Dynamical theory of scattering, invisible configurations of the $\zeta\, e^{2iax}$ potential & common zeros of Bessel functions

Ali Mostafazadeh

Koç University, Istanbul

Supported by Turkish Academy of Sciences & TÜBİTAK (Proj. No: 114F357)

Dynamical theory of scattering, invisible configurations of the ζe^{2iax} potential & common zeros of Bessel functions

Mathematical Aspects of Physics
with non-selfadjoint operators

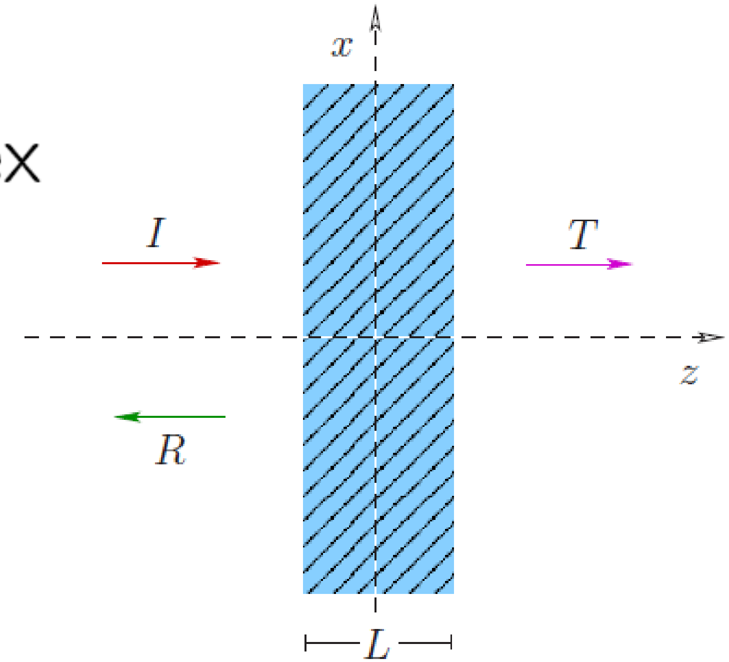
Dynamical theory of scattering, invisible configurations of the ζe^{2iax} potential & common zeros of Bessel functions

Mathematical Aspects of Physics
with non-selfadjoint operators

EM Waves in Media with Planar Symmetry:

$$\mathfrak{n}^2(z)\partial_t^2\vec{E} - c^2\partial_z^2\vec{E} = 0$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index

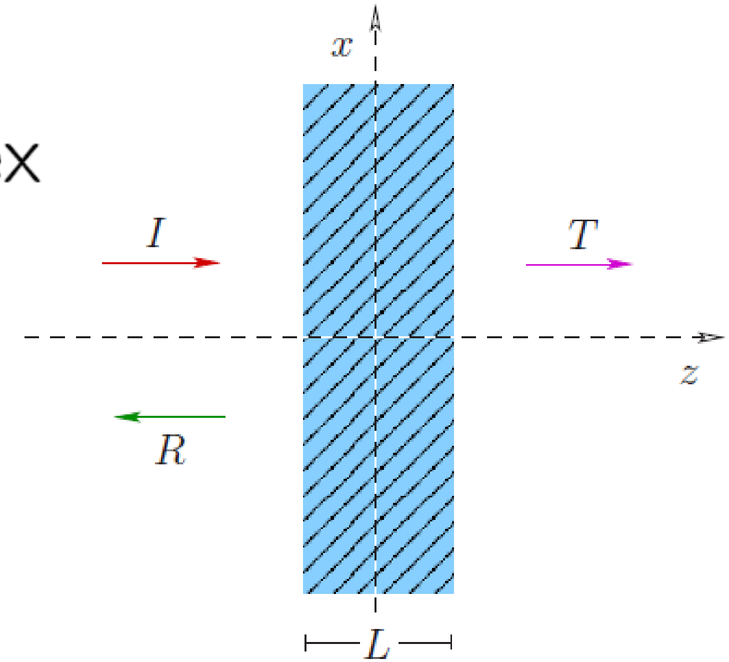


EM Waves in Media with Planar Symmetry:

$$\mathfrak{n}^2(z)\partial_t^2\vec{E} - c^2\partial_z^2\vec{E} = 0$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index

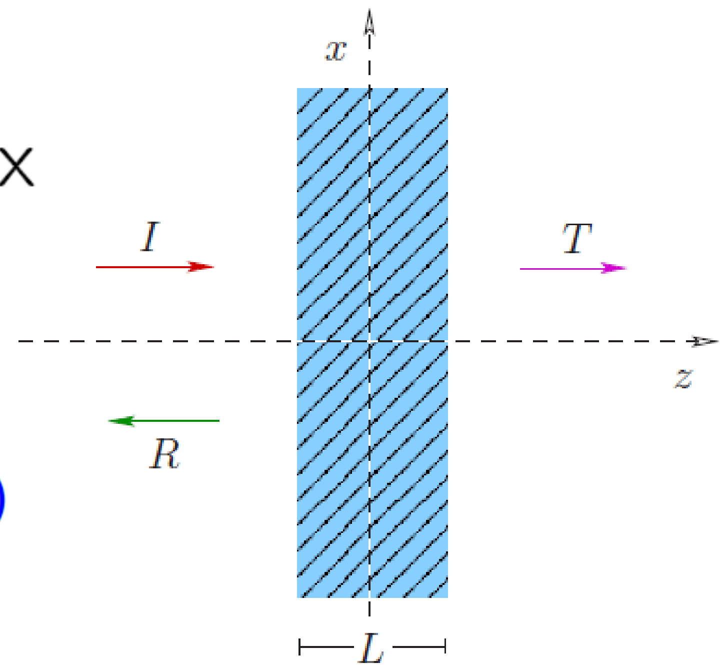
$$\vec{E}(z,t) = E\, e^{-i\omega t}\,\psi(z)\,\hat{\imath}$$



EM Waves in Media with Planar Symmetry:

$$\mathfrak{n}^2(z)\partial_t^2\vec{E} - c^2\partial_z^2\vec{E} = 0$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index



$$\vec{E}(z,t) = E\,e^{-i\omega t}\,\psi(z)\,\hat{\imath}$$

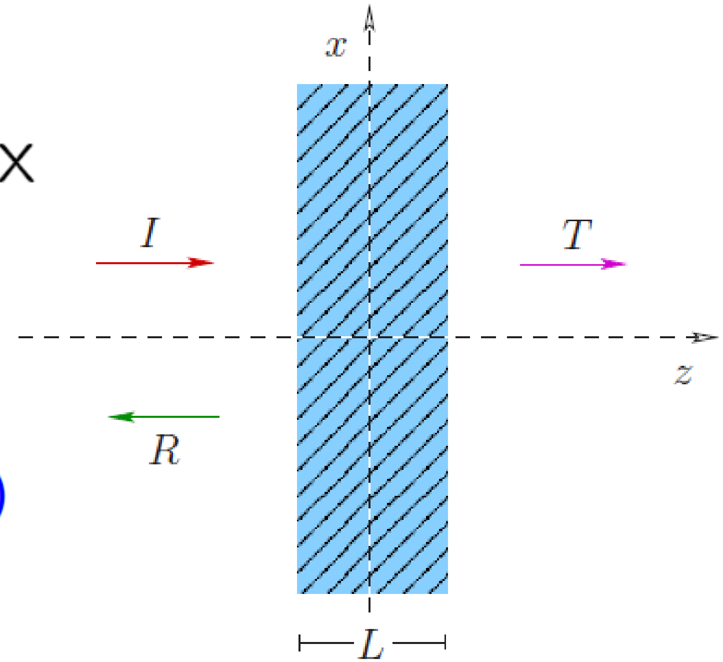
$$-\psi''(z) + v(z)\psi(z) = k^2\psi(z)$$

$$v(z) := k^2[1 - \mathfrak{n}^2(z)]$$

EM Waves in Media with Planar Symmetry:

$$\mathfrak{n}^2(z)\partial_t^2\vec{E} - c^2\partial_z^2\vec{E} = 0$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index



$$\vec{E}(z,t) = E\,e^{-i\omega t}\,\psi(z)\,\hat{\imath}$$

$$-\psi''(z) + v(z)\psi(z) = k^2\psi(z)$$

$$v(z) := k^2[1 - \mathfrak{n}^2(z)]$$

For $|z| > \frac{L}{2}$, $\mathfrak{n}(z) = 1 \;\Rightarrow\; v(z) = 0$

$v(z)$ is a complex scattering potential.

