Weyl functions and Darboux transformations for non-self-adjoint systems

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We consider Dirac systems, skew-self-adjoint Dirac systems and dynamical Dirac-Weyl systems as non-self-adjoint examples.

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The physically important Dirac-type (or simply Dirac) system is given by the equation

$$\frac{d}{dx}y(x,z) = \mathrm{i}(zj + jV(x))y(x,z) \quad (x \geq 0), \tag{1}$$

where $V(x)$ is an $m \times m$ locally summable matrix function,

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \quad m_1 + m_2 =: m, \tag{2}$$

I_{m_k} is the $m_k \times m_k$ identity matrix and $v(x)$ is an $m_1 \times m_2$ m.-f.

The cases $m_1 = m_2 = 1$ and $m_1 = m_2 \geq 1$ are well-known. In those cases, there is a self-adjoint Dirac operator corresponding to the system (1) and the spectral function τ of this operator in uniquely determined by the Weyl function $\varphi(z)$, which belongs to Herglotz class.

Recall the formula (Herglotz-Nevanlinna representation):

$$\varphi(z) = \mu z + \nu + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\tau(t),$$

where $\mu \geq 0$, $\nu = \nu^*$, $\int_{-\infty}^{\infty}(1+t^2)^{-1}d\tau(t) < \infty$.

$$\text{Dirac system} \quad y' = \mathrm{i}\big(zj + jV(x)\big)y, \quad j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}.$$

When $m_1 \neq m_2$, the Dirac system above has deficiency indices m_1 and m_2, which means that the corresponding Dirac operator has not self-adjoint extensions.

However, the main ideas of the Weyl theory work in the case $m_1 \neq m_2$ as well, and we have generalized the main direct and inverse results for the case $m_1 \neq m_2$. These results are essential in the initial-boundary value problems for important nonlinear integrable wave equations.

The skew-self-adjoint Dirac system $y' = \big(\mathrm{i}zj + jV(x)\big)y$ is equally important for the integrable wave equations theory.

When $m_1 = m_2 = 1$, the corresponding Dirac operator H generated by the expression $-\mathrm{i}j\frac{d}{dx} + \mathrm{i}V$ is complex self-adjoint in the Krejcirik-Siegl terminology.

Namely, $H = J_c H J_c^{-1}$, where $J_c f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overline{f}$.

Inverse problems to recover system from the Weyl functions constitute the most complicated part of the Weyl theory.

The procedures *of general-type* and *of explicit solving* inverse problems are developed for Dirac systems.

We shall demonstrate our methods
by presenting general-type solution of the inverse problem
for the Dirac system $y' = \mathrm{i}\big(zj + jV(x)\big)y$,
and by presenting explicit solution of the inverse problem
for the skew-self-adjoint Dirac system.

Explicit methods (our GBDT approach) may be used also
for the construction of the explicit solutions
of the dynamical Dirac-Weyl system

$$\psi_x = \mathrm{i}j(-\psi_y + \mathrm{i}v(x)\sigma_2\psi), \quad v = \overline{v}, \quad \sigma_2 := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix},$$

where $m_1 = m_2 = 1$ and $\psi_x := \frac{\partial}{\partial x}\psi$.
Such systems are essential in the graphene theory.

The insertion and removal of non-real eigenvalues via GBDT is also of interest.

I. Dirac system:

$$y' = \mathrm{i}\big(zj + jV(x)\big)y, \quad j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}. \qquad (3)$$

Let $u(x, z)$ be the fundamental solution of (3) normalized by

$$u(0, z) = I_m, \qquad m = m_1 + m_2.$$

Definition 1. A Weyl (Weyl-Titchmarsh) function of Dirac system (3) on $[0, \infty)$, where the potential V is locally integrable, is an $m_2 \times \; m_1$ matrix function φ such that

$$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x, z)^* u(x, z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty, \quad z \in \mathbb{C}_+.$$

Note that the same definition of the Weyl function works for the skew-self-adjoint Dirac system.

