Eigenvalue bounds for non-self-adjoint Schrödinger operators with the inverse square potential

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CIRM conference on

Mathematical aspects of the physics

with non-self-adjoint operators

Marseille, June 5, 2017

- $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^n)$, $V \in L^{n/2+\gamma}(\mathbb{R}^n;\mathbb{R})$ with $\gamma \geq 0$.
- $\sigma_{\text{ess}}(H) = \sigma(-\Delta) = [0,\infty)$, $\sigma_{\text{d}}(H) \subset (-\infty, 0)$.

Theorem (Keller '61, Lieb-Thirring '76, Lieb '76, Rosenbljum '76, Cwikel '77, Weidl '96)

- Keller inequality: For $\gamma \geq 1/2$ $(n=1)$ and $\gamma > 0$ $(n \geq 2)$,

$$(-E_1)^\gamma \leq C_{\gamma,n} \int V_-(x)^{n/2+\gamma} dx,$$

where $E_1 = \inf \sigma_{\text{d}}(H)$ and $V = V_+ - V_-$.

- Lieb-Thirring and Cwikel-Lieb-Rosenblum inequalities:

For $\gamma \geq 1/2$ $(n=1)$, $\gamma > 0$ $(n=2)$ and $\gamma \geq 0$ $(n \geq 3)$,

$$\sum_{E \in \sigma_{\text{d}}(H)} |E|^\gamma \leq L_{\gamma,n} \int V_-(x)^{n/2+\gamma} dx.$$

(1) Keller inequality for complex-valued potentials.

- $H = -\Delta + V(x)$, $V \in L^{n/2+\gamma}(\mathbb{R}^n; \mathbb{C})$.

Theorem (Abramov-Aslanyan-Davies '01, Frank '11, Frank-Simon '15)

(1) Let $\gamma = 1/2$ $(n = 1)$, $0 < \gamma \leq 1/2$ $(n \geq 2)$. Then any eigenvalue $E \in \sigma_{\mathrm{p}}(H)$ $(E \in \sigma_{\mathrm{d}}(H)$ if $n = 1)$ satisfies

$$|E|^\gamma \leq C_{\gamma,n} \int |V(x)|^{n/2+\gamma} dx. \qquad (*)$$

If in addition V is radial, $(*)$ also holds for $1/2 < \gamma < n/2$, $n \geq 2$.

(2) When $\gamma = 0$ and $n \geq 3$, H has no eigenvalue if $||V||_{L^{n/2}} \ll 1$.

- Conjecture: $(*)$ holds for $0 < \gamma \leq n/2$ (Laptev-Safronov '09).
- Lieb-Thirring type inequalities for complex-valued potentials: Demuth-Hansmann-Katriel '09, '13, Frank-Sabin '14,...

Two generalizations

(2) Hardy-Lieb-Thirring inequality for real-valued potentials

- $H = -\Delta - \frac{(n-2)^2}{4}|x|^{-2} + V(x)$, $V \in L^{n/2+\gamma}(\mathbb{R}^n; \mathbb{R})$

Theorem (Ekholm-Frank '06)

Let $n \geq 3$ and $\gamma > 0$. Then

$$\sum_{E \in \sigma_{\mathrm{d}}(H)} |E|^\gamma \leq L'_{\gamma,n} \int V_-(x)^{n/2+\gamma} dx.$$

- This inequality fails in general when $\gamma = 0$.
- $\frac{(n-2)^2}{4}$ is critical in the sense that it is the best constant in Hardy's inequality $\frac{(n-2)^2}{4} \int |x|^{-2}|f|^2 dx \leq \int |\nabla f|^2 dx$
- HLT inequality suggests that only the part of the potential which is stronger than $-\frac{(n-2)^2}{4}|x|^{-2}$ creates negative eigenvalues.

Question: Hardy-Keller and Hardy-Lieb-Thirring type inequalities for complex-valued potentials.

