Spectral properties of certain non-selfadjoint differential operators with non-local boundary conditions: Examples in search of a theory

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Outline

Stochastic Processes, elliptic operators and non-local boundary conditions

Spectral Analytic Results

Apology:

I am probabilist

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Goal:
Advertize a class of differential operators with non-local boundary conditions, which until now mainly was considered by probabilists.
Try to convince you that this class of problems has some surprising properties.

Motivation

When considering a diffusion process in a bounded domain, i.e. an elliptic operator of the form

$$\mathcal{L}u := \sum_{i,j=1}^{d} a_{ij} D_i D_j u + \sum_{j=1}^{d} b_j D_j u,$$

one needs to put boundary conditions in order to specify a well defined process (in analytic terms a one parameter semigroup).

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- Dirichlet boundary conditions (killing at the boundary)
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- mixtures of Dirichlet and Neumann.

There are many other ways to produce a well-defined stochastic process.

Diffusions with jump boundary

Intuitive description of the process: Let $D \subset \mathbb{R}^d$ be a smooth bounded domain, $\mu(y, \cdot)$ a probability measure on D for every $y \in \partial D$

1) Start a diffusion $(X_t)_t$ at $x \in D$ and wait until it hits the boundary ∂D of D at a point $y \in \partial D$.

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3) Start an i.i.d. copy of $(X_t)_t$ at x_1 and wait until it hits the boundary ∂D of D.

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3) Start an i.i.d. copy of $(X_t)_t$ at x_1 and wait until it hits the boundary ∂D of D.
4) Move to stage 2).

The stochastic process

Let $W^{\rho,0}$ be a diffusion process on D corresponding to $\mathcal{L}$ killed at the boundary with initial distribution ρ. Define

$$\tau_1 = \sigma_1 = \inf\{t \geq 0 \mid W_t^{\rho,0} \in \partial D\}, \Theta_1 = W^{\rho,0}(\sigma_1)$$

and

$$\sigma_{n+1} = \inf\{t \geq 0 \mid W^{\mu_{\Theta_n},n} \in \partial D\}, \Theta_{n+1} = W^{\mu_{\Theta_n},n}(\sigma_{n+1})$$

$$\tau_{n+1} = \tau_n + \sigma_{n+1}.$$

Then the process just described intuitively is given by

$$\boxed{X_t = \sum_{n=0}^{\infty} \mathbf{1}_{t \in [\tau_n, \tau_{n+1})} W^{\mu_{\Theta_n},n}(t - \tau_n)}$$

Some history and motivation

- Processes of this type appeared in Feller's famous analysis of all possible extensions of a given one-dimensional diffusion processes in an interval up to the first hitting time of the boundary.

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- Processes of this type appeared in Feller's famous analysis of all possible extensions of a given one-dimensional diffusion processes in an interval up to the first hitting time of the boundary.
- Used in a number of applications such as e.g. neuroscience. [*] Membrane potential of a single neuron described by diffusion, if this process hits a certain level, it fires and the membrane potential is set back to zero.
- Also appear in mathematical finance in order to describe double knock out barrier options, in statistical inference of survival analysis, versions of the google page rank algorithm...

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Analytic Aspects

Assumptions:
Let

$$\mathcal{L}u := \sum_{i,j=1}^{d} a_{ij} D_i D_j u + \sum_{j=1}^{d} b_j D_j u,$$

where $a_{ij} \in C^{2,\alpha}(\mathbb{R}^d)$ and $b \in C^{1,\alpha}(\mathbb{R}^d)$ and

$$\frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq \eta |\xi|^2.$$

We furthermore assume that $\mu : \partial D \to \mathcal{P}(D)$ be continuous with respect to the weak topology on $\mathcal{P}(D)$.

Analytic Aspects

It is easy to see that for every $f \in L^\infty(D)$

$$\mathbb{E}_x[f(X_t)] = \int_D p^\mu(t,x,y) f(y)\, dy$$

Theorem (Ben-Ari/Pinsky SPA 2009)

The mapping

$$L^1(D) \ni g \mapsto \int_D g(x) p^\mu(t,x,y)\, dx \in L^1(D)$$

defines a strongly continuous semigroup.

