On the stability of linearized Euler's equations in compressible flows

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based on works in collaboration with

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Context and motivation

Aeroacoustics : sound propagation in flows

⟶ Many applications in aeronautics

The mathematical models

Euler
$$\begin{cases} (\partial_t + M \cdot \nabla) v + (v \cdot \nabla) M + \nabla p = 0 \\ (\partial_t + M \cdot \nabla) p + \nabla \cdot v = 0 \end{cases}$$

No flow : $\mathrm{M} = 0$

Wave equation
$$\begin{cases} \partial_t v + \nabla p = 0 \\ \partial_t p + \nabla \cdot v = 0 \end{cases}$$

Uniform flow : $\nabla M = 0$

Convected wave equation
$$\begin{cases} (\partial_t + M \cdot \nabla) v + \nabla p = 0 \\ (\partial_t + M \cdot \nabla) p + \nabla \cdot v = 0 \end{cases}$$

The mathematical models

Euler

$$\begin{cases} (\partial_t + M \cdot \nabla) v + (v \cdot \nabla) M + \nabla p = 0 \\ \nabla \cdot v = 0 \end{cases}$$

(incompressible fluids)

The mathematical models

Euler
$$\begin{cases} (\partial_t + M\cdot\nabla)v + (v\cdot\nabla)M + \nabla p = 0 \\ \\ (\partial_t + M\cdot\nabla)p + \nabla\cdot v = 0 \end{cases}$$

U is the perturbation of Lagrangian displacement

$$v = (\partial_t + M\cdot\nabla)U + (U\cdot\nabla)M \qquad p = \nabla\cdot U$$

Galbrun
$$(\partial_t + M\cdot\nabla)^2 U - \nabla(\nabla\cdot U) = 0$$

The mathematical models

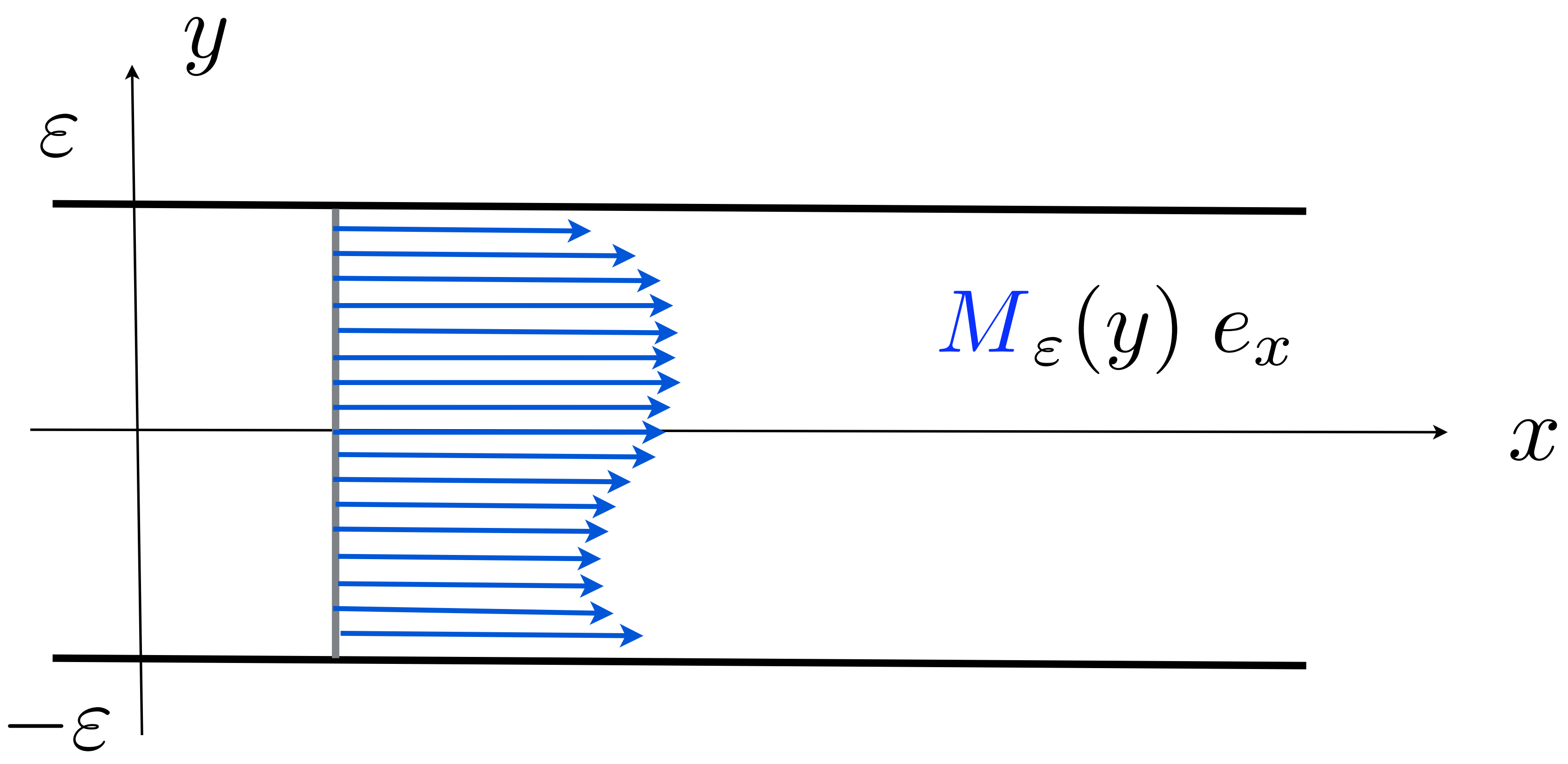
Galbrun $\quad (\partial_t + M \cdot \nabla)^2 U - \nabla(\nabla \cdot U) = 0$

