Spectrum of $\mathcal{PT}$ symmetric operators

C. Trunk (TU Ilmenau)

joint work with F. Leben (TU Ilmenau)

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Bender, Böttcher, Physical Review Letters '98:

The Hamiltonian studied by Bessis is just one example of a huge and remarkable class of non-Hermitian Hamiltonians whose energy levels are real and positive. The purpose of this Letter is to understand the fundamental properties of such a theory by examining the class of quantum-mechanical Hamiltonians

$$H = p^2 + m^2 x^2 - (ix)^N \qquad (N \text{ real}) . \tag{1}$$

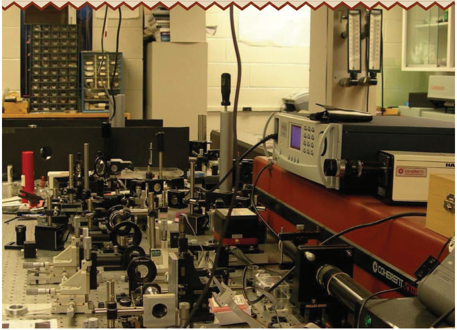
As a function of N and mass m^2 we find various phases with transition points at which entirely real spectra begin to develop complex eigenvalues.

Carl Bender in Europhysics News 2016:

PT SYMMETRY IN QUANTUM PHYSICS: FROM A MATHEMATICAL CURIOSITY TO OPTICAL EXPERIMENTS

Carl M. Bender – Washington University in St. Louis, St. Louis, MO 63130, USA – DOI: http://dx.doi.org/10.1051/epn/2016201

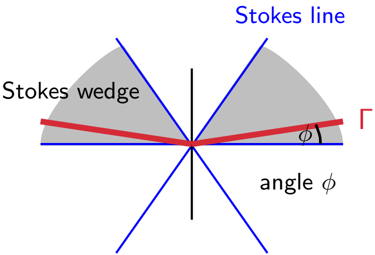
Space-time reflection symmetry, or PT symmetry, first proposed in quantum mechanics by Bender and Boettcher in 1998 [1], has become an active research area in fundamental physics. More than two thousand papers have been published on the subject and papers have appeared in two dozen categories of the arXiv. Over two dozen international conferences and symposia specifically devoted to PT symmetry have been held and many PhD theses have been written.



Aim: L^2 spectral th. for Bender-Böttcher potential

$$\ell(y) := -y''(z) - (iz)^{N+2} y(z), \quad N \in \mathbb{N}, \quad z \in \Gamma,$$

where Γ is a contour in two Sokes wedges. Today: Two half-rays.



Stokes lines: Argument in
$\{\frac{-N-2}{8+2N}\pi, \frac{-N+2}{8+2N}\pi, \frac{-N+6}{8+2N}\pi, \ldots\}$

Stokes wedges: Γ is in two wedges, sym. and tends $\to \infty$.

$$\boxed{\text{Bender, Böttcher: } \mathcal{PT}\ell = \ell\mathcal{PT} \text{ formally.}}$$

Recall: $(\mathcal{P}f)(z) = f(-\overline{z})$ and $(\mathcal{T}f)(z) = \overline{f(z)}$.

Back to the real line

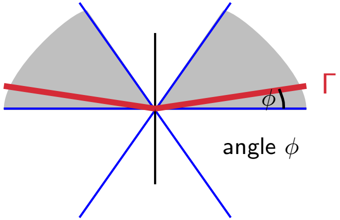
([Bender et al.'06], [Jones, Mateo '06], [Mostafazadeh '05 and '10])

$$-y''(z) - (iz)^{N+2} y(z) = \lambda y(z), \quad N \in \mathbb{N}, \quad z \in \Gamma. \tag{1}$$