Analytic Aspects

Setting

$$W(D) := \bigcap_{1<p<\infty} W^{2,p}_{loc}(D)$$

and define an operator in $L^\infty(D)$:

$$\mathcal{D}(L_\mu) := \Big\{ u \in C(\bar{D}) \cap W(D) \,|\mathcal{L}u \in L^\infty(D)$$

$$u(z) := \int_D u(x)\mu(z, dx) \, \forall z \in \partial D \Big\}$$

$$L_\mu u := \mathcal{L}u \quad (u \in \mathcal{D}(L_\mu)$$

Analytic Aspects

Theorem (Arendt et al. JFA 2017)

- *L_μ is the generator of a holomorphic semigroup $T_\mu(t)$ on $L^\infty(D)$.*
- *The operators $T_\mu(t)$ are positive contractions.*
- *$T_\mu(t)$ is compact for every $t > 0$*
- *The semigroup $(T_\mu(t))_{t\geq 0}$ is strong Feller.*
- *There exists a positive projection P of rank one and constants $\varepsilon > 0$ and $M \geq 1$, such that*

  $$\|T_\mu(t) - P\| \leq Me^{-\varepsilon t}$$

  for all $t > 0$.

Further Results and Consequences

- First observe that

$$T_{\mu}(t)\mathbf{1}_A(x) = \mathbb{P}_x(X_t \in A).$$

[*] Important Contributions due to Grigorescu/Kang, Ben-Ari/Pinsky and Leung/Li/Rakesh

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- Let us assume now that $\mu(y,\cdot) = \mu$ for every $y \in \partial D$. By the last item of the previous theorem the semigroup T_μ has an invariant distribution ν^μ:

$$\frac{1}{C}\int_D g^D(z,x)\,d\mu(z)dx.$$

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- The process is uniformly ergodic in the sense that

$$0 > \gamma(\mu) := \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} \big\| \mathbb{P}_x\big(X_t \in \cdot\,\big) - \nu^\mu \big\|_{TV}$$

- $\gamma(\mu) := \sup\{\Re\lambda \mid 0 \neq \lambda \text{ is an eigenvalue for } \mathcal{L}_\mu\}$

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Further Spectral Results [Ben-Ari, Pinsky]

If the differential operator $\mathcal{L}$ is reversible, i.e. if $b := a\nabla Q$ and

$$\mathcal{L} := \frac{1}{2}e^{-2Q}\nabla \cdot e^{2Q}\nabla$$

and without loss of generality $\int_D e^{2Q}\,dx = 1$ and

$$\mu := e^{2Q(x)}\,dx$$

then

- All eigenvalues of L_μ are real

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then

- All eigenvalues of L_μ are real
- Let $(\lambda_i^D)_{i \in \mathbb{N}_0}$ denotes the sequence of the Dirichlet realization of $\mathcal{L}$ in $L^2(D, \mu)$. Then

$$\boxed{\lambda_1^D \leq \gamma(\mu) < \lambda_0^D.}$$

Formula for the resolvent

Observe that

$$T_\mu(t)f(x) = T^D(t)f(x) + \int_0^t \mathbb{P}_x(\tau_D \in ds) \int_D T_\mu(t-s)f(x)\,\mu(dx)$$

where $T^D(t)$ denotes the semigroup with Dirichlet boundary conditions.

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$$\boxed{R_z f(x) = R_z^D f(x) + \frac{\mathbb{E}_x\left[e^{-z\tau_D}\right]}{1 - \int_D \mathbb{E}_y\left[e^{-z\tau_D}\right]\mu(dy)} \int_D R_z^D f(y)\mu(dy)}$$

In the case $\mu = e^{2Q(x)}dx$ or if μ coincides with the normalized ground state one can show that the resolvent is analytic outside the real axis.

Another surprising (?) result

Theorem (Ben-Ari/Pinsky JFA 2007)

Consider the operator $\frac{1}{2}\Delta$ *in the d-dim. cube* $(0,1)^d$ *and let* μ *be the Lebesgue measure. Then*

- *If* $d \leq 10$ *then* $\gamma(\mu) = \lambda_1^D$.
- *If* $d > 10$ *then* $\lambda_1^D < \gamma(\mu) < \lambda_0^D$.

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Theorem (Ben-Ari/Pinsky JFA 2007)

In general if the spectral gap corresponds to a real eigenvalue, then

$$\gamma(\mu) < \lambda_0^D.$$

The case of one dimension

Let us consider the problem

$$\frac{1}{2}u'' = \lambda u \quad \text{in } (a, b)$$

$$u(a) = \int_a^b u(y)\mu_a(dy) \quad \text{and} \quad u(b) = \int_a^b u(y)\mu_b(dy)$$

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Theorem (PAMS 2008, Li-Leung-Rakesh)