Our basic formulas: Dirac system $y' = \mathrm{i}\big(zj + jV(x)\big)y$,

Weyl f-ns $\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x,z)^* u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty$.

Some connections with the previous talks. The interesting papers on NSA Dirac systems by Cuenin–Laptev–Tretter and by Cuenin–Siegl have been already mentioned yesterday.

Certain subclasses of Dirac systems are equivalent to Schrödinger equations.

Moreover, interesting interconnections between boundary problems and Weyl theory have been discussed in the previous talk. *We could add here* the interconnection (see A.S., J. Math. Phys, 2015) between the response function r of the dynamical Dirac

$$\mathrm{i}u_t + Ju_x + \mathcal{V}u = 0 \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} P & Q \\ Q & -P \end{bmatrix},$$

where $v(x) = \mathrm{i}Q(x) - P(x)$, and Weyl function for spectral case:

$$\varphi(z) = \widehat{r}(z)/(\widehat{r}(z) + 2\mathrm{i}), \quad \widehat{r}(z) = \int_0^\infty \mathrm{e}^{\mathrm{i}zt} r(t)dt.$$

Further we give complete description of the Weyl functions of Dirac systems $y' = \mathrm{i}\big(zj + jV(x)\big)y$ with locally square-integrable potentials V and solve the corresponding inverse problem.

The solution of the inverse problem with the only requirement on V to be locally square-integrable is new even in the case $m_1 = m_2$.

Theorem 2. Weyl function $\varphi(z)$ always exists in $\mathbb{C}_+$. It is unique, holomorphic and contractive (i.e., $\varphi(z)^*\varphi(z) \leq I_{m_1}$).

Moreover, when the potential V is locally square-integrable, Φ_1 given on $\mathbb{R}$ by

$$\Phi_1\Big(\frac{x}{2}\Big) = \frac{1}{\pi}\mathrm{e}^{x\eta}\mathrm{l.i.m.}_{a\to\infty}\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}x\zeta}\frac{\varphi(\zeta+\mathrm{i}\eta)}{2\mathrm{i}(\zeta+\mathrm{i}\eta)}d\zeta \quad (\eta>0) \quad (4)$$

is absolutely continuous, $\Phi_1(x) \equiv 0$ for $x \leq 0$, Φ_1' is locally square-integrable on $\mathbb{R}$, and the operators

$$S_\xi = I - \frac{1}{2}\int_0^\xi \int_{|x-t|}^{x+t} \Phi_1'\left(\frac{\zeta+x-t}{2}\right)\Phi_1'\left(\frac{\zeta+t-x}{2}\right)^* d\zeta \cdot dt$$

are positive and boundedly invertible in $L^2(0,\xi)$ $(0<\xi<\infty)$.

Theorem 3. The properties of the function $\varphi(z)$ described in Theorem 2 are not only necessary. They are also sufficient for φ to be the Weyl function of some Dirac system on $[0, \infty)$ such that the potential V of this Dirac system is locally square-integrable.

Note that the procedure to recover the potential $V(x)$ from the Weyl function $\varphi(z)$ is described by the formulas

$$V(x) = \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}, \quad v(x) = \mathrm{i}\beta'(x)j\gamma(x)^*.$$

Here β and γ are the blocks rows of $u(0,z)$, and $\beta(x)$ is given by

$$\beta(x) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} + \int_0^x \left(S_x^{-1}\Phi_1'\right)(t)^* \begin{bmatrix} \Phi_1(t) & I_{m_2} \end{bmatrix} dt.$$

The block row $\gamma(x)$ is uniquely recovered from $\beta(x)$ using the equalities:

$$\gamma(0) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix}, \quad \gamma' j \gamma^* \equiv 0, \quad \gamma j \beta^* \equiv 0.$$

Recall that Φ_1 and S_x were expressed via φ on the previous frame.

NB. It is of interest that Φ_1' is the Dirac system analog of the well-known A-amplitude introduced by B. Simon and coauthors.