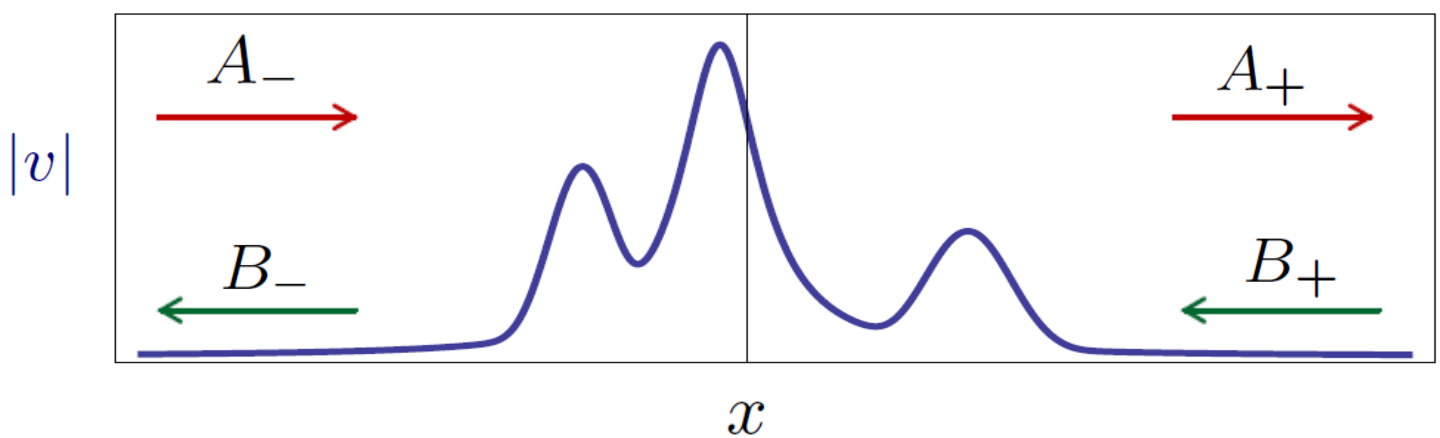
Some Basic Concepts:

- A function $v:\mathbb{R}\to\mathbb{C}$ is a **scattering potential** if every solution of

$$-\psi''(x)+v(x)\psi(x)=k^2\psi(x), \qquad k\in\mathbb{R}^+,$$

satisfies

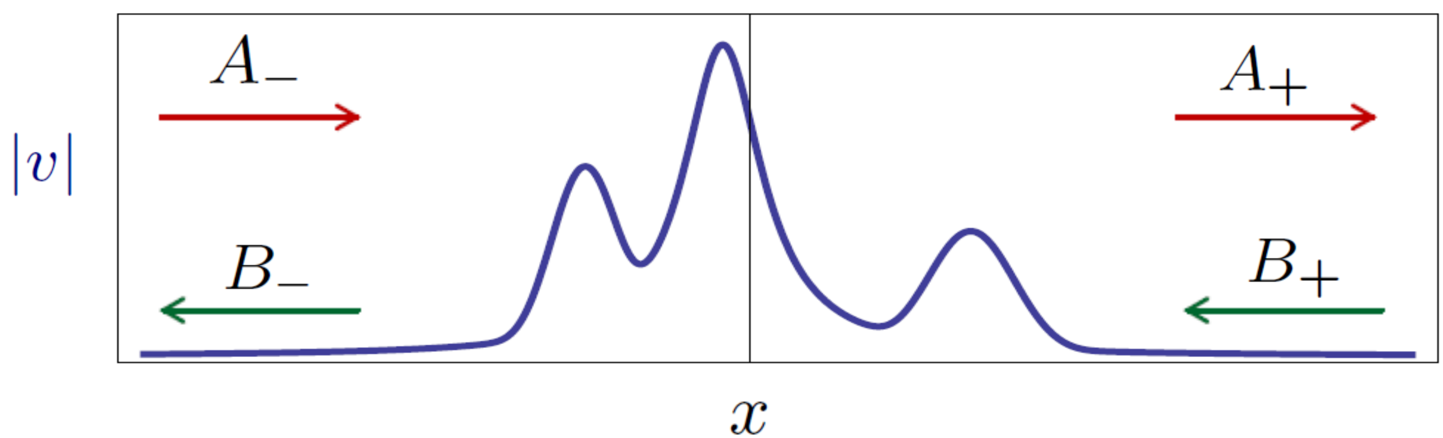
$$\psi(x)\to A_\pm(k)e^{ikx}+B_\pm(k)e^{-ikx} \quad \text{as} \quad x\to\pm\infty.$$



$$\psi(x) \to A_\pm(k)e^{ikx} + B_\pm(k)e^{-ikx} \quad \text{as} \quad x \to \pm\infty.$$

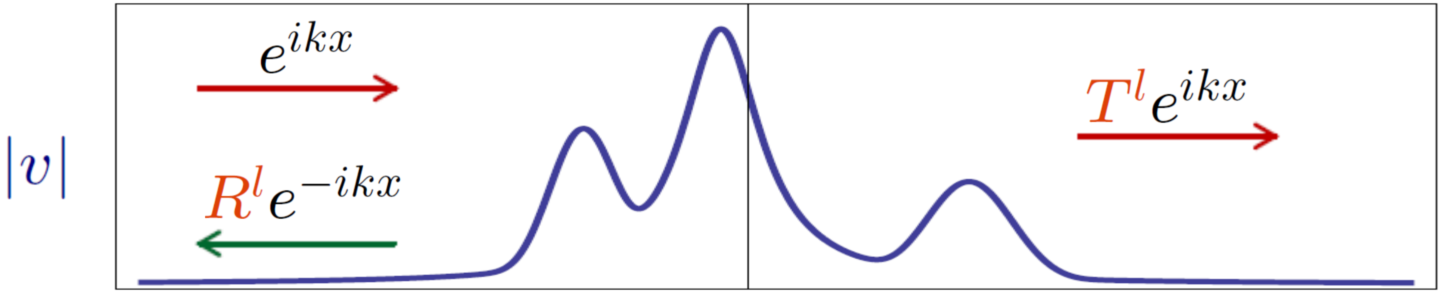
- **Transfer matrix** of v, by definition, satisfies

$$\begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix} = \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{bmatrix} \begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix}.$$

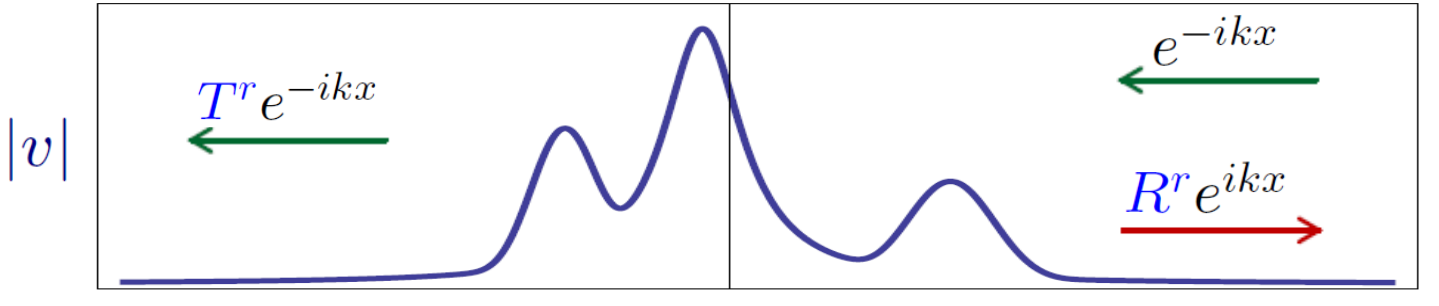


- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$



$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$



$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

Theorem: $T^r = T^l$, $\quad \det \mathbf{M}(k) = 1$, &

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T := T^{l/r} = \frac{1}{M_{22}}.$$

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

Theorem: $T^r = T^l$, $\quad \det \mathbf{M}(k) = 1$, &

$$R^l = -\frac{M_{22}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T := T^{l/r} = \frac{1}{M_{22}}.$$

$$\mathbf{M} = \begin{bmatrix} T - R^l R^r / T & R^r / T \\ -R^l / T & 1/T \end{bmatrix}$$

