On spectral divisors for a class of non-selfadjoint operator polynomials

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Joint work with Christian Engström

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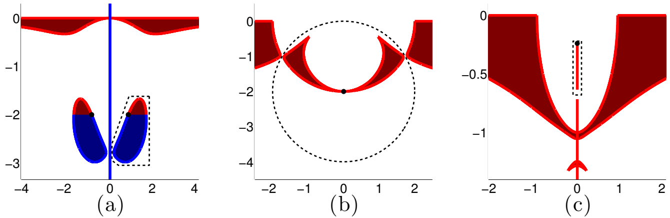
Main results

- Accumulation of a branch of eigenvalues of operator polynomials and minimality and completeness of the associated vectors.
- Sufficient conditions for accumulation for a rational operator function with applications to Maxwells equations.



Outline

The operator polynomial

- Let $n, k \in \mathbb{N}$, where $k \leqslant n$, and consider the polynomial

$$P(\lambda) := \sum_{i=0}^{n} P_i \lambda^i.$$

- $P_0 = (I + K)H$, where K is compact and $H \in S_p(\mathcal{H})$ is normal. (H has spectrum on a finite number of rays from the origin).
- P_i is compact and $\ker H \subset \ker P_i$ for $i = 1, \ldots, k-1$.
- $P_k = I + \widetilde{K}$, where $\widetilde{K}$ is compact.

Previous interest in the problem

- Pioneers in this area include Krein, Gohberg, Langer, Matsaev and Russo.
- In the case $\ker H = \{0\}$ and $k = 1$ this was studied in the book *Introduction to the spectral theory of polynomial operator pencils* by A. Markus.
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Spectral divisor

- Assume that there exists operator polynomials $Q(\cdot)$ and $R(\cdot)$ such that

$$P(\lambda) = Q(\lambda)R(\lambda) = \left(\sum_{i=0}^{n-k} \lambda^i Q_i\right)\left(\lambda^k + \sum_{i=0}^{k-1} \lambda^i R_i\right),$$

  and there exist some open bounded $0 \in \Gamma \subset \mathbb{C}$ such that $\sigma(R) = \Gamma \cap \sigma(P)$ and $\sigma(Q) \subset \mathbb{C} \backslash \Gamma$.
- Then R is called a spectral divisor of P of order k.
- From the definition of P and the spectral divisor it follows that:
  - i. $R_0 = (I + \widehat{K})H$, where $\widehat{K}$ is compact.
  - ii. R_i is compact and $\ker H \subset \ker R_i$ for $i = 1, \ldots, k-1$.
- P has a spectral divisor of order k if:
  - i. $\overline{W(P)} \cap \partial\Gamma = \varnothing$.
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How to deal with $\ker H \neq \{0\}$

- Define $\mathcal{H}_0 := \ker H$, $\mathcal{H}_1 := \mathcal{H} \perp \mathcal{H}_0$ and let $V_i^* : \mathcal{H} \to \mathcal{H}_i$ denote the partial isometry given by

$$V_i^* u = \begin{cases} u, & u \in \mathcal{H}_i \\ 0, & u \in \mathcal{H}_i^\perp \end{cases}.$$

- $\ker H \subset \ker R_i$ for $i < k$ thus, in the block representation $\mathcal{H} \simeq \mathcal{H}_1 \oplus \mathcal{H}_0$, the operator R_i for $i < k$ has the structure

$$R_i = \begin{bmatrix} C_i & 0 \\ D_i & 0 \end{bmatrix}.$$

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- R admits the factorization

$$R(\lambda) = \begin{bmatrix} C(\lambda) & 0 \\ \sum_{i=0}^{k-1} \lambda^i D_i & \lambda^k \end{bmatrix} = \begin{bmatrix} C(\lambda) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_1} & 0 \\ \sum_{i=0}^{k-1} \lambda^i D_i & \lambda^k \end{bmatrix},$$

  where $C(\lambda) := \lambda^k + \sum_{i=0}^{k-1} \lambda^i C_i$.
- R is said to be equivalent to C on $\mathbb{C}\setminus\{0\}$ after extension
  i. $\sigma(R)\setminus\{0\} = \sigma(C)\setminus\{0\}$.
  ii. There is a one-to-one-correspondence between the eigenvectors and associated vectors of R and of C.
- The operator polynomial C satisfies:
  i. $C_0 = (I + K_1)H_1$, where K_1 is compact and $\ker H_1 = \{0\}$,
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- Assume $k = 1$, then if $I + K_1$ is invertible it follows that $C(\lambda) = \lambda + (I + K_1)H_1$ and the eigenvectors and associated vectors in $\mathcal{H}_1$ are complete and minimal.
- It follows that the eigenvectors and associated vectors to $\sigma(P) \cap \Gamma$ are complete and minimal in $\mathcal{H}$.
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How to deal with $k > 1$

- If $k > 1$, then $C(\lambda)$ is a compact perturbation of

$$\widetilde{C}(\lambda) := \lambda^k - (I + K_1)H_1,$$

which has a complete set of associated vectors.

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The studied rational operator function

$$T(\omega) := A - \omega^2 - \frac{\omega^2}{c - \mathrm{i}d\omega - \omega^2} B, \quad \mathrm{dom}\, T(\omega) = \mathrm{dom}\, A,$$

where $\omega \in \mathcal{C} := \mathbb{C} \backslash \{\delta_+, \delta_-\}, \quad \theta := \sqrt{c - \frac{d^2}{4}}, \quad \delta_\pm := \pm\theta - \mathrm{i}\frac{d}{2}.$

- $T(\omega)$ linear operator in a Hilbert space $\mathcal{H}$.
- A is a self-adjoint operator in $\mathcal{H}$. Further $\overline{W(A)} \subset (\alpha_0, \infty)$ and the resolvent of A is in $S_p(\mathcal{H})$ for some p.
- $B \geqslant 0$ is a bounded self-adjoint operator in $\mathcal{H}$.
- The constants satisfy $c \geqslant 0$ and $d > 0$.

