Non-accretive Schrödinger operators and exponential decay of their eigenfunctions

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(joint work with **D. Krejčiřík**, **J. Royer**, and **P. Siegl**)

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IRMAR

1 Context and motivations
- From the self-adjoint world...
- ... to the non-self-adjoint world

2 A representation theorem by Almog-Helffer
- How to define a nice operator?
- How to apply the theorem?

3 Spectrum and non-self-adjoint Agmon estimates
- Statements
- Ideas of the proofs

Let us consider the electromagnetic Schrödinger operator

$$\boxed{(-i\nabla + \mathbf{A})^2 + V} \qquad \text{in} \qquad L^2(\Omega)\,,$$

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subject to Dirichlet boundary conditions on $\partial\Omega$ and where

- Ω is an arbitrary open subset of $\mathbb{R}^d$,
- the functions $V : \Omega \to \boxed{\mathbb{C}}$ and $\mathbf{A} : \Omega \to \mathbb{R}^d$ are the scalar (electric) and (magnetic) vector potentials and they satisfy

$$(V, \mathbf{A}) \in \mathcal{C}^1(\bar{\Omega}; \mathbb{C}) \times \mathcal{C}^2(\bar{\Omega}; \mathbb{R}^d)\,.$$

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As in many talks of this conference, the important point is that V takes complex values.

The aim of this talk is to describe the spectrum of this operator.

An example that we can keep in mind is

$$\boxed{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2 + ix^3} \qquad \text{in} \qquad L^2(\mathbb{R}) \,.$$

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- is not bounded from below,
- has a non-empty resolvent set,
- has compact resolvent,
- has a numerical range equal to $\mathbb{C}$,

$$\{Q(u,u)\,, \quad u \in C_0^\infty(\mathbb{R})\,, \quad \|u\|_{L^2(\mathbb{R})} = 1\} = \mathbb{C}\,.$$

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- If $d = 3$ and if V is real-valued, the self-adjoint Dirichlet realisation is the Hamiltonian of a quantum particle constrained to a nanostructure Ω and submitted to an external electromagnetic field $(-\operatorname{grad} V, -\operatorname{rot} \mathbf{A})$.

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- If $d = 3$ and if V is real-valued, the self-adjoint Dirichlet realisation is the Hamiltonian of a quantum particle constrained to a nanostructure Ω and submitted to an external electromagnetic field $(-\operatorname{grad} V, -\operatorname{rot} \mathbf{A})$.
- The literature on the subject is enormous and we may consult, for instance, the bibliography of the recent book

  Bound States of the Magnetic Schrödinger Operator
  EMS Tracts (27) (2017),

  focused on magnetic spectral effects.

Why considering non-self-adjoint operators?

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If you want to learn how to live without the spectral theorem, you might be interested in

Elements of spectral theory without the spectral theorem

(**D. Krejčiřík** and **P. Siegl**, 2015).

A Lax-Milgram theorem

Theorem (Almog-Helffer, CPDE, 2015)

Let $\mathcal{V}$ be a Hilbert space. Let Q be a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$. Assume that there exist $\Phi_1, \Phi_2 \in \mathcal{L}(\mathcal{V})$ and $\alpha > 0$ such that for all $u \in \mathcal{V}$ we have

$$|Q(u,u)| + |Q(\Phi_1(u), u)| \geq \alpha \|u\|_{\mathcal{V}}^2\,,$$
$$|Q(u,u)| + |Q(u, \Phi_2(u))| \geq \alpha \|u\|_{\mathcal{V}}^2\,.$$

The operator $\mathscr{A}$ defined by

$$\forall u, v \in \mathcal{V}, \quad Q(u,v) = \langle \mathscr{A} u, v \rangle_{\mathcal{V}}$$

is a continuous isomorphism of $\mathcal{V}$ onto $\mathcal{V}$ with bounded inverse.

Theorem (Almog-Helffer, CPDE, 2015)

Assume moreover that H *is a Hilbert space such that* $\mathcal{V}$ *is continuously embedded and dense in* H *and that* Φ_1 *and* Φ_2 *extend to bounded operators on* H. *Then the operator* $\mathscr{L}$ *defined by*

$$\forall u \in \mathrm{Dom}(\mathscr{L}), \quad \forall v \in \mathcal{V}, \qquad Q(u,v) =: \langle \mathscr{L} u, v \rangle_{\mathsf{H}} \,,$$

$$\mathrm{Dom}(\mathscr{L}) := \{u \in \mathcal{V} : \text{ the map } v \mapsto Q(u,v) \text{ is continuous on } \mathcal{V} \text{ for the norm of } \mathsf{H}\} \,,$$

satisfies the following properties:

i. $\mathscr{L}$ *is bijective from* $\mathrm{Dom}(\mathscr{L})$ *onto* H,

ii. $\mathrm{Dom}(\mathscr{L})$ *is dense in* $\mathcal{V}$ *and in* H,

iii. $\mathscr{L}$ *is closed.*

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- The usual Lax-Milgram theorem is obtained by replacing Φ_1 and Φ_2 by 0.
- Even if the proof is essentially the same as for the usual theorem, the idea to add the "multipliers" Φ_j has fruitful consequences in the applications.
- This generalization of the Lax-Migram theorem is itself a generalization of the "$\mathbb{T}$-coercivity" used in **Time harmonic wave diffraction problems in materials with sign-shifting coefficients** (**A. S. Bonnet-Ben Dhia**, **P. Ciarlet, Jr.**, and **C. M. Zwölf**, 2010).