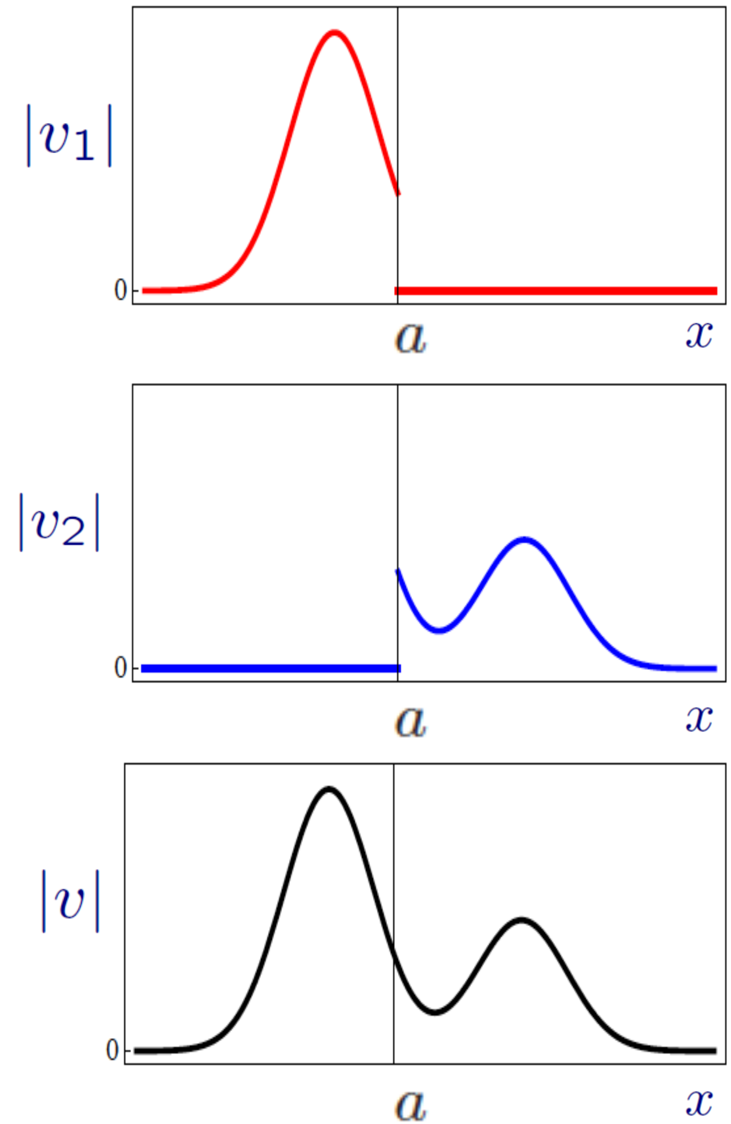
Composition Property of $\mathbb{M}$

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$

$$v_2(x) = 0 \quad \text{for} \quad x < a$$

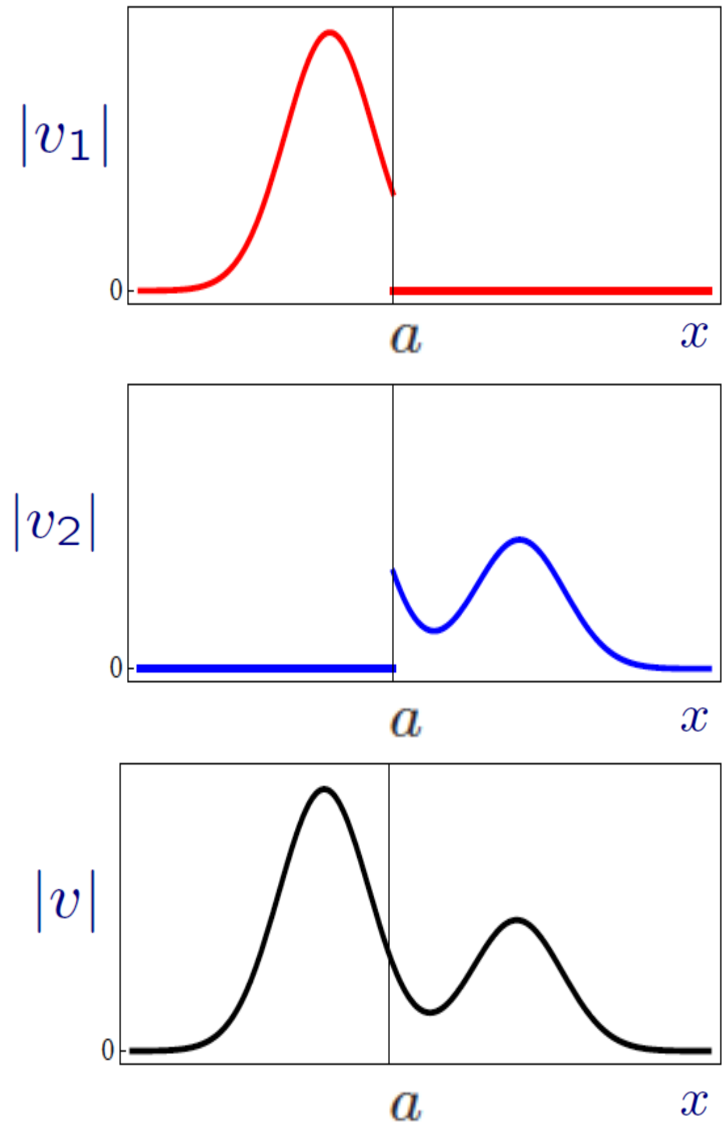
$$v(x) = v_1(x) + v_2(x).$$



Composition Property of $\mathbf{M}$

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$
$$v_2(x) = 0 \quad \text{for} \quad x < a$$
$$v(x) = v_1(x) + v_2(x).$$



$\mathbf{M}_1$: Transfer matrix of v_1

$\mathbf{M}_2$: Transfer matrix of v_2

$\mathbf{M}$: Transfer matrix of $v = v_1 + v_2$

Composition Property of M

Let v_1 and v_2 be scattering potentials such that

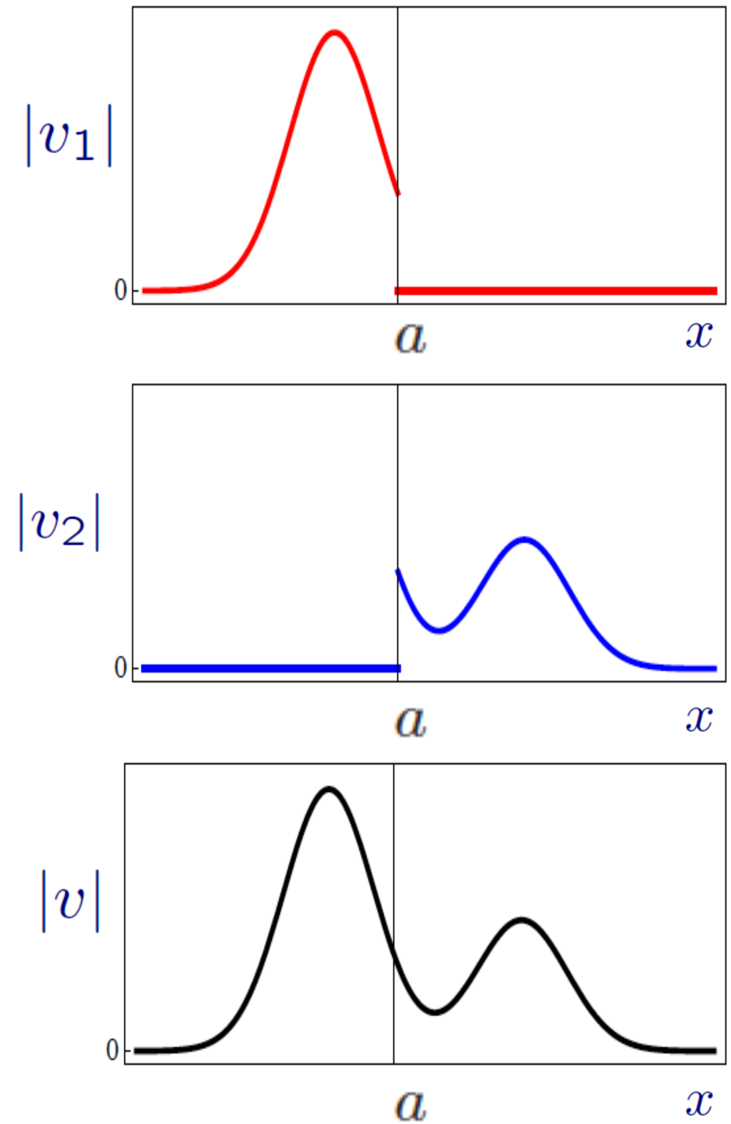
$$v_1(x) = 0 \quad \text{for} \quad x > a,$$
$$v_2(x) = 0 \quad \text{for} \quad x < a$$
$$v(x) = v_1(x) + v_2(x).$$

M_1: Transfer matrix of v_1

M_2: Transfer matrix of v_2

M: Transfer matrix of $v = v_1 + v_2$

Then $\mathrm{M} = \mathrm{M}_2 \mathrm{M}_1$.