*All eigenvalues are **real** and non-positive. Furthermore,*

$$\sup_{\mu_a, \mu_b} \gamma(\mu_a, \mu_b) = \lambda_0^D = -\frac{\pi^2}{2(b-a)^2}$$

and if $\mu = \mu_a = \mu_b$ *we have*

$$\boxed{\gamma(\mu, \mu) = \lambda_1^D = -\frac{2\pi^2}{(b-a)^2}}$$

The spectrum is real

This is shown in the following way: $z^2/2 = \lambda$, $(a, b) = (0, 1)$, general solution of eigenvalue problem

$$u(t) = A\cos(zt) + B\sin(zt),$$

Boundary conditions can be satisfied if and only if

$$\begin{aligned} F(z) := \sin(z) - \int_0^1 \sin(zt)\,\mu_1(dt) - \int_0^1 \sin(z(1-s))\mu_0(ds) \\ + \int_0^1 \int_0^1 \sin(z(t-s))\mu_0(ds)\mu_1(dt) \end{aligned}$$

is zero. Approximate the integrals by a Riemann sum and follow a strategy of Pólya †

† Über die Nullstellen gewisser ganzer Funktionen, Math. Z. 2 (1918)

Theorem (Li-Leung, unpublished)

$$\sup_{\mu_a,\mu_b} \gamma(\mu_a,\mu_b) = \underline{\lambda_0^D} = -\frac{\pi^2}{2(b-a)^2},$$

if $\mu = \mu_a = \mu_b$ *we have*

$$\gamma(\mu,\mu) = \underline{\lambda_1^D} = -\frac{2\pi^2}{(b-a)^2}{}^{a}$$

and

$$\inf_{\mu_a,\mu_b} \gamma(\mu_a,\mu_b) = \underline{\lambda_2^D}.$$

[a] first two results have probabilistic proofs K/Wübker EJP 2011

Simple bound

The case $\mu_a = \mu = \mu_b$: Instead of considering

$$\sup_{x \in (a,b)} \left\| \mathbb{P}_x\left(X_t \in \cdot\right) - \nu^\mu \right\|_{TV}$$

we look at

$$\sup_{x,y \in (a,b)} \left\| \mathbb{P}_x\left(X_t \in \cdot\right) - \mathbb{P}_y\left(X_t \in \cdot\right) \right\|_{TV}$$

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Observe that for two points $x < y$ symmetric with respect to $c = \frac{a+b}{2}$ we have for every A

$$\|\mathbb{P}_x\big(X_t \in \cdot\big) - \mathbb{P}_y\big(X_t \in \cdot\big)\|_{TV} = \mathbb{P}_x\big(\tau_{(a,c)} > t\big) = e^{-tH^D_{(a,c)}}\mathbf{1}_{(a,c)}(x),$$

where $H^D_{(a,c)}$ denotes the operator $\frac{1}{2}\frac{d^2}{dx^2}$ in (a, c) with Dirichlet boundary conditions.

Upper bound

Definition

- A coupling of the process $(X_t)_{t\geq 0}$ is a pair of processes $((X_t^1, X_t^2))_{t\geq}$, which are defined on the same probability space, such that the marginals X^1 and X^2 have the same distribution as $(X_t)_{t\geq 0}$.

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- The coupling is called successful, if the random time

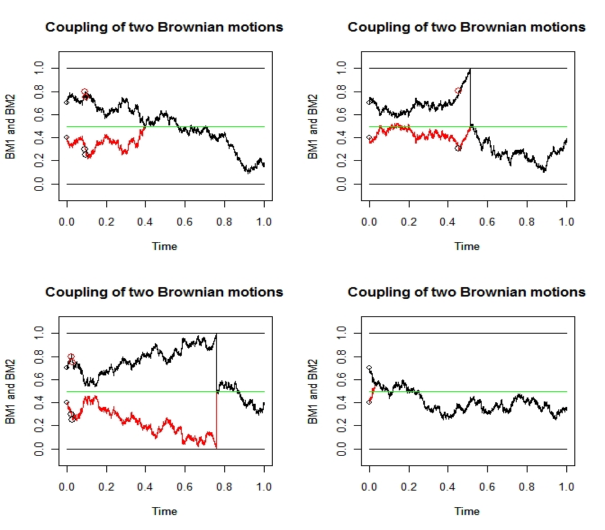
$$\tau = \inf\{t \geq 0 \mid \forall s \geq t : \ X_s^1 = X_s^2\}$$

is finite almost surely.

Lemma (Coupling inequality)

$$d_t(x, y) := \left\|\mathbb{P}_x(X_t \in \cdot) - \mathbb{P}_y(X_t \in \cdot)\right\|_{TV} \leq 2\mathbb{P}_{xy}(\tau > t)$$

Method of Proof: Coupling



Theorem (Li-Leung-Rakesh)

Suppose that $d > 1$ is odd and μ is an absolutely continuous probability measure on the open unit ball $B \subset \mathbb{R}^d$ with an L^2-density. If

$$(0,1) \ni r \mapsto r^{-d}\mu(\{x \in \mathbb{R}^d \mid |x| < r\})$$

is an increasing function of r, then the eigenvalues are real.

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In the higher dim. situation there can be complex eigenvalues, but the examples solved by Li-Leung-Rakesh always give that the eigenvalue giving the spectral gap is real.

Open problem: Is it possible to characterize all jump distributions giving rise to purely real spectrum?

Brownian motion with constant drift and random jumps

In many cases

$$\boxed{\lambda_1^D \leq \gamma(\mu) < \lambda_0^D}$$

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Is it true that the eigenvalue corresponding to the spectral gap is real?[‡]

[‡]Questions formulated by Ben-Ari/Pinsky JFA 2007

This is not true!

Spectral Analysis of the 1d case with deterministic jumps

Let us consider in the interval $(0, 1)$

$$L^{\sigma,b} = \frac{\sigma^2}{2}\frac{d^2}{dx^2} + b\frac{d}{dx}$$

with $\mu = \delta_{1/2}$.

[§]see K/Wübker JFA 2011, Ben-Ari ECP 2014

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Then

- $\lambda_0^{D,\sigma,b} \to \infty$ as $b \to \infty$.
- $\nu^{\delta_{1/2},\sigma,b} \to 2 \cdot \mathbf{1}_{[1/2,1)}(x)\, dx$ as $b \to \infty$.

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- $\nu^{\delta_{1/2},\sigma,b} \to 2 \cdot \mathbf{1}_{[1/2,1)}(x)\,dx$ as $b \to \infty$.
- $$\boxed{\gamma(\delta_{1/2}) = \begin{cases} 2\sigma^2\pi^2 + \frac{b^2}{2} & \text{if } |b| \le \sqrt{3}2\sigma^2\pi^2 \\ 8\sigma^2\pi^2 & \text{otherwise} \end{cases}}$$ [§]

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Further detailed spectral properties

Let us finally come back to the simplest case

$$\mathcal{L} = -\frac{d^2}{dx^2} \text{ in } L^2((-\pi/2, \pi/2)), \mu = \delta_{a\pi/2}$$

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$$\mathcal{D}(L_\mu) = \{\psi \in H^2((-\pi/2, \pi/2)) \mid \psi(-\pi/2) = \psi(a\pi/2) = \psi(\pi/2)\}.$$

- L_μ is densely defined and closed
- L_μ is quasi-accretive,

$$\Re(\psi, L_\mu \psi) \geq -\frac{1}{16}\|\psi\|^2$$

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$$\sigma(L_\mu) = \left\{ \left(\frac{4m}{1-a}\right)^2, (2m)^2, \left(\frac{4m}{1+a}\right)^2 \mid m \in \mathbb{N}_0 \right\}$$

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- *If $a \in \mathbb{Q}$ then the eigenfunctions do not form a minimal complete set*
- *If $a \notin \mathbb{Q}$ then the eigenfunctions form a minimal complete set, but do not form a conditional (Schauder) basis.*

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- *If $a \notin \mathbb{Q}$ then the eigenfunctions form a minimal complete set, but do not form a conditional (Schauder) basis.*

Open problem: If $a \in \mathbb{Q}$ do the eigenfunctions together with the generalized eigenfunctions form a conditional basis?

Theorem (K/DK 2016,quasi-selfadjointness)

There exists an rather explicit positive bounded injective operator Θ such that

$$H^*\Theta = \Theta H.$$

(Θ is not bounded invertible)

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- These operators turn out to have interesting spectral properties.
- There does not seem to exist a general approach, which can be used to answer basic spectral questions such as,

Thank you for your attention!

- A complete set $(\psi_j)_j$ in a Hilbert space is minimal complete if the removal of any term makes it incomplete. A minmal set is complete if there exists a sequence $(\varphi_j)_j$ such that (ψ_j, φ_j) is biorthogonal.
- A minimal complete set $(\psi_j)_j$ is a conditional basis if for all f in the Hilbert space there exists uniquely $(\alpha_j)_j \subset \mathbb{C}$ such that
$$f = \sum_j \alpha_j \psi_j$$
- $(\psi_i)_j$ normalized sequence in a Hilbert space $\mathcal{H}$ is an unconditional (Riesz) basis if it is a conditional basis and for all $f \in \mathcal{H}$
$$C^{-1}\|f\|^2 \leq \sum_j |(\psi, f)|^2 \leq C\|f\|^2$$