Boundary conditions

at rigid walls : $\quad U \cdot n = 0$

Non slipping condition

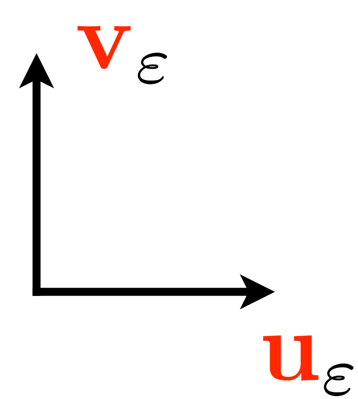
The problem under consideration : Acoustic wave propagation in a thin duct



Galbrun's equations in a 2D thin duct

$$(\widetilde{\mathcal{P}})_\varepsilon \begin{cases} (\partial_t + M_\varepsilon \partial_x)^2 \,\mathbf{u}_\varepsilon - \partial_x(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \\ \\ (\partial_t + M_\varepsilon \partial_x)^2 \,\mathbf{v}_\varepsilon - \partial_y(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \end{cases}$$

$$U_\varepsilon = (\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon)^t$$



Galbrun's equations in a 2D thin duct

$$(\widetilde{\mathcal{P}})_\varepsilon \left\{ \begin{array}{l} (\partial_t + M_\varepsilon \partial_x)^2\, \mathbf{u}_\varepsilon - \partial_x(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \\ \\ (\partial_t + M_\varepsilon \partial_x)^2\, \mathbf{v}_\varepsilon - \partial_y(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \end{array} \right.$$

$$\mathbf{v}_\varepsilon(x, \pm\varepsilon, t) = 0$$

The problem is well-posed as soon as

$$M_\varepsilon \in W^{1,\infty}(-1,1)$$

Galbrun's equations in a 2D thin duct

$$(\widetilde{\mathcal{P}})_\varepsilon \begin{cases} (\partial_t + M_\varepsilon \partial_x)^2 \, \mathbf{u}_\varepsilon - \partial_x(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \\ (\partial_t + M_\varepsilon \partial_x)^2 \, \mathbf{v}_\varepsilon - \partial_y(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \end{cases}$$