The first procedure to recover the continuous potential of Dirac system from the spectral function was given in the seminal paper

M.G. Krein, *Continuous analogues of propositions on polynomials orthogonal on the unit circle* (Russian), Dokl. Akad. Nauk SSSR **105** (1955) 637–640.

(M.G. Krein considered the case $m_1 = m_2 = 1$, where a one to one correspondence between Weyl and spectral functions exists.)

The inverse problems for Dirac systems where m_1 does not necessarily equals m_2 and V are locally bounded were solved in

B. Fritzsche, B. Kirstein, I.Ya. Roitberg and A.L. Sakhnovich, *Recovery of Dirac system from the rectangular Weyl matrix function*, Inverse Problems **28** (2012), 015010, 18 p.

and in

B. Fritzsche, B. Kirstein, I.Ya. Roitberg and A.L. Sakhnovich, *Skew-self-adjoint Dirac systems with a rectangular matrix potential: Weyl theory, direct and inverse problems*, Integral Equations Operator Theory **74** (2012) 163–187.

For the results of this talk on the complete characterization of the Weyl f-ns (and solving of the inverse problem) for Dirac systems with locally square-summable potentials see :

1. A.L. Sakhnovich, *Inverse problem for Dirac systems with locally square-summable potentials and rectangular Weyl functions*, Journal of Spectral Theory **5**:3 (2015), 547–569;
2. A.L. Sakhnovich, *On accelerants and their analogs, and on the characterization of the rectangular Weyl functions for Dirac systems with locally square-integrable potentials on a semi-axis*, arXiv:1611.00550 (Oper. Theory Adv. Appl., volume dedicated to H. Langer to appear).

Applications of these results to Schrödinger-Type Operators with Distributional Matrix-Valued Potentials are given in

J. Eckhardt, F. Gesztesy, R. Nichols, A. Sakhnovich, and G. Teschl: *Inverse Spectral Problems for Schrödinger-Type Operators with Distributional Matrix-Valued Potentials*, Differential Integral Equations **28**: 5/6 (2015), 505–522.

Next, we discuss the explicit solving of inverse problems for *discrete and continuous skew-self-adjoint* Dirac systems.

See B. Fritzsche, M.A. Kaashoek, B. Kirstein, and A.S., Math. Nachr. **289:**14-15 (2016) 1792–1819.

See also:

a) I. Gohberg, M.A. Kaashoek, and A.S., J. Diff. Eqs **146** (1998);

b) M.A. Kaashoek and A.S., J. Funct. Anal. **228** (2005)

for the case $m_1 = m_2$.

Recall that skew-self-adjoint Dirac system has the form

$$y'(x,z) = (\mathrm{i}zj + jV(x))y(x,z), \qquad x \geq 0. \tag{5}$$

The fundamental solution of (5) (normalized by I_m at $x = 0$) is denoted by $u(x,z)$. The Weyl function $\varphi(z)$ is defined by the inequality which coincides with the inequality from Definition 1:

$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x,z)^* u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty.$

We assume that $v(x)$ is bounded, i.e. $\sup_{x\in[0,\infty)} \|v(x)\| \leq M$ and consider $\varphi(z)$ in the semi-plane $\mathbb{C}_M = \{z : \Im(z) > M\}$.

Then, the Weyl function $\varphi(z)$ exists and is unique. Moreover, $\varphi(z)$ is holomorphic and contractive in $\mathbb{C}_M$.

GBDT (generalized version of Darboux transformation).

Initial system: $y' = G(x,z)y, \quad G(x,z) = -\sum_{k=0}^{r} z^k q_k(x)$.
The initial fundamental solution is denoted by $u_0(x,z)$.