Choose

$$\Gamma := \{x e^{i\phi \operatorname{sgn} x} : x \in \mathbb{R}\}.$$

Set $w(x) := y(z(x))$ with $\boxed{z(x) := x e^{i\phi \operatorname{sgn} x}.}$



Limit point/circle for Bender-Böttcher potential

Bender-Böttcher potential, mapped back to the real line:

$$\begin{cases} -w''(x) - (ix)^{N+2} e^{(N+4)i\phi} w(x), & x > 0, \qquad (2) \\ -w''(x) - (ix)^{N+2} e^{-(N+4)i\phi} w(x) & x < 0. \qquad (3) \end{cases}$$

According to the "Sims/BrownMcCormackEvansPlum alternative" and the Sibuya/Olver asymptotics:

Theorem (cf. Florian's talk)

- $\phi \notin \{\frac{-N-2}{8+2N}\pi, \frac{-N+2}{8+2N}\pi, \ldots\}$, *then Limit Point Case (I).*
- $\phi \in \{\frac{-N-2}{8+2N}\pi, \frac{-N+2}{8+2N}\pi, \ldots\}$, *then Limit Circle Case (III) .*

Which implies

$$\begin{aligned} \Gamma \text{ in Stokes wedge} \quad &\cong \quad \text{Limit Point Case (I)} \\ \Gamma \text{ in Stokes line} \quad &\cong \quad \text{Limit Circle Case (III)} \end{aligned}$$

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Which implies

$$\frac{\Gamma \text{ in Stokes wedge}}{\Gamma \text{ in Stokes line}} \cong \frac{\text{Limit Point Case (I)}}{\text{Limit Circle Case (III)}}$$

Full line operator A + conditions at zero

Define op. A for $\phi \notin \{\frac{-N-2}{8+2N}\pi, \frac{-N+2}{8+2N}\pi, \ldots\}$, Limit Point Case (I).

$$Aw := \begin{cases} -e^{-2i\phi}w''(x) - (ix)^{N+2}e^{(N+2)i\phi}w(x), & x > 0 \\ -e^{2i\phi}w''(x) - (ix)^{N+2}e^{-(N+2)i\phi}w(x), & x < 0 \end{cases}$$

with domain

$$\operatorname{dom} A := \left\{ w, Aw \in L^2(\mathbb{R}) : \begin{array}{c} w|_{\mathbb{R}^\pm}, w'|_{\mathbb{R}^\pm} \in AC(\mathbb{R}^\pm), \\ w(0+) = w(0-) \\ w'(0+) = \alpha w'(0-) \end{array} \right\}$$

Theorem (cf. Florian's talk)

- *y' from the originally problem is cont. if and only if $\alpha = e^{2i\phi}$.*
- *A is $\mathcal{PT}$-symmetric if and only if $|\alpha| = 1$.*
- *A is $[\cdot,\cdot]$-selfadjoint if and only if $\alpha = e^{4i\phi}$. Some mismatch.*

Recall: Selfadjointness in Krein spaces

$\mathcal{H}$ with a hermitian sesquilinear form $[\cdot,\cdot]$ is a *Krein space* if

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

and $(\mathcal{H}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces.

Here:

$$(L^2(\mathbb{R}), [\cdot,\cdot]) \quad \text{with} \quad [\cdot,\cdot] := (\mathcal{P}\,\cdot\,,\,\cdot\,)$$

is a Krein space.

- Define the *Adjoint* A^+ with respect to $[\cdot,\cdot]$.
- A $[\cdot,\cdot]$*-selfadjoint* if $A^+ = A$.