- $H_a = -\Delta - a|x|^{-2}$ on $\mathbb{R}^n$, $V \in L^{n/2+\gamma}(\mathbb{R}^n; \mathbb{C})$ with $\gamma > 0$
- V is H_a- form compact (cf. Frank '09)

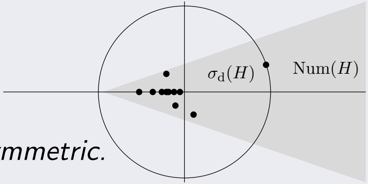
Theorem 1 (H. '16, preprint in arxiv)

Let $n \geq 3$, $a \leq (n-2)^2/4$ *and*

- $0 < \gamma \leq 1/2$

  or

- $0 < \gamma < n/2$ *and* V *is radially symmetric.*



Then, any $E \in \sigma_{\mathrm{d}}(H_a + V)$ *satisfies*

$$|E|^\gamma \leq C_{\gamma,n,a} \int |V(x)|^{n/2+\gamma} dx.$$

Remarks

$$|E|^\gamma \le C_{\gamma,n,a} \int |V(x)|^{n/2+\gamma} dx. \quad 0 < \gamma \le 1/2$$

- When $\gamma = 0$ and $a = (n-2)^2/4$, $H_a + V$ may have eigenvalues as soon as $V < 0$ even if $||V||_{L^{n/2}} \ll 1$ and V: real-valued.
- For $\gamma > 1/2$, one has
$$\mathrm{dist}(E, [0,\infty))^{1/2} |E|^{\gamma - 1/2} \le C_{\gamma,n,a} \int |V(x)|^{n/2+\gamma} dx.$$
The case $-\Delta + V$ is due to Frank '15.
- The inverse-square potential $-a|x|^{-2}$ can be replaced by more larger class of potentials V_0 such that either
  (i) $V_0 \in L^{n/2}_{weak}(\mathbb{R}^n; \mathbb{R})$ with $|x|V_0 \in L^n_{weak}$, $x \cdot \nabla V \in L^{n/2}_{weak}$ and
$$-\Delta + V_0 \ge -\delta\Delta, \quad -\Delta - \partial_r(rV_0) \ge -\delta\Delta \quad (\delta > 0,\ r = |x|)$$
  (ii) or $V_0 \in L^{n/2}(\mathbb{R}^n; \mathbb{R})$.

- Decompose $V = |V|^{\frac{1}{2}} \cdot \mathrm{sgn} V |V|^{\frac{1}{2}} =: V_1 V_2$.
- The Birman-Schwinger principle implies that

$$E \in \sigma_{\mathrm{d}}(H) \Leftrightarrow -1 \in \sigma_{\mathrm{d}}(V_1(H_a - E)^{-1} V_2) \Rightarrow ||V_1(H_a - E)^{-1} V_2|| \geq 1$$

- It suffices to show $||V_1(H_a - E)^{-1} V_2||^{\frac{n}{2}+\gamma} \lesssim |E|^{-\gamma} \int |V|^{\frac{n}{2}+\gamma} dx$. This weighted resolvent estimate follows from Hölder's inequality and the following uniform Sobolev estimate with $\frac{1}{p} = \frac{1}{n+2\gamma} + \frac{1}{2}$.

Theorem 2 (Uniform Sobolev estimates (H. '16))

Let $n \geq 3$, $H_a = -\Delta - a|x|^{-2}$ and $a \leq (n-2)^2/4$. Then

$$||(H_a - z)^{-1}||_{L^p \to L^{p'}} \leq C_{p,n,a} |z|^{-\frac{n+2}{2} + \frac{n}{p}}, \quad z \in \mathbb{C} \setminus [0, \infty)$$

for all $\frac{2n}{n+2} < p \leq \frac{2(n+1)}{n+3}$. (The case $p = \frac{2n}{n+2}$ can fail.)

Proof of Theorem 2 (1)

Theorem 2

$$||(H_a - z)^{-1}||_{L^p \to L^{p'}} \le C_{p,n,a}|z|^{-\frac{n+2}{2}+\frac{n}{p}}, \quad \frac{2n}{n+2} < p \le \frac{2(n+1)}{n+3}$$

- Let $a = (n-2)^2/4$ (critical case) and $P_r : L^2(\mathbb{R}^n) \to L^2_{\rm rad}(\mathbb{R}^n)$ be the projection, $P_r^\perp = 1 - P_r$. Decompose

$$H_a = P_r H_a P_r + P_r^\perp H_a P_r^\perp$$

- The radial part $P_r H_a P_r$ is unitarily equivalent to the radial part of $-\Delta_{\mathbb{R}^2}$ and the estimates for $P_r(H_a - z)^{-1} P_r$ follow from the following estimates for the 2D free resolvent $(-\Delta_{\mathbb{R}^2} - z)^{-1}$:

$$||(-\Delta_{\mathbb{R}^2} - z)^{-1}||_{L_r^{p_2} L_\theta^2(\mathbb{R}^2) \to L_r^{p_2'} L_\theta^2(\mathbb{R}^2)} \lesssim |z|^{-2-2/p_2}, \quad 1 < p_2 < 4/3.$$

(Frank-Simon '15)

- For the "non-radial" part $P_r^\perp(H_a - z)^{-1}P_r^\perp$, we use an iterated resolvent equation

$$R_a = R_0 + aR_0|x|^{-1}\cdot|x|^{-1}R_0 + a^2R_0|x|^{-1}\cdot|x|^{-1}R_a|x|^{-1}\cdot|x|^{-1}R_0$$

  where $R_0 = (-\Delta_{\mathbb{R}^n} - z)^{-1}$ and $R_a = (H_a - z)^{-1}$.
- Desired estimates for $P_r^\perp R_a P_r^\perp$ follow from $P_r^\perp \in \mathbb{B}(L^p)$ and
  - (i) $R_0 \in \mathbb{B}(L^p, L^{p'})$, $R_0|x|^{-1} \in \mathbb{B}(L^2, L^{p'})$ (and its dual)
    Cf. Kenig-Ruiz-Sogge '87, Gutiérrez '04
  - (ii) $P_r^\perp|x|^{-1}R_a|x|^{-1}P_r^\perp \in \mathbb{B}(L^2)$
- On $\mathrm{Ran}P_r^\perp$, $-c_1\Delta_{\mathbb{R}^n} \le H_a \le -c_2\Delta_{\mathbb{R}^n}$ and we can use a multiplier method (integration by parts) to obtain (ii).
  cf. Burq, Planchon, Stalker & Tahvildar-Zadeh '04,
  Barceló, Vega & Zubeldia '13.

Remarks and future topics

Remark:

- Theorem 2 (uniform Sobolev estimates) can be also applied to study the Schrödinger equation $(i\partial_t - H_a)u = F$, $u|_{t=0} = \psi$.
  - Endpoint Strichartz estimates for $a < (n-2)^2/4$

    $$||u||_{L^2(\mathbb{R};L^{\frac{2n}{n-2}}(\mathbb{R}^n))} \lesssim ||\psi||_{L^2} + ||F||_{L^2(\mathbb{R};L^{\frac{2n}{n+2}}(\mathbb{R}^n))}$$

    (The case $F \equiv 0$ is due to B-P-S-T '03, '04)
  - Non-endpoint Strichartz estimates and weak-type endpoint Strichartz estimates for $a = (n-2)^2/4$.

Future topic:

- Hardy-Lieb-Thirring inequality for complex-valued potentials
- Other models with scaling-critical perturbation such as the Aharonov-Bohm Hamiltonian or Dirac operators with Coulomb singularities.

Thank you very much for your attention