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Obtaining bounded polynomial

- Define the "equivalent" bounded polynomial $P(\cdot)$ as

$$P(\lambda) := \lambda\left(\lambda + \sqrt{4c - d^2}\right)(A - \alpha_0)^{-\frac{1}{2}}\,T(\lambda + \delta_+)(A - \alpha_0)^{-\frac{1}{2}}.$$

- $P(\lambda)$ is on the desired form where $k = 1$ if $d \neq 2\sqrt{c}$ and $k = 2$ if $d = 2\sqrt{c}$.
- P has a spectral divisor of order k if there is some open bounded $\Gamma \in \mathbb{C}$ such that and $0 \in \Gamma$ and
  i. $\overline{W(P)} \cap \partial\Gamma = \varnothing$.
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Definition of the enclosure of $W(T)$

- We can in general not compute $W(T)$. However, we can derive a computable enclosure of $\overline{W(T)}$.
- Define for $u \in \operatorname{dom} T$ the functions,

$$t_{\alpha_u,\beta_u}(\omega) := \frac{(T(\omega)u,u)}{\|u\|^2} = \alpha_u - \omega^2 - \frac{\omega^2}{c - id\omega - \omega^2}\beta_u.$$

- Let $\Omega := \overline{W(A)} \times \overline{W(B)}$ and define the enclosure of $\overline{W(T)}$

$$W_\Omega(T) := \{\delta_+, \delta_-\} \cup \{\omega \in \mathbb{C}\backslash\{\delta_+, \delta_-\} : t_{\alpha,\beta}(\omega) = 0 : (\alpha,\beta) \in \Omega\}.$$

- Hence, what we want to show is:
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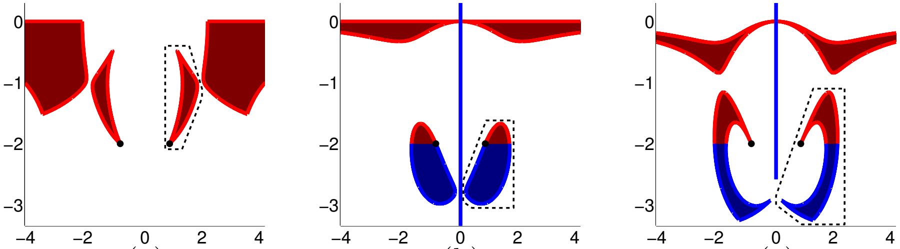
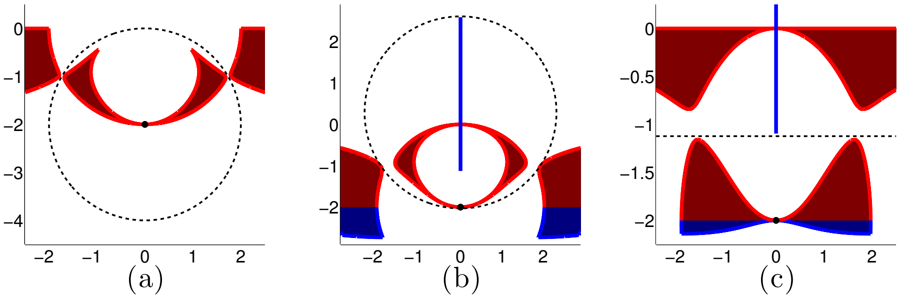


(a) (b) (c)

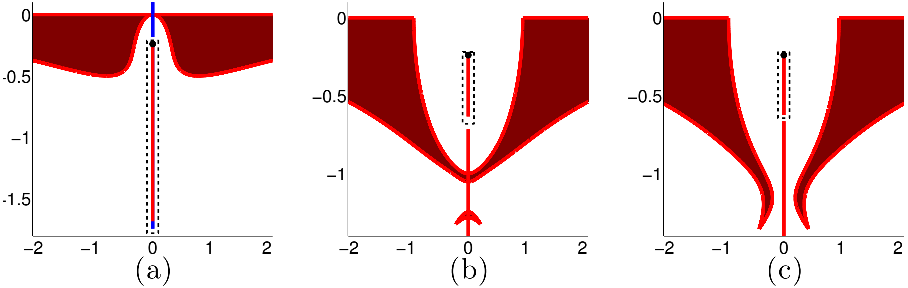
Examples when there is a accumulation to δ_+, (and to δ_- due to symmetry with respect to the imaginary axis).

$d = 2\sqrt{c}$



Examples when there is a accumulation to $-id/2 = \delta_+ = \delta_-$.

$d > 2\sqrt{c}$



Examples when there is a accumulation to δ_+. However, in (b) we can not guarantee an accumulation to δ_-.

Final thoughts and generalizations

- Several rational terms are common in models of electromagnetic materials:

$$T(\omega) = A - \omega^2 - \sum_{j=1}^{m} \frac{\omega^2}{c_j - \mathrm{i}d_j\omega - \omega^2} B_j \quad \mathrm{dom}\ T(\omega) = \mathrm{dom}\, A.$$

- Proof of accumulation can be done similarly.
- However, finding the lowest possible lower bound of A for which we can guarantee the existence of spectral divisors, will be extremely technical.

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Thank you for your attention

Excellence, and why it matter.