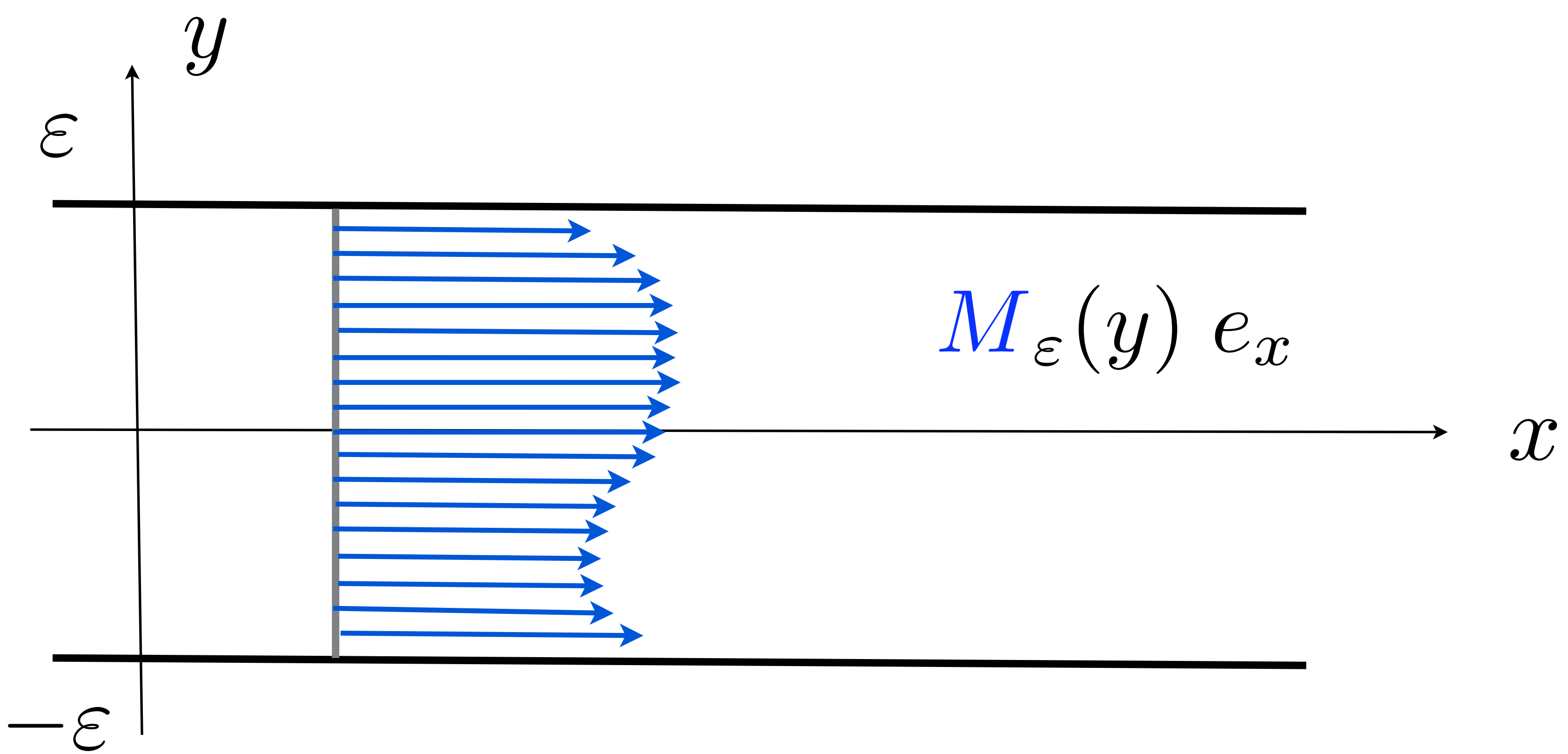
Question : is this evolution problem stable or not ?

What conditions on the profile for stability or instability ?