A triple of two $n \times n$ ($n \in \mathbb{N}$) matrices A and $S(0) > 0$, and of $n \times m$ matrix $\Lambda(0)$ such that

$$AS(0) - S(0)A^* = \mathrm{i}\Lambda(0)\mathcal{J}\Lambda(0)^*$$

determines a transformed system.
In the case of the skew-self-adjoint Dirac we have
$r = 1$, $q_1 \equiv -\mathrm{i}j$, $q_0(x) = V_0(x)j$, $\mathcal{J} = I_m$.
The matrix functions $\Lambda(x)$ and $S(x)$ are introduced by the equations:

$$\Lambda'(x) = A\Lambda(x)q_1 + \Lambda(x)q_0(x), \quad S'(x) = \Lambda(x)j\Lambda(x)^*.$$

Theorem 4. The *fundamental solution* $u(x,z)$ of the transformed system $y' = (\mathrm{i}zj + jV(x))y$, where
$V(x) = V_0(x) + \Lambda(x)^*S(x)^{-1}\Lambda(x) - j\Lambda(x)^*S(x)^{-1}\Lambda(x)j$,
has the form $u(x,z) = w_A(x,z)u_0(x,z)w_A(0,z)^{-1} \ \big(u(0,z) = I_m\big)$,
where the Darboux matrix w_A is given by

$$w_A(x,z) := I_m - \mathrm{i}\Lambda(x)^*S(x)^{-1}(A - zI_n)^{-1}\Lambda(x).$$

Recall basic formulas: $u(x,z) = w_A(x,z)u_0(x,z)w_A(0,z)^{-1}$,
$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x,z)^* u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty.$

Explicit solving of the direct problem.
When the initial system is trivial, i.e., $V_0(x) \equiv 0$ we have
$u_0(x,z) = \mathrm{e}^{\mathrm{i}zxj}, \quad u(x,z) = w_A(x,z)\mathrm{e}^{\mathrm{i}zxj}w_A(0,z)^{-1}$,
and $V(x)$ and w_A are expressed explicitly in terms of the triple
$\{A, S(0), \Lambda(0)\}$.
More precisely, we split $\Lambda(0)$ into the blocks $\Lambda(0) = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}$ and

$$V(x) = \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}, \quad v(x) = 2\vartheta_1^* \mathrm{e}^{\mathrm{i}xA^*} S(x)^{-1} \mathrm{e}^{\mathrm{i}xA}\vartheta_2, \qquad (6)$$

$$S(x) = S(0) + \int_0^x \Lambda(t) j \Lambda(t)^* dt, \quad \Lambda(x) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}xA}\vartheta_1 & \mathrm{e}^{\mathrm{i}xA}\vartheta_2 \end{bmatrix}.$$

The invertibility of $S(x)$ follows from $S(0) > 0$ and the identity
$AS(x) - S(x)A^* = \mathrm{i}\Lambda(x)j\Lambda(x)^*$.

Then, $\varphi(z)$ is a strictly proper rational function also expressed in terms of the triple (see the next frame).

Recall that $S(0) > 0$, $\Lambda(0) = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}$;

$$V(x) = \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}, \quad v(x) = 2\vartheta_1^* \mathrm{e}^{\mathrm{i}xA^*} S(x)^{-1} \mathrm{e}^{\mathrm{i}xA} \vartheta_2, \qquad (7)$$

$$S(x) = S(0) + \int_0^x \Lambda(t) j \Lambda(t)^* dt, \quad \Lambda(x) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}xA}\vartheta_1 & \mathrm{e}^{\mathrm{i}xA}\vartheta_2 \end{bmatrix}.$$

Theorem 5. Let the potential V of the system $y' = (\mathrm{i}zj + jV(x))y$ be given by (7).

Then, the Weyl function $\varphi(z)$ is a strictly proper rational $m_2 \times m_1$ matrix function given by the formula

$$\varphi(z) = \mathrm{i}\vartheta_2^* S(0)^{-1} (zI_n - \theta)^{-1} \vartheta_1, \quad \theta := A - \mathrm{i}\vartheta_1 \vartheta_1^* S(0)^{-1}. \qquad (8)$$

NB. All strictly proper rational $m_2 \times m_1$ matrix functions admit representation (8).

Theorem 6. Let $\varphi(z)$ be a strictly proper rational $m_2 \times m_1$ matrix function. Then, $\varphi(z)$ is the Weyl function of some system

$y' = (\mathrm{i}zj + jV(x))y.$

The potential $V(x)$ is uniquely recovered in two steps.