Composition Property of $\mathbf{M}$

Theorem: Dissect v into pieces $v_1, v_2, \cdots, v_n$ with

$I_j :=$ support of v_j,

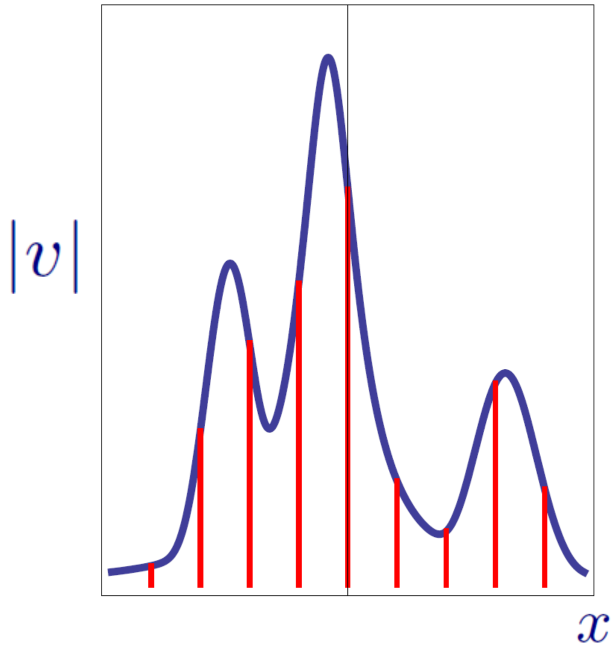
$\mathbf{M}_j :=$ transfer matrix of v_j,

such that

1) I_j is to the left of I_{j+1};

2) $v = v_1 + v_2 + \cdots + v_n$.

Then $\mathbf{M} = \mathbf{M}_n \mathbf{M}_{n-1} \cdots \mathbf{M}_1$.



Composition Property of $\mathbf{M}$

Theorem: Dissect v into pieces $v_1, v_2, \cdots, v_n$ with

$I_j :=$ support of v_j,

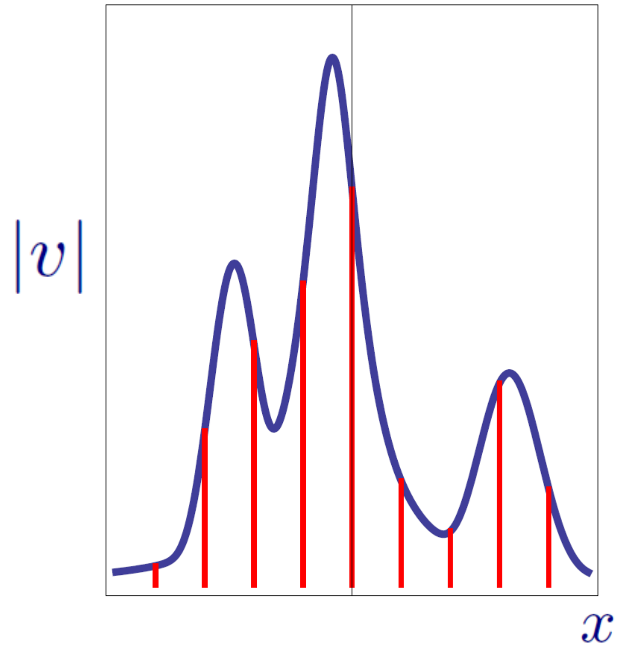
$\mathbf{M}_j :=$ transfer matrix of v_j,

such that

1) I_j is to the left of I_{j+1};

2) $v = v_1 + v_2 + \cdots + v_n$.

Then $\mathbf{M} = \mathbf{M}_n \mathbf{M}_{n-1} \cdots \mathbf{M}_1$.



Scattering properties of v can be obtained from those of v_j. $\Rightarrow$ Numerous Applications

Unidirectional Reflectionlessness:

$$R^l = 0 \neq R^r \text{ or } R^r = 0 \neq R^l$$

Unidirectional Invisibility:

$$R^l = 0 \neq R^r \text{ or } R^r = 0 \neq R^l \;\&\; T = 1$$

Unidirectional Reflectionlessness:

$$R^l = 0 \neq R^r \text{ or } R^r = 0 \neq R^l$$

Unidirectional Invisibility:

$$R^l = 0 \neq R^r \text{ or } R^r = 0 \neq R^l \;\&\; T = 1$$

For real potentials, $|R^l| = |R^r|$. They cannot be unidirectionally reflectionless or invisible.

Single-Mode Inverse Scattering

Given $k_0 \in \mathbb{R}^+$, $R_0^{l/r} \in \mathbb{C}$, & $T_0 \in \mathbb{C} \setminus \{0\}$, find a $v(x)$ such that $R^{l/r}(k_0) = R_0^{l/r}$ & $T(k_0) = T_0$.

Single-Mode Inverse Scattering

Given $k_0 \in \mathbb{R}^+$, $R_0^{l/r} \in \mathbb{C}$, & $T_0 \in \mathbb{C} \setminus \{0\}$, find a $v(x)$ such that $R^{l/r}(k_0) = R_0^{l/r}$ & $T(k_0) = T_0$.

For practical purposes, one is usually interested in finding **finite-range potentials**.

Single-mode inverse scattering for **finite-range potentials** is a key to **optical design**.

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

Left-invisible: $\mathbf{M} = \begin{bmatrix} 1 & R^r \\ 0 & 1 \end{bmatrix}$

Right-invisible: $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -R^l & 1 \end{bmatrix}$

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

Left-invisible: $\mathbf{M} = \begin{bmatrix} 1 & R^r \\ 0 & 1 \end{bmatrix}$

Right-invisible: $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -R^l & 1 \end{bmatrix}$

$v_1 \prec v_2$ means the support of v_1 is to the left of the support of v_2.

Suppose we can find $v_1 \prec v_2 \prec v_3$ such that at $k = k_0$ their transfer matrix have the form

$$v_1: \qquad \mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ -R_1^l & 1 \end{bmatrix}, \qquad R_1^l := R_0^l + \frac{(1-T_0)T_0}{R_0^r},$$

$$v_2: \qquad \mathbf{M}_2 = \begin{bmatrix} 1 & R_2^r \\ 0 & 1 \end{bmatrix}, \qquad R_2^r := \frac{R_0^r}{T_0},$$

$$v_3: \qquad \mathbf{M}_3 = \begin{bmatrix} 1 & 0 \\ -R_3^l & 1 \end{bmatrix}, \qquad R_3^l := \frac{T_0 - 1}{R_0^r}.$$

Suppose we can find $v_1 \prec v_2 \prec v_3$ such that at $k = k_0$ their transfer matrix have the form

$$v_1 : \qquad \mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ -R_1^l & 1 \end{bmatrix}, \qquad R_1^l := R_0^l + \frac{(1 - T_0)T_0}{R_0^r},$$

$$v_2 : \qquad \mathbf{M}_2 = \begin{bmatrix} 1 & R_2^r \\ 0 & 1 \end{bmatrix}, \qquad R_2^r := \frac{R_0^r}{T_0},$$

$$v_3 : \qquad \mathbf{M}_3 = \begin{bmatrix} 1 & 0 \\ -R_3^l & 1 \end{bmatrix}, \qquad R_3^l := \frac{T_0 - 1}{R_0^r}.$$

Then:

$$v = v_1 + v_2 + v_3 : \qquad \mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$$

Suppose we can find $v_1 \prec v_2 \prec v_3$ such that at $k = k_0$ their transfer matrix have the form

$$v_1 : \qquad \mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ -R_1^l & 1 \end{bmatrix}, \qquad R_1^l := R_0^l + \frac{(1-T_0)T_0}{R_0^r},$$

$$v_2 : \qquad \mathbf{M}_2 = \begin{bmatrix} 1 & R_2^r \\ 0 & 1 \end{bmatrix}, \qquad R_2^r := \frac{R_0^r}{T_0},$$

$$v_3 : \qquad \mathbf{M}_3 = \begin{bmatrix} 1 & 0 \\ -R_3^l & 1 \end{bmatrix}, \qquad R_3^l := \frac{T_0 - 1}{R_0^r}.$$

Then:

$$v = v_1 + v_2 + v_3 : \qquad \mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} T_0 - \dfrac{R_0^l R_0^r}{T_0} & \dfrac{R_0^r}{T_0} \\ -\dfrac{R_0^l}{T_0} & \dfrac{1}{T_0} \end{bmatrix}$$

Suppose we can find $v_1 \prec v_2 \prec v_3$ such that at $k = k_0$ their transfer matrix have the form

$$v_1: \qquad \mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ -R_1^l & 1 \end{bmatrix}, \qquad R_1^l := R_0^l + \frac{(1-T_0)T_0}{R_0^r},$$

$$v_2: \qquad \mathbf{M}_2 = \begin{bmatrix} 1 & R_2^r \\ 0 & 1 \end{bmatrix}, \qquad R_2^r := \frac{R_0^r}{T_0},$$

$$v_3: \qquad \mathbf{M}_3 = \begin{bmatrix} 1 & 0 \\ -R_3^l & 1 \end{bmatrix}, \qquad R_3^l := \frac{T_0 - 1}{R_0^r}.$$

Then:

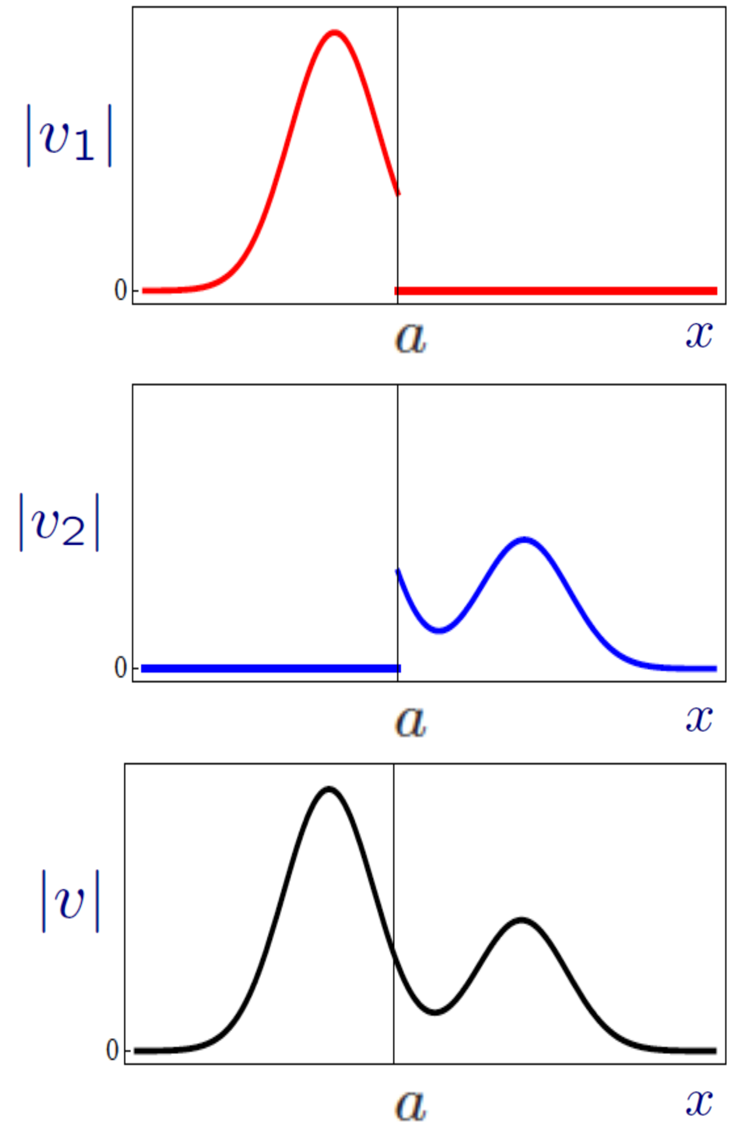
$$v = v_1 + v_2 + v_3: \qquad \mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} T_0 - \dfrac{R_0^l R_0^r}{T_0} & \dfrac{R_0^r}{T_0} \\ -\dfrac{R_0^l}{T_0} & \dfrac{1}{T_0} \end{bmatrix}$$

Tunable finite-range unidir. invisible potentials

$\Rightarrow$ **Single-mode inverse scattering**

Composition Property of M

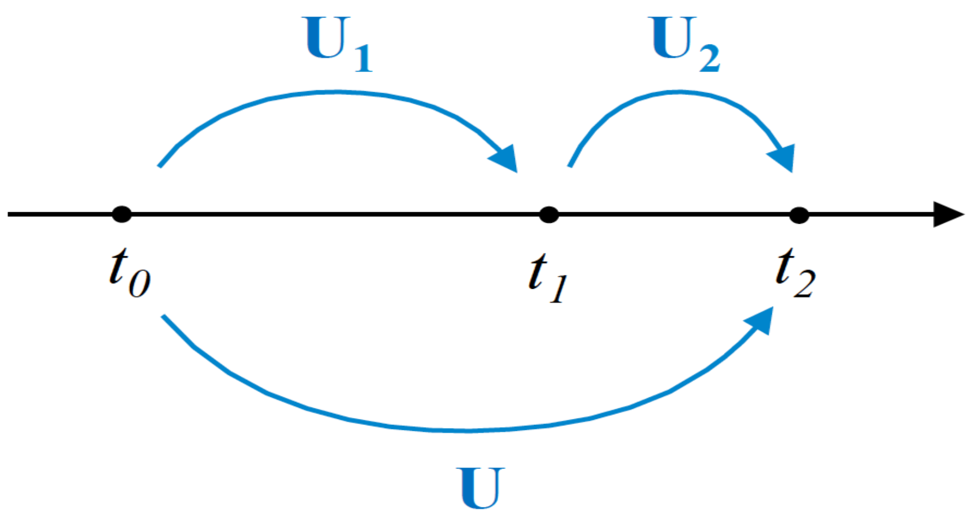
$$\mathrm{M} = \mathrm{M}_2\,\mathrm{M}_1$$



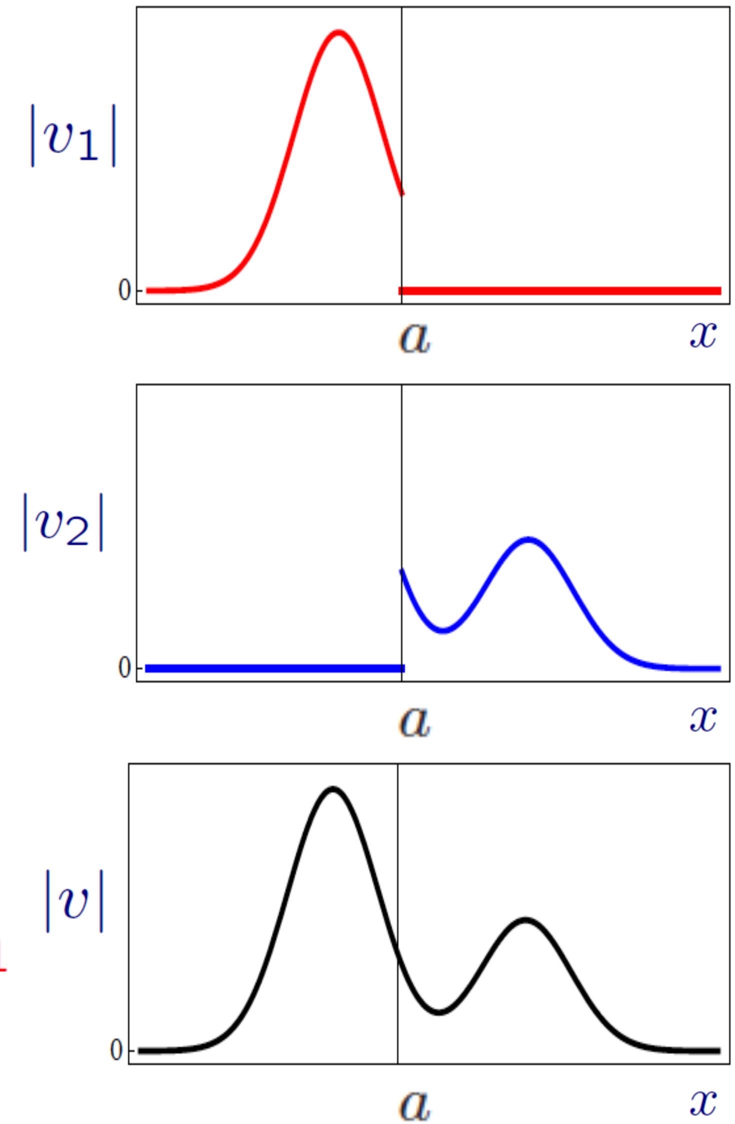
Composition Property of $\mathbf{M}$

$\mathrm{M} = \mathrm{M}_2\,\mathrm{M}_1$

This is the same as the composition rule for **evolution operators** in QM.



$$\left.\begin{array}{r}\mathrm{U}_1 := \mathrm{U}(t_1, t_0)\\ \mathrm{U}_2 := \mathrm{U}(t_2, t_1)\\ \mathrm{U} := \mathrm{U}(t_2, t_0)\end{array}\right\} \Rightarrow \mathrm{U} = \mathrm{U}_2\mathrm{U}_1$$



Dynamical Formulation of Scattering

Theorem: Let $v:\mathbb{R}\to\mathbb{C}$ has support $[a,b]$. Then

$\mathbf{M} = \mathrm{U}(b,a)$ where

$$i\frac{d}{dx}\mathrm{U}(x,a) = \mathbf{H}(x)\mathrm{U}(x,a), \qquad \mathrm{U}(a,a) = 1$$

$$\mathbf{H}(x) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}$$

[Ann. Phys. (NY), **341**, 77 (2014)]

Dynamical Formulation of Scattering

Theorem: Let $v:\mathbb{R}\to\mathbb{C}$ has support $[a,b]$. Then $\mathbf{M}=\mathrm{U}(b,a)$ where

$$i\frac{d}{dx}\mathrm{U}(x,a)=\mathbf{H}(x)\mathrm{U}(x,a),\qquad \mathrm{U}(a,a)=1$$

$$\mathbf{H}(x):=\frac{v(x)}{2k}\begin{bmatrix}1 & e^{-2ikx}\\ -e^{2ikx} & -1\end{bmatrix}$$

Make b into a variable: $b\to y\in[a,b]$.