Full line operator A + conditions at zero

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- *A is $\mathcal{PT}$-symmetric if and only if $|\alpha| = 1$.*
- *A is $[\cdot,\cdot]$-selfadjoint $\Leftrightarrow$ $\alpha = e^{4i\phi}$.* $\boxed{\textit{From now on: } \alpha = e^{4i\phi}}$

Full line operator A: Essential spectrum

Theorem

Let $\alpha = e^{4i\phi}$. Then A is $\mathcal{PT}$-symmetric, $[\cdot,\cdot]$-selfadjoint, and

1. $\rho(A) \neq \emptyset$,
2. $\sigma_{ess}(A) = \emptyset$,
3. *Either $\sigma(A) = \emptyset$ or $\sigma(A)$ consists only of isolated eigenvalues acc. to ∞ with*

$$\dim \mathit{ker}\,(A - \lambda) = 1 \text{ (this is due to limit point).}$$

Key Idea: Use A_+ and A_- from Florian's talk (half-axis operators with Dirichlet b.c.) and transport their good properties to A via

$$\dim \left((A-\lambda)^{-1} - \begin{pmatrix} A_- - \lambda & 0 \\ 0 & A_+ - \lambda \end{pmatrix}^{-1} \right) = 1$$

and, hence $\sigma_{ess}(A) = \emptyset$

Eigenvalues of the full line operator A

Let w_+ (w_-) be a solution of the eigenvalue eq. on $\mathbb{R}^+$ (resp $\mathbb{R}^-$).

$$-e^{-2i\phi}w_+''(x) - (ix)^{N+2}e^{(N+2)i\phi}w_+(x) = \lambda w_+(x), \quad x > 0$$

$$-e^{2i\phi}w_-''(x) - (ix)^{N+2}e^{-(N+2)i\phi}w_-(x) = \lambda w_-(x), \quad x < 0$$

Lemma

Then $w_\pm$ are unique up to a constant due limit point case (I) and

$$\lambda \in \sigma_p(A) \quad \Leftrightarrow \quad \frac{w_+'(0)}{w_+(0)} = e^{4i\phi}\frac{w_-'(0)}{w_-(0)},$$

Proof.

$$w(x) := \begin{cases} w_+(x), & x > 0 \\ \frac{w_+(0)}{w_-(0)}w_-(x), & x < 0 \end{cases}$$

$\square$

Find enclosures for the eigenvalues of A

Utilize:

$$\lambda \in \sigma_p(A) \quad \Leftrightarrow \quad \frac{w_+'(0)}{w_+(0)} = e^{4i\phi} \frac{w_-'(0)}{w_-(0)}, \tag{2}$$

WKB-asymptotics: For a solution w of

$$w''(x) = (p(x) + q(x))\, w(x), \quad x > 0$$

we have

$$w(x) = p(x)^{1/4} \exp\left(\pm \int_0^x p(y)^{1/2} dy\right) (1 + R(x))$$

with

$$|R(x)| \leq \exp\left(\int_0^x |\mathcal{E}(y)| dy\right) - 1.$$

$\mathcal{E}$ contains p, q. In our case $p =$const and q contains λ. Observe: We can let $\lambda \to \infty$ and get an estimate for $w(0)$!. Apply to (2)

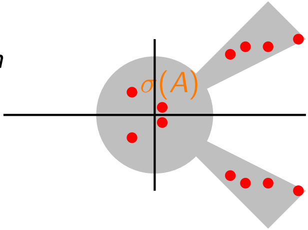
Main result: Spectrum of A

Theorem

Let $\alpha = e^{4i\phi}$. Then A is $\mathcal{PT}$-sym., $[\cdot,\cdot]$-s.a., and $\sigma_{ess}(A) = \emptyset$.
$\sigma(A) =$ eigenvalues in a circle and two small sectors acc. to ∞.

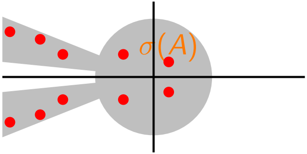
E.g., if $N = 2$ and $\phi = \frac{\pi}{12}$. Then $\sigma(A)$ is in a neighbourhood of

$$\{xe^{\pm\frac{\pi i}{3}} : x > M\} \cup K_M(0)$$



If $N = 4$ and $\phi = \frac{\pi}{7}$. Then $\sigma(A)$ is in a neighbourhood of

$$\{xe^{\pm\frac{6\pi i}{7}} : x > M\} \cup K_M(0)$$



Thank You !