Block-Diagonalization of unbounded operator matrices

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Joint works with K. A. Makarov, A. Seelmann;
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- Diagonalization of Operator matrices

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- Diagonalization of Operator matrices
- Graphene

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- Diagonalization of Operator matrices
- Graphene
- Diagonalization of Operator matrices via forms

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- Diagonalization of Operator matrices
- Graphene
- Diagonalization of Operator matrices via forms
- The Stokes operator

Diagonalization

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ Hilbert space, $B = \begin{pmatrix} A_0 & W_1 \\ W_0 & A_1 \end{pmatrix} = A + V$,

A self-adjoint.

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1. Complementary invariant graph subspaces $\mathcal{H} = \mathcal{G}_0 + \mathcal{G}_1$
   $\mathcal{G}_0 = \{f + X_0 f \mid f \in \mathcal{H}_0\}, \quad \mathcal{G}_1 = \{X_1 g + g \mid g \in \mathcal{H}_1\}$
   $X_0 \colon \mathcal{H}_0 \to \mathcal{H}_1$, $X_1 \colon \mathcal{H}_1 \to \mathcal{H}_0$ bounded operator.

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Main Theorem [MSS 16]

Set $Y = \begin{pmatrix} 0 & X_1 \\ X_0 & 0 \end{pmatrix}$, $\mathcal{D} = \{f \in \mathrm{Dom}(B) \mid Yf \in \mathrm{Dom}(B)\}$

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$\Rightarrow$ Block-diagonalization

Diagonlizations

Riccati equation

$\mathcal{G}_0$ and $\mathcal{G}_1$ invariant for $A + V \iff Y$ satisfies

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What is $\mathcal{D}$? Operator equality?

Operator Inclusions

$\mathcal{G}_0$, $\mathcal{G}_1$ invariant for B

- $\mathrm{Dom}(B) = (\mathrm{Dom}(B) \cap \mathcal{G}_0) + (\mathrm{Dom}(B) \cap \mathcal{G}_1)$
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One inclusion implies the other if $A + V$ and $A - YV$ have a common point in the resolvent set.

Alternative Diagonalization

Riccati equation

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can be written as

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Two Diagonalizations [MSS]

Theorem 2

If Dom(A) $\subset$ Dom(V) (diagonal dominant), $\mathcal{G}_{0,1}$ invariant graph subspaces for $B = A + V$ and B and $A - YV$ closed with common point λ in the resolvent,

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is a graph subspace that reduces B. X contraction.

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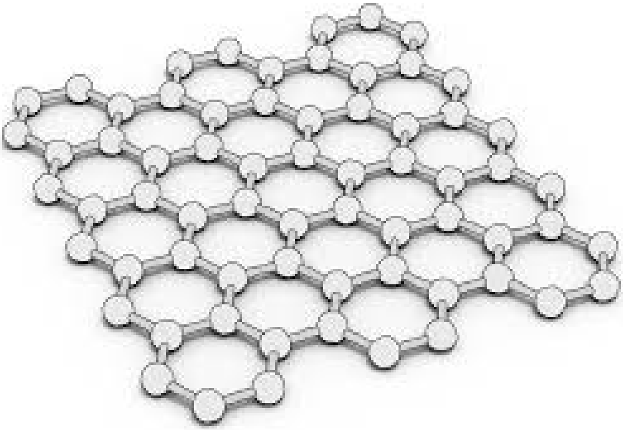
both diagonalizations hold and $A + VY, A - YV$ are mutually adjoint.

Grapene



Two dimensional structure of carbon
Survey: Geim, Novoselov 2007.

Graphene

Hamiltonian for massless Dirac fermions in the presence of an impurity in graphene, 2 dimensional single layer structure

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$\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$, with σ_x, σ_y the 2×2 Pauli matrices

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$\hbar \nu_F := 1$. Not diagonal dominant

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Θ Fourier multiplier with the unimodular symbol $\theta(\mathbf{k})$.
Subordinated spectra if U is a compactly supported bounded potential with $\|U\|_\infty$ small enough.

How to diagonalize upper-dominant Matrices? [GKMSV]

$$S = \begin{pmatrix} A_+ & W^* \\ W & -A_- \end{pmatrix} = A + V \quad \text{on } \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

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Upper dominant Matrix! Diagonalize the form $\mathfrak{s} = \mathfrak{a} + \mathfrak{v}$ on

$$\mathrm{Dom}[\mathfrak{s}] = \mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}(|A|^{1/2}),$$

$$\mathfrak{a}[x, y] = \langle |A|^{1/2} x, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} |A|^{1/2} y \rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}(|A|^{1/2}),$$

How to diagonalize upper-dominant Matrices? [GKMSV]

$$S = \begin{pmatrix} A_+ & W^* \\ W & -A_- \end{pmatrix} = A + V \quad \text{on } \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

$A_\pm \geq 0$ in $\mathcal{H}_\pm$ and $\mathrm{Dom}(A_+^{1/2}) \subset \mathrm{Dom}(W)$.
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and

$$\mathfrak{v}[x,y] = \langle W x_+, y_- \rangle + \langle x_-, W y_+ \rangle \quad \text{on} \quad \mathrm{Dom}[\mathfrak{v}] = \mathrm{Dom}[\mathfrak{a}],$$

$$x = x_+ \oplus x_-, \quad y = y_+ \oplus y_-, \quad x_\pm, y_\pm \in \mathrm{Dom}(|A_\pm|^{1/2}) \subset \mathcal{H}_\pm \,.$$

Diagonalization of forms

The form $\mathfrak{s}$ is associated with a unique self-adjoint operator S, $\mathrm{Dom}(S) \subseteq \mathrm{Dom}(|A|^{1/2})$,

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One-To-One correspondence between these forms and operators.

$$\mathcal{L}_\pm = \mathrm{Ran}\,\mathsf{E}_S(\mathbb{R}_\pm \setminus \{0\}) \oplus \big(\,\mathrm{Ker}(S) \cap \mathcal{H}_\pm\,\big)$$

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are reducing graph subspaces of contractions $X, -X^*$ for the form and the Operator. $\to$ Unitary block diagonalization for S if $I \pm Y$, $|I \pm Y|$ bijective on $\mathrm{Dom}[\mathfrak{s}]$.

Stokes Operator on Lipschitz domain Ω.

Stationary Stokes System: Slow flow of incompressible viscous fluid

$$-\mathbf{\Delta} u + \operatorname{grad} p = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0.$$

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using the semibounded form

$$\mathfrak{b}[u \oplus p] = \sum_{j=1}^{n} \int_\Omega |D_j u(x)|^2 dx + 2\mathrm{Re} \int_\Omega p(x) \overline{\mathrm{div}\, u(x)} dx.$$

Spectral Block-Diagonalisation of S

Lemma [*GKMSV*]

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Form variant of the Operator Riccati Equation.

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Theorem 3: Unitary Spectral Block-Diagonalisation [*GKMSV*]

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3. $S_- \geq -1$ sharp estimate.

Physical interpretation:

Introduce viscosity ν, characteristic velocity v_* and rescale

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$\rightarrow$ Geometric interpretation of Ladyzhenskaya's Stability Result

Ladyzhenskaya:

v^s stationary solution of the 2D-Navier-Stokes equation

$$\frac{\partial}{\partial t} v_s + (v^s \cdot \mathrm{grad}) v^s - \nu \boldsymbol{\Delta} v^s + \frac{1}{\rho} \mathrm{grad}\, p = f, \quad \mathrm{div} v^s = 0$$

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Thank You!