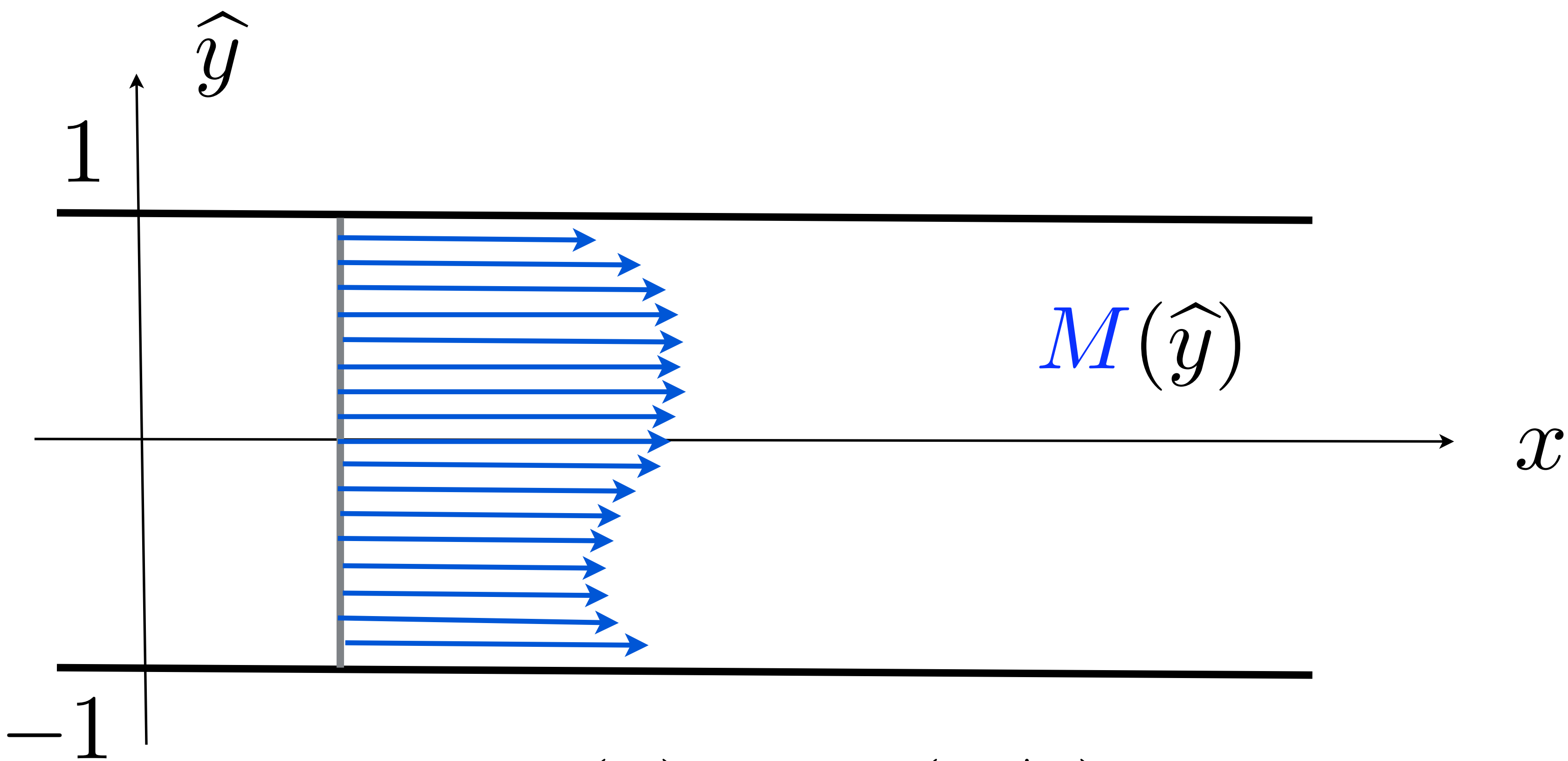
Most known results concern the incompressible case:
Rayleigh (1879), Fjortoft (1950), Drazin (2004), Schmid-Henningson...

Our approach to the problem

Asymptotic analysis for small ε



A preliminary analysis : Acoustic wave propagation in a thin duct



$$M_\varepsilon(y) = M(y/\varepsilon)$$

Galbrun's equations in a 2D thin duct

$$(\widetilde{\mathcal{P}})_\varepsilon \begin{cases} (\partial_t + M_\varepsilon \partial_x)^2\, \mathbf{u}_\varepsilon - \partial_x(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \\ \\ (\partial_t + M_\varepsilon \partial_x)^2\, