)$-th and $(2k-2)$-th bands remains real for $V_k < V < V_k + \varepsilon$ for some positive ε, where $k = 2, 3, ...$ Using the last theorems we prove that the spectrum of $L(V)$ has the following shape



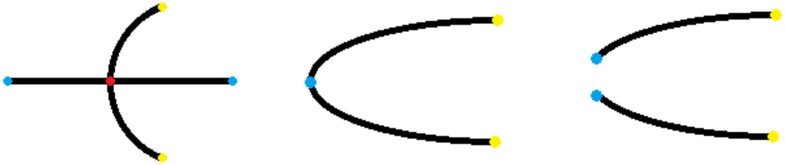
Pic. 5. $1/2 < V < V_2$. All antiperiodic eigenvalues are nonreal and periodic eigenvalues are real



Pic. 6. $V = V_2$.



Pic. 7. $V_2 < V < \sqrt{5}/2$. All periodic eigenvalues (except first and second) are real, the first and second periodic eigenvalues $\lambda_1(0)$ and $\lambda_2(0)$ are simple and nonreal and $\lambda_2(0) = \overline{\lambda_1(0)}$. Thus the first and second bands have different shapes in the following 3 cases:



Pic. 8. Case 1: $1/2 < V < V_2$, Case 2: $V = V_2$, Case 3: $V_2 < V < \sqrt{5}/2$.

In Case 1, the real part of the first component $\Omega_1 = \Gamma_1 \cup \Gamma_2$ is the closed interval $I_1 =: [\lambda_1(0), \lambda_2(0)]$. We prove that if V approaches V_2 from the left, then the eigenvalues $\lambda_1(0)$ and $\lambda_2(0)$ get close to each other and the length of the interval I_1 approaches zero.

As a result if $V = V_2$, then we get the equality $\lambda_1(0) = \lambda_2(0)$ which means that the first and second bands Γ_1 and Γ_2 have only one real point which is their common point

$$I_1 = \operatorname{Re}\Omega_1 = \operatorname{Re}\Gamma_1 = \operatorname{Re}\Gamma_2 = \{\lambda_1(0)\}.$$

The other parts of the bands Γ_1 and Γ_2 are nonreal and symmetric with respect to the real line. Then we prove that if $V > V_2$, then the eigenvalues $\lambda_1(0)$ and $\lambda_2(0)$ get off the real line and hence I_1 becomes the empty set. As a results, the first and second bands Γ_1 and Γ_2 became the nonreal curves symmetric with respect to the real line.

Definition

A real number $V_2 \in [0, \sqrt{5}/2)$ is called the second critical point or the degeneration point for the first periodic eigenvalue if the first real eigenvalue of $L_0(V_2)$ is a double eigenvalue or equivalently if the first real component of $\sigma(L(V_2))$ is a point.

Now let us describe briefly the shapes of all bands. I prove that the spectrum of $L(V)$ in Case 1 has the following properties:
Pr. 1. *The real part $\sigma(H(a)) \cap R$ of the spectrum of $H(a)$ consist of the intervals*

$$I_1 = [\lambda_1(0), \lambda_2(0)], \ I_2 = [\lambda_3(0), \lambda_4(0)], ..., I_n = [\lambda_{2n-1}(0), \lambda_{2n}(0)], ... \tag{9}$$

Pr. 2. *For each $n = 1, 2, ...,$ the interval I_n is the real part of* $\Omega_n =: \Gamma_{2n-1} \cup \Gamma_{2n}$.
Pr. 3. *The bands Γ_{2n-1} and Γ_{2n} have only one common point Λ_n which is interior point of I_n. Moreover, Λ_n is a double eigenvalue of $L_{t_n}(V)$ for some $t_n \in (0, \pi)$ and a spectral singularity of $L(V)$ and hence*

$$\Gamma_{2n-1} \cap \Gamma_{2n} = \Lambda_n = \lambda_{2n-1}(t_n) = \lambda_{2n}(t_n) \in \mathbb{R}.$$

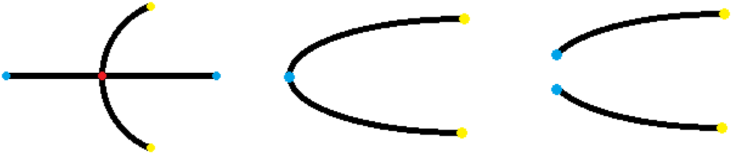
Pr. 4. *The real parts of the bands Γ_{2n-1} and Γ_{2n} are respectively the intervals* $[\lambda_{2n-1}(0), \Lambda_n] = \{\lambda_{2n-1}(t) : t \in [0, t_n]\}$ and $[\Lambda_n, \lambda_{2n}(0)] = \{\lambda_{2n}(t) : t \in [0, t_n]\}$

Pr. 5. *The nonreal parts of* Γ_{2n-1} *and* Γ_{2n} *are respectively the analytic curves*

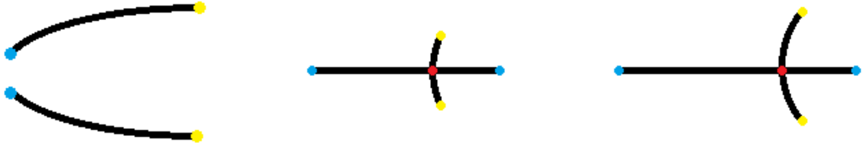
$$\gamma_{2n-1}(a) = \{\lambda_{2n-1}(t) : t \in (t_n, \pi]\} \ , \ \gamma_{2n}(a) = \{\lambda_{2n}(t) : t \in (t_n, \pi]\}$$

and $\gamma_{2n}(a) = \left\{\overline{\lambda} : \lambda \in \gamma_{2n-1}(a)\right\}$
Thus the bands Γ_{2n-1} and Γ_{2n} are joined by Λ_n and hence they form together the connected subset of the spectrum. The spectrum $\sigma(L(a))$ consist of the connected sets $\Omega_1 =: \Gamma_1 \cup \Gamma_2$, $\Omega_2 =: \Gamma_3 \cup \Gamma_4$, ...Moreover in Case 1 $(1/2 < V < V_2)$ we prove that
Pr. 6 *The sets* $\Omega_1, \Omega_2, ...$ *are connected separated subset of* the spectrum.
Moreover, in all cases the intervals (9) are pairwise disjoint sets. Therefore they are called the real components of $\sigma(L(V))$ if $V \in (1/2, \sqrt{5}/2)$.

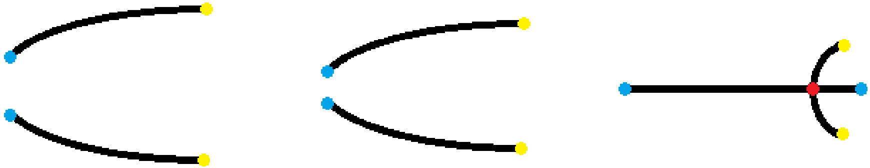
Note that in Case 2 ($V = V_2$) and Case 3 ($V_2 < V < \sqrt{5}/2$) the shapes of the components Ω_2, Ω_3, ... are as in Case 1. In this way one can prove that there exists k-th critical point, denoted by V_k, such that for $\frac{1}{2} < V < V_k$, $V = V_k$ and $V_k < V < V_k + \varepsilon$ the set $\Omega_{k-1} =: \Gamma_{2k-3} \cup \Gamma_{2k-2}$ have the shape as Ω_1 in Case 1, Case 2 and Case 3 respectively.



Pic.9 The shape of Ω_{k-1} in the cases $\frac{1}{2} < V < V_k$, $V = V_k$ and $V_k < V < V_{k+1}$



Pic.10. The shape of the spectrum for $V_2 < V < V_3$



Pic.11. The shape of the spectrum for $V_3 < V < V_4$

THANK YOU