First step. Assuming that $\varphi(z) = \mathcal{C}(zI_n - \mathcal{A})^{-1}\mathcal{B}$ is a minimal realization of φ and choosing a positive solution $X > 0$ of

$$X\mathcal{C}^*\mathcal{C}X + \mathrm{i}(\mathcal{A}X - X\mathcal{A}^*) - \mathcal{B}\mathcal{B}^* = 0, \tag{9}$$

we put

$$A = \mathcal{A} + \mathrm{i}\mathcal{B}\mathcal{B}^*X^{-1}, \quad S_0 = X, \quad \vartheta_1 = \mathcal{B}, \quad \vartheta_2 = \mathrm{i}X\mathcal{C}^*. \tag{10}$$

Second step. Using the quadruple $\{A, S_0, \vartheta_1, \vartheta_2\}$ generate V:

$$V(x) = \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}, \quad v(x) = 2\vartheta_1^* \mathrm{e}^{\mathrm{i}xA^*} S(x)^{-1} \mathrm{e}^{\mathrm{i}xA} \vartheta_2, \tag{11}$$

$$S(x) = S(0) + \int_0^x \Lambda(t) j \Lambda(t)^* dt, \quad \Lambda(x) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}xA}\vartheta_1 & \mathrm{e}^{\mathrm{i}xA}\vartheta_2 \end{bmatrix}.$$

The procedure in Theorem 6 (explicit solution of the inverse problem) is stable.
See B. Fritzsche, B. Kirstein, I.Ya. Roitberg and A.S., arXiv:1510.00793 v.2.

Scheme of the proof. Using stability Theorem 5.4 in H. Langer, A.C.M. Ran, and D. Temme, LAA **261** (1997) and some results in Lancaster-Rodman book we show that there is a unique solution $X > 0$ of the Riccati eq-n on the previous frame and this solution is stable.

In the proof of stability of the second step (construction of V) we show that $V(x) \to 0$ $(x \to \infty)$, whereas only boundedness of V was proved in our earlier works on the skew-self-adjoint case.

We discussed earlier Darboux transformation for the skew-self-adjoint Dirac. An important intermediate relation here is

$$\left(\Lambda^* S^{-1}\right)' = \mathrm{i}j\Lambda^* S^{-1} A + jV\Lambda^* S^{-1}. \qquad (12)$$

Thus, for the eigenfunction f of A: $Af = \lambda f$ we have

$$\left(\Lambda(x)^* S(x)^{-1} f\right)' = \left(\mathrm{i}\lambda j + jV(x)\right)\Lambda(x)^* S(x)^{-1} f.$$

In this way, the eigenvalue λ is inserted under natural conditions. We may choose its algebraic multiplicity as well.

The same equation (12) is used in order to construct solutions

$$\psi(x) = \Lambda(x)^* S(x)^{-1} \mathrm{e}^{-yA}$$

of the Dirac-Weyl system

$$\psi_x = \mathrm{i}j(-\psi_y + \mathrm{i}v(x)\sigma_2\psi), \quad v = \overline{v}, \quad \sigma_2 := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}.$$

Discrete skew-selfadjoint Dirac system is given by the formula:

$$y_{k+1}(z) = \left(I_m + \frac{\mathrm{i}}{z} C_k\right) y_k(z), \quad C_k = U_k^* j U_k \quad (k \geq 0),$$

where the matrices U_k are unitary.

This system is an auxiliary linear system for the discrete isotropic Heisenberg magnet model and its generalization.

The procedure to recover the potential $\{C_k\}$ from the rational Weyl function is similar to the continuous case and is stable as well.

The same Riccati equation as in the continuous case appears in the first step of the procedure, namely

$$X\mathcal{C}^*\mathcal{C}X + \mathrm{i}(\mathcal{A}X - X\mathcal{A}^*) - \mathcal{B}\mathcal{B}^* = 0.$$

However, the proof of the second step of the procedure is more complicated than in the continuous case.