What is the sesquilinear form in the present context?

The variational space is

$$\mathscr{V} := \left\{ u \in H^1_{\mathbf{A},0}(\Omega) \ : \ m_{\mathbf{B},V}^{\frac{1}{2}} u \in L^2(\Omega) \right\}, \quad m_{\mathbf{B},V} := \sqrt{1 + |\mathbf{B}|^2 + |V|^2}$$

equipped with the norm

$$\|u\|_{\mathscr{V}} := \sqrt{\|u\|^2_{H^1_{\mathbf{A}}(\Omega)} + \int_\Omega m_{\mathbf{B},V} \, |u|^2 \, \mathrm{d}x} \, .$$

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On this space, the sesquilinear form is

$$Q(u,v) := \langle (-i\nabla + \mathbf{A})u, (-i\nabla + \mathbf{A})v \rangle + \int_\Omega V u \bar{v} \, \mathrm{d}x \, .$$

What are the multipliers?

The multipliers are

$$\Phi_1 = \Phi_2 = \Phi := \frac{\operatorname{Im} V}{m_{\mathbf{B},V}}\,.$$

It is essentially the sign of the imaginary part of the potential. They were already used by Almog and Helffer to apply their theorem to a large class of operators including for example

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2 + ix^3 \qquad \text{in} \qquad L^2(\mathbb{R})\,.$$

How do we get the α?

We have extended the class of allowed electro-magnetic fields.

Theorem (Krejčiřík-R-Royer-Siegl, Isr. J. Math., 2017)

We assume that

$$|\nabla V(x)| + |\nabla \mathbf{B}(x)| = o\left(m_{\mathbf{B},V}^{\frac{3}{2}}(x)\right),$$
$$(\operatorname{Re} V)_{-}(x) = o\left(m_{\mathbf{B},V}(x)\right),$$

as $|x| \to +\infty$.
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In the previous work by Almog and Helffer, the power $\frac{3}{2}$ was replaced by 1 and the small o by a big $\mathcal{O}$. About this power, see also Helffer-Mohammed (Annales Inst. Fourier, 1988) in the self-adjoint case.

Comments

- Our theorems apply for instance to wilder electric potentials:

$$\boxed{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - e^{x^2} + ie^{x^4} \qquad \text{in} \qquad L^2(\mathbb{R})}\ .$$

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- The coercivity is not an effect of the real part of V.
- The imaginary part of V and the magnetic field "play at the same level": the intensity of the magnetic field, as the imaginary part of V, can compensate the negative part of the electric potential.

Compact resolvent

As a consequence of our analysis and using our weaker assumptions, we can extend a result of Almog-Helffer:

Proposition

Under our assumptions, if, moreover,

$$\lim_{|x|\to+\infty} m_{\mathbf{B},V}(x) = +\infty\,,$$

then $\mathscr{L}$ is an operator with compact resolvent.

Comments

- The possibility to define a nice (non-accretive) non-self-adjoint operator and its properties related to compactness are strongly connected to coercivity (and its generalizations).

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- When the operator has compact resolvent, the spectrum is discrete and the functions belonging to the algebraic eigenspaces are in $L^2(\mathbb{R})$ by definition. Their existence is related to coercivity and so, we can reasonably expect to relate their decay to the coercivity.

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- The possibility to define a nice (non-accretive) non-self-adjoint operator and its properties related to compactness are strongly connected to coercivity (and its generalizations).
- When the operator has compact resolvent, the spectrum is discrete and the functions belonging to the algebraic eigenspaces are in $L^2(\mathbb{R})$ by definition. Their existence is related to coercivity and so, we can reasonably expect to relate their decay to the coercivity.
- In the last part of this talk, we will discuss these decay estimates and we will show that the definition of the operator, its compactness properties and the decay of its eigenfunctions can be deduced from the same weighted coercivity estimate.

Here come our main theorems.
For $c \in \mathbb{R}$, we let

$$\rho_c := \{\mu \in \mathbb{C} \, : \, -c - \operatorname{Re}\mu - |\operatorname{Im}\mu| > 0\} \, .$$

We also introduce

$$m_\infty := \liminf_{|x|\to+\infty} m_{\mathbf{B},V}(x) \, ,$$

and assume $m_\infty > 0$.