$\mathrm{U}(y,a)=$ Transfer matrix of

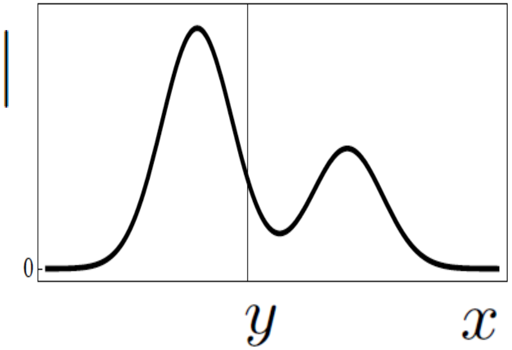
$$v_y(x):=\begin{cases} v(x) & \text{for} \quad x\le y\\ 0 & \text{for} \quad x>y\end{cases}$$

[Ann. Phys. (NY), **341**, 77 (2014)]

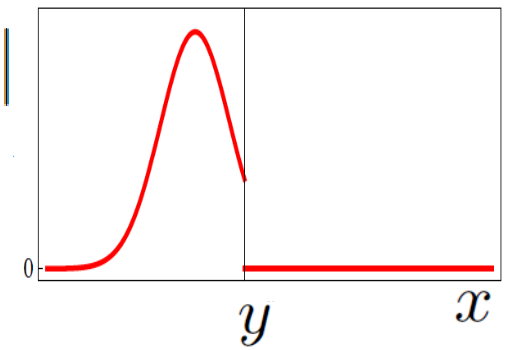
$\mathbf{U}(y,a) =$ Transfer matrix of

$$v_y(x) := \begin{cases} v(x) & \text{for} \quad x \le y \\ 0 & \text{for} \quad x > y \end{cases}$$

$|v(x)|$



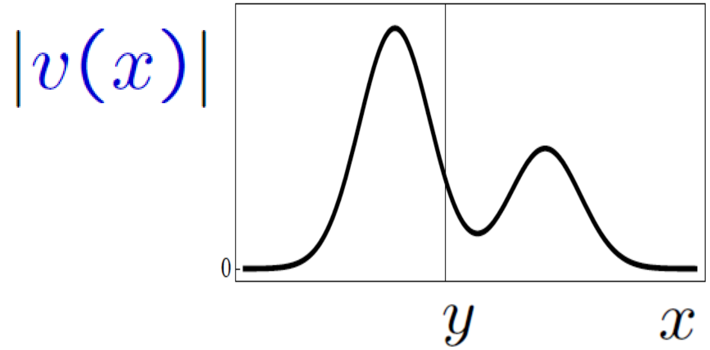
$|v_y(x)|$

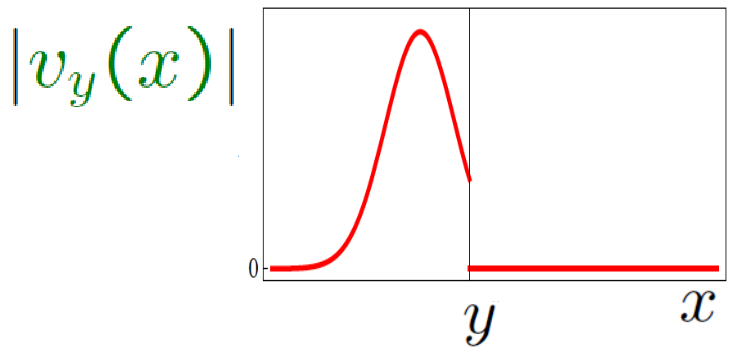


[Ann. Phys. (NY), **341**, 77 (2014)]

$\mathbf{U}(y,a) =$ Transfer matrix of

$$v_y(x) := \begin{cases} v(x) & \text{for} \quad x \le y \\ 0 & \text{for} \quad x > y \end{cases}$$





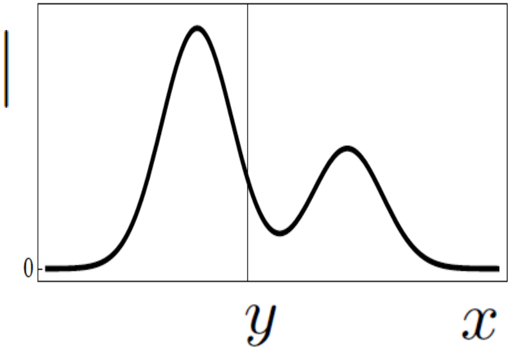
$$\mathbf{M} = \begin{bmatrix} T - \frac{R^l R^r}{T} & \frac{R^r}{T} \\ -\frac{R^l}{T} & \frac{1}{T} \end{bmatrix}$$

[Ann. Phys. (NY), **341**, 77 (2014)]

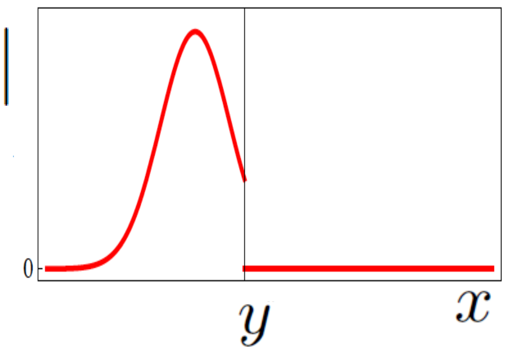
$\mathbf{U}(y,a) =$ Transfer matrix of

$$v_y(x) := \begin{cases} v(x) & \text{for} \quad x \le y \\ 0 & \text{for} \quad x > y \end{cases}$$

$|v(x)|$



$|v_y(x)|$



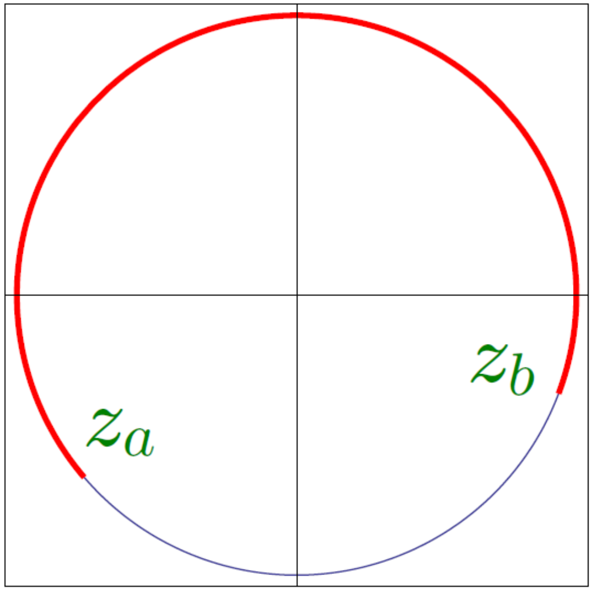
$$\mathbf{M} = \begin{bmatrix} T - \frac{R^l R^r}{T} & \frac{R^r}{T} \\ -\frac{R^l}{T} & \frac{1}{T} \end{bmatrix}$$

$$i\frac{d}{dy}\mathbf{U}(y,a) = \mathbf{H}(y)\mathbf{U}(y,a)$$

$\Rightarrow \; \exists$ Dynamical Eqs. for $R^{l/r}$ & T.

[Ann. Phys. (NY), **341**, 77 (2014)]

$$z := e^{-2ikx} \in \mathcal{C} := \{e^{-2ikx} \,|\, x \in [a,b]\}$$



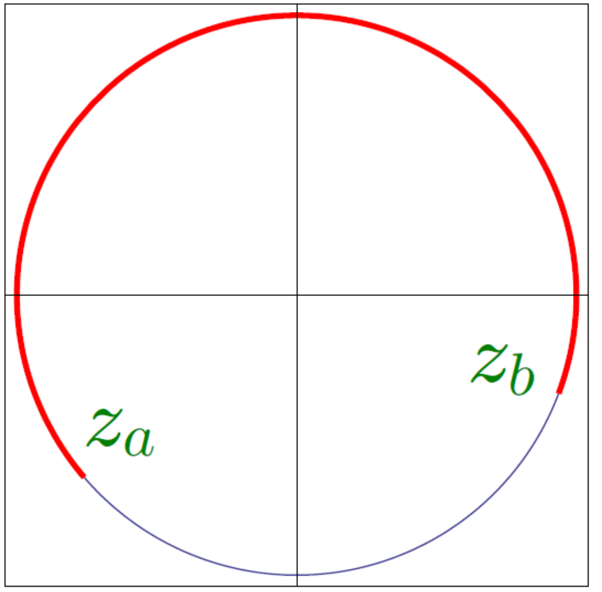
[Ann. Phys. (NY), **341**, 77 (2014)]

$$z := e^{-2ikx} \in \mathcal{C} := \{e^{-2ikx} \,|\, x \in [a,b]\}$$

$$R^{\mathsf{l}}(k) = -\int_{\mathcal{C}} dz\, \frac{S_k''(z)}{S_k(z)S_k'(z)^2},$$

$$R^{\mathsf{r}}(k) = \frac{S_k(z_b)}{S_k'(z_b)} - z_b,$$

$$T(k) = \frac{1}{S_k'(z_b)},$$



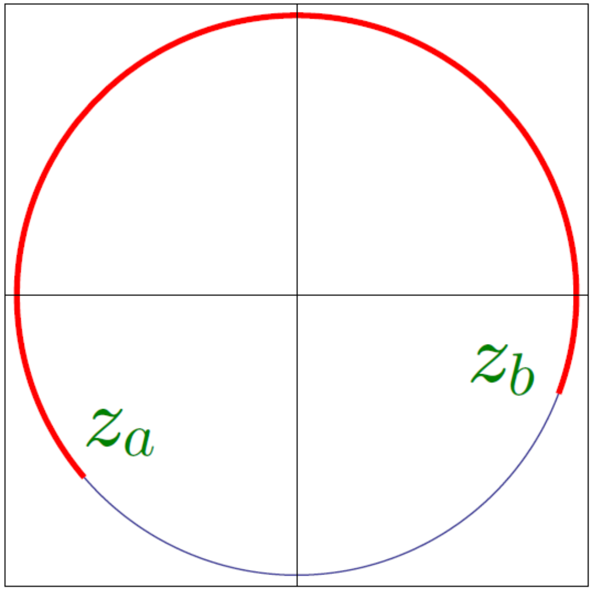
[Ann. Phys. (NY), **341**, 77 (2014)]

$$z := e^{-2ikx} \in \mathcal{C} := \{e^{-2ikx} \,|\, x \in [a,b]\}$$

$$R^{\mathsf{l}}(k) = -\int_{\mathcal{C}} dz \, \frac{S_k''(z)}{S_k(z) S_k'(z)^2},$$

$$R^{\mathsf{r}}(k) = \frac{S_k(z_b)}{S_k'(z_b)} - z_b,$$

$$T(k) = \frac{1}{S_k'(z_b)},$$



$$z^2 S_k''(z) + \left[\frac{\breve{v}(z)}{4k^2}\right] S_k(z) = 0, \qquad z \in \mathcal{C},$$

$$S_k(z_a) = z_a, \qquad S_k'(z_a) = 1,$$

$$\breve{v}(z) := v\left(\tfrac{i \ln z}{2k}\right) \quad \Rightarrow \quad v(x) = \breve{v}(e^{-2ikx})$$

Finite-range right-invisible potential at $k = k_0$:

$$R^r(k_0) = \frac{S_{k_0}(z_b)}{S'_{k_0}(z_b)} - z_b = 0, \qquad T(k_0) = \frac{1}{S'_{k_0}(z_b)} = 1$$

Finite-range right-invisible potential at $k = k_0$:

$$R^r(k_0) = \frac{S_{k_0}(z_b)}{S'_{k_0}(z_b)} - z_b = 0, \qquad T(k_0) = \frac{1}{S'_{k_0}(z_b)} = 1$$

$$\Leftrightarrow \qquad S_{k_0}(z_b) = z_b, \qquad S'_{k_0}(z_b) = 1 \qquad (1)$$

Finite-range right-invisible potential at $k = k_0$:

$$R^r(k_0) = \frac{S_{k_0}(z_b)}{S'_{k_0}(z_b)} - z_b = 0, \qquad T(k_0) = \frac{1}{S'_{k_0}(z_b)} = 1$$

$$\Leftrightarrow \qquad S_{k_0}(z_b) = z_b, \qquad S'_{k_0}(z_b) = 1 \qquad (1)$$

$$z^2 S''_{k_0}(z) + \left[\frac{\breve{v}(z)}{4k^2}\right] S_{k_0}(z) = 0, \qquad (2)$$

$$S_{k_0}(z_a) = z_a, \qquad S'_{k_0}(z_a) = 1, \qquad (3)$$

Finite-range right-invisible potential at $k = k_0$:

$$R^r(k_0) = \frac{S_{k_0}(z_b)}{S'_{k_0}(z_b)} - z_b = 0, \qquad T(k_0) = \frac{1}{S'_{k_0}(z_b)} = 1$$

$$\Leftrightarrow \qquad S_{k_0}(z_b) = z_b, \qquad S'_{k_0}(z_b) = 1 \qquad (1)$$

$$z^2 S''_{k_0}(z) + \left[\frac{\check{v}(z)}{4k^2}\right] S_{k_0}(z) = 0, \qquad (2)$$

$$S_{k_0}(z_a) = z_a, \qquad S'_{k_0}(z_a) = 1, \qquad (3)$$

- Find a twice diff. $S_{k_0} : \mathcal{C} \to \mathbb{C}$ satisfying (1) & (3).
- Plug it in (2) & solve for $\check{v}(z)$.
- Recall that $v(x) = \check{v}(e^{-2ik_0 x})$.

Finite-range right-invisible potential at $k=k_0$:

$$R^r(k_0)=\frac{S_{k_0}(z_b)}{S'_{k_0}(z_b)}-z_b=0,\qquad T(k_0)=\frac{1}{S'_{k_0}(z_b)}=1$$

$$\Leftrightarrow\qquad S_{k_0}(z_b)=z_b,\qquad S'_{k_0}(z_b)=1\qquad(1)$$

$$z^2S''_{k_0}(z)+\left[\frac{\breve{v}(z)}{4k^2}\right]S_{k_0}(z)=0,\qquad(2)$$

$$S_{k_0}(z_a)=z_a,\qquad S'_{k_0}(z_a)=1,\qquad(3)$$

- Find a twice diff. $S_{k_0}:\mathcal{C}\to\mathbb{C}$ satisfying (1) & (3).
- Plug it in (2) & solve for $\breve{v}(z)$.
- Recall that $v(x)=\breve{v}(e^{-2ik_0x})$.

$\Rightarrow$ **Tunable finite-range right-invisible potentials.**

[Phys. Rev. A **90**, 023833 (2014)]

Theorem: Let $R_v^{l/r}$:= reflection amplitude of v and T_v:= transmission amplitude of v. Then

$$R_v^l = \frac{T_v^2 \overline{R_{\bar{v}}^r}}{R_v^r \overline{R_{\bar{v}}^r} - 1}.$$

[J. Phys. A **49**, 445302 (2016)]

Theorem: Let $R_v^{l/r}$:= reflection amplitude of v and T_v:= transmission amplitude of v. Then

$$R_v^l = \frac{T_v^2 \overline{R_{\overline{v}}^r}}{R_v^r \overline{R_{\overline{v}}^r} - 1}.$$

Corollary: Suppose that $\exists \mu > 0$ such that

$$e^{\mu|x|}|v(x)| < \infty \quad \text{for} \quad x \to \pm\infty.$$

Then $\forall k \in \mathbb{R}^+$, $R_v^{l/r}(k) = 0$ iff $R_{\overline{v}}^{r/l}(k) = 0$.

---

v is left-invisible $\Leftrightarrow$ $\overline{v}$ is right-invisible.

[J. Phys. A **49**, 445302 (2016)]

Principal example of unidirectional invisibility:

$$v(x) = \begin{cases} \mathfrak{z}\, e^{-2ik_0 x} & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L], \end{cases} \qquad \mathfrak{z} \in \mathbb{C} \setminus \{0\}$$

is perturbatively right-invisible for

$$k = k_0 = \frac{\pi m}{L}, \qquad m \in \mathbb{Z}^+.$$

Perturbatively:= To first order in $|\mathfrak{z}|/k_0^2$.

Poladian, PRE **54**, 2963 (1996).
Greenberg & Orenstein, Opt. Lett. **29**, 451 (2004).
Kulishov, et al, Opt. Exp. **13**, 3068 (2005).
Lin et al, PRL **106**, 213901 (2011)
Phys. Rev. A **89**, 012709 (2014)

Solving scattering problem for

$$v(x) = \begin{cases} \mathfrak{z}\, e^{-2ik_0 x} & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L], \end{cases} \qquad \mathfrak{z} \in \mathbb{C} \setminus \{0\}$$

$$k_0 = \frac{\pi m}{L}, \quad m \in \mathbb{Z}^+$$

Solving scattering problem for

$$v(x) = \begin{cases} \mathfrak{z}\, e^{-2ik_0 x} & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L], \end{cases} \qquad \mathfrak{z} \in \mathbb{C} \setminus \{0\}$$

$$k_0 = \frac{\pi m}{L}, \quad m \in \mathbb{Z}^+$$

$$\varepsilon(x) = \mathfrak{n}(x)^2 = 1 - \frac{\mathfrak{a}^2}{\gamma^2} e^{-2ik_0 x} \qquad x \in [0, L]$$

$$\mathfrak{a} := \frac{\sqrt{\mathfrak{z}}}{k} \in \mathbb{C} \setminus \{0\}, \quad \gamma := \frac{k}{k_0} \in \mathbb{R}^+$$

Solving scattering problem for

$$v(x) = \begin{cases} \mathfrak{z}\, e^{-2ik_0x} & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L], \end{cases} \qquad \mathfrak{z} \in \mathbb{C} \setminus \{0\}$$

$$k_0 = \frac{\pi m}{L}, \qquad m \in \mathbb{Z}^+$$

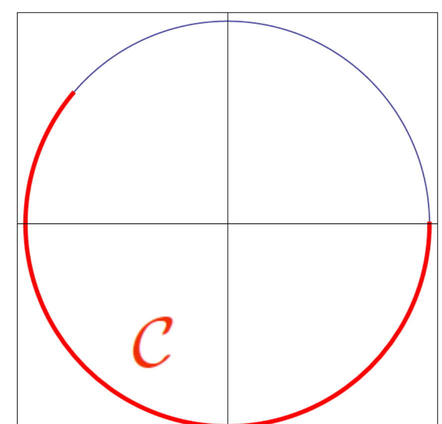
$$\varepsilon(x) = \mathfrak{n}(x)^2 = 1 - \frac{\mathfrak{a}^2}{\gamma^2} e^{-2ik_0x} \qquad x \in [0, L]$$

$$\mathfrak{a} := \frac{\sqrt{\mathfrak{z}}}{k} \in \mathbb{C} \setminus \{0\}, \qquad \gamma := \frac{k}{k_0} \in \mathbb{R}^+$$

$$z := e^{-2ikx} \in \mathcal{C}$$

$$S_k''(z) + \frac{\mathfrak{a}^2 z^{-2+\frac{1}{\gamma}}}{4\gamma^2} S_k(z) = 0$$

$$S_k(1) = S_k'(1) = 1$$



$$S_k(z) = \frac{-\pi \mathfrak{a}\sqrt{z}}{2\sin(\pi\gamma)} \left[J_{-\gamma-1}(\mathfrak{a}) J_{\gamma}(\mathfrak{a}\, z^{\frac{1}{2\gamma}}) + J_{\gamma+1}(\mathfrak{a}) J_{-\gamma}(\mathfrak{a}\, z^{\frac{1}{2\gamma}}) \right]$$

$$R^r(k) \;=\; \frac{S_k(e^{-2ikL})}{S_k'(e^{-2ikL})} - e^{-2ikL}, \quad T(k) \;=\; \frac{1}{S_k'(e^{-2ikL})}$$

$$kL = \gamma k_0 L = \pi m \gamma$$

$$S_k(z) = \frac{-\pi \mathfrak{a} \sqrt{z}}{2\sin(\pi\gamma)} \left[J_{-\gamma-1}(\mathfrak{a}) J_\gamma(\mathfrak{a}\, z^{\frac{1}{2\gamma}}) + J_{\gamma+1}(\mathfrak{a}) J_{-\gamma}(\mathfrak{a}\, z^{\frac{1}{2\gamma}}) \right]$$

$$R^r(k) = \frac{S_k(e^{-2ikL})}{S'_k(e^{-2ikL})} - e^{-2ikL}, \quad T(k) = \frac{1}{S'_k(e^{-2ikL})}$$

$$kL = \gamma k_0 L = \pi m \gamma$$

$$R^r(k) = \frac{-i\pi\mathfrak{a}\,\mu^* J_{-\gamma-1}(\mathfrak{a}) J_{\gamma+1}(\mathfrak{a})}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a}) J_{\gamma+1}(\mathfrak{a})},$$

$$T(k) = \frac{2\gamma}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a}) J_{\gamma+1}(\mathfrak{a})},$$

$$\mu := \frac{1 - e^{2\pi i m\gamma}}{2i\sin(\pi\gamma)}$$

$$S_k(z) = \frac{-\pi\mathfrak{a}\sqrt{z}}{2\sin(\pi\gamma)}\left[J_{-\gamma-1}(\mathfrak{a})J_\gamma(\mathfrak{a}\,z^{\frac{1}{2\gamma}}) + J_{\gamma+1}(\mathfrak{a})J_{-\gamma}(\mathfrak{a}\,z^{\frac{1}{2\gamma}})\right]$$

$$R^{\mathrm{r}}(k) = \frac{S_k(e^{-2ikL})}{S_k'(e^{-2ikL})} - e^{-2ikL}, \quad T(k) = \frac{1}{S_k'(e^{-2ikL})}$$

$$kL = \gamma k_0 L = \pi m \gamma$$

$$R^{\mathrm{r}}(k) = \frac{-i\pi\mathfrak{a}\,\mu^* J_{-\gamma-1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})},$$

$$T(k) = \frac{2\gamma}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})},$$

$$R^l_v = \frac{T_v^2\overline{R^r_{\bar{v}}}}{R^r_v\overline{R^r_{\bar{v}}} - 1}.$$

$$\overline{R^{\mathsf{r}}_{\bar{v}}(k)} = \frac{i\pi\mathfrak{a}\,\mu J_{-\gamma+1}(\mathfrak{a})J_{\gamma-1}(\mathfrak{a})}{2\gamma + i\pi\mathfrak{a}^2\mu^* J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}$$

$$R^{\mathsf{r}}(k) = \frac{-i\pi\mathfrak{a}\,\mu^* J_{-\gamma-1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})},$$

$$T(k) = \frac{2\gamma}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})},$$

[J. Phys. A **49**, 445302 (2016)]

$$\overline{R^{\mathsf{r}}_{\overline{v}}(k)} = \frac{i\pi\mathfrak{a}\,\mu J_{-\gamma+1}(\mathfrak{a})J_{\gamma-1}(\mathfrak{a})}{2\gamma + i\pi\mathfrak{a}^2\mu^* J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}$$

$$R^{\mathsf{r}}(k) = \frac{-i\pi\mathfrak{a}\,\mu^* J_{-\gamma-1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})},$$

$$T(k) = \frac{2\gamma}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})},$$

Classification of the undir. invisible configurations

$\Leftrightarrow$ **common zeros of Bessel functions** J_ν **with** $\nu \in \mathbb{R}$!

[J. Phys. A **49**, 445302 (2016)]

$$\overline{R^{\mathsf{r}}_{\bar{v}}(k)} = \frac{i\pi\mathfrak{a}\,\mu J_{-\gamma+1}(\mathfrak{a})J_{\gamma-1}(\mathfrak{a})}{2\gamma + i\pi\mathfrak{a}^2\mu^* J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}$$

$$\begin{aligned} R^{\mathsf{r}}(k) &= \frac{-i\pi\mathfrak{a}\,\mu^* J_{-\gamma-1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}, \\ T(k) &= \frac{2\gamma}{2\gamma - i\pi\mathfrak{a}^2\mu\, J_{-\gamma+1}(\mathfrak{a})J_{\gamma+1}(\mathfrak{a})}, \end{aligned}$$

Classification of the undir. invisible configurations

$\Leftrightarrow$ **common zeros of Bessel functions J_ν with $\nu \in \mathbb{R}$!**

$J_{\gamma+1}$ and $J_{\gamma-1}$ have no nonzero common zeros.

Is this true for $J_{\gamma+1}$ and $J_{-\gamma+1}$?

[J. Phys. A **49**, 445302 (2016)]

Concluding Remarks

– Unidirectional invisibility

– Single-mode inverse scattering

– Dynamical formulation of scattering

– Truncated e^{-2ik_0x} potential & Bessel fn. zeros

Concluding Remarks

– **Dynamical formulation of scattering in dim.** ≥ 2

[F.Loran & A.M. Phys. Rev. A **93**, 042707 (2016)]

$\Rightarrow$ **Unidirectional invisibility in dim.** ≥ 2

[F.Loran & A.M. Proc. R. Soc. A **472**, 20160250 (2016)]

Thank you for your attention

Concluding Remarks

– **Dynamical formulation of scattering in dim.≥ 2**
[F.Loran & A.M. Phys. Rev. A **93**, 042707 (2016)]

$\Rightarrow$ **Unidirectional invisibility in dim.≥ 2**
[F.Loran & A.M. Proc. R. Soc. A **472**, 20160250 (2016)]

$\Rightarrow$ **Criterion for perfect invisibility in dim.≥ 2**

Theorem: Let $\alpha > 0$ and $v(x, y)$ be such that

$$\tilde{v}(x, \mathfrak{K}_y) = 0 \quad \text{for all} \quad \mathfrak{K}_y \leq 2\alpha.$$

Then $v(x, y)$ is omnidirectionally invisible for all $k \in [0, \alpha]$.

[F.Loran & A.M., arXiv:1705.00500]

Thank you for your attention