Theorem (Krejčiřík-R-Royer-Siegl, Isr. J. Math., 2017)

Under our assumptions, there exist $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}$ such that we have

$$\rho_{\gamma_2} \subset \rho(\mathscr{L}) .$$

Moreover,

$$\rho_{\gamma_2} \subset \rho_{\gamma_2 - \gamma_1 \check{m}_\infty} \subset \mathrm{Fred}_0(\mathscr{L})$$

for all $\check{m}_\infty \in (0, m_\infty)$.
In particular, the spectrum of $\mathscr{L}$ contained in $\rho_{\gamma_2 - \gamma_1 \check{m}_\infty}$, if it exists, is formed by isolated eigenvalues with finite algebraic multiplicity.

Comments

- This theorem is a non-self-adjoint generalization of the Persson theorem (Mathematica Scandinavica, 1960). In the self-adjoint case, it states more precisely that m_∞ is related to the bottom of the essential spectrum.

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- In practice, γ_1 and γ_2 can be computed explicitly.

Once we know that the spectrum is discrete in some regions of the complex plane, it is natural to try to estimate the decay of the corresponding eigenfunctions.

Theorem (Krejčiřík-R-Royer-Siegl, Isr. J. Math., 2017)

Let us assume that

$$\mathsf{sp}(\mathscr{L}) \cap \rho_{\gamma_2 - \gamma_1 \check{m}_\infty} \neq \emptyset$$

and consider λ in this set. Let us define the metric

$$g(x) := (\gamma_1 m_{\mathbf{B},V}(x) - \mathrm{Re}\,(\lambda) - |\mathrm{Im}\,(\lambda)| - \gamma_2)_+ \, \mathrm{d}x^2 \,,$$

and the corresponding Agmon distance (to any fixed point of Ω) $\mathrm{d}_{\mathrm{Ag}}(x)$. It satisfies

$$|\nabla \mathrm{d}_{\mathrm{Ag}}|^2 = (\gamma_1 m_{\mathbf{B},V} - \mathrm{Re}\,(\lambda) - |\mathrm{Im}\,(\lambda)| - \gamma_2)_+ \,.$$

Theorem (continued)

Pick up any $\varepsilon \in (0,1)$. If ψ is an eigenfunction associated with λ, we have

$$e^{\frac{1-\varepsilon}{3}\mathrm{d}_{\mathrm{Ag}}}\,\psi \in L^2(\Omega)\,.$$

The same conclusion holds for all ψ in the algebraic eigenspace associated with λ.

Comments

- This theorem is a non-self-adjoint generalization of the Agmon estimates (Lecture Notes in Math., 1985). In the self-adjoint case, if λ is an eigenvalue strictly below the bottom of the essential spectrum, the corresponding eigenfunctions have an exponential decay (which can be measured in terms of the distance between λ and the essential spectrum).

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- If we apply our theorem to the example

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2 + ix^3 \qquad \text{in} \qquad L^2(\mathbb{R})\,,$$

we get that, for all eigenvalue λ, there exists $\delta > 0$ such that, for all ψ in the characteristic space of λ,

$$e^{\delta|x|^{\frac{5}{2}}}\psi \in L^2(\mathbb{R})\,.$$

The main idea is to exploit the weighted coercivity (use the real part of the form and the imaginary part of the weighted form) and to combine it with the principle behind the Agmon estimates.

For any complex number μ, consider the shifted form

$$Q_\mu(u, v) := Q(u, v) - \mu \langle u, v \rangle \ .$$

Let

$$\Phi = \frac{\operatorname{Im} V}{m_{\mathbf{B},V}}\,, \qquad \Psi = \frac{\mathbf{B}}{m_{\mathbf{B},V}}\,.$$

All the above mentioned results can be deduced from the following explicit estimate.

Theorem (Weighted coercivity)

For every $\mu \in \mathbb{C}$, $W \in W^{1,\infty}(\Omega;\mathbb{R})$ and all $u \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\operatorname{Re}\left[Q_\mu(u, e^{2W}u)\right] + \operatorname{Im}\left[Q_\mu(u, \Phi e^{2W}u)\right] \geq \frac{1}{2}\left\|(-i\nabla + \mathbf{A})e^{W}u\right\|^2$$
$$+ \int_\Omega \left|e^{W}u\right|^2 \left[\frac{V_2^2 + \frac{1}{12d}\left|\mathbf{B}\right|^2}{m_{\mathbf{B},V}} + V_1 - \operatorname{Re}\mu - |\operatorname{Im}\mu|\right.$$
$$\left. - 9\left(|\nabla\Phi|^2 + |\nabla\Psi|^2 + |\nabla W|^2\right)\right] \mathrm{d}x\,,$$

where $V_1 = \operatorname{Re} V$ *and* $V_2 = \operatorname{Im} V$.

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$$\operatorname{Re}\left[Q_\mu(u, e^{2W}u)\right] + \operatorname{Im}\left[Q_\mu(u, \Phi e^{2W}u)\right] \geq \int_\Omega (\gamma_1 m_{\mathbf{B},V} - \gamma_2 - \operatorname{Re}\mu - |\operatorname{Im}\mu| - 9|\nabla W|^2)e^{2W}|u|^2 \,\mathrm{d}x\,,$$

where $V_1 = \operatorname{Re} V$ *and* $V_2 = \operatorname{Im} V$*.*

Free afternoon?

Credits: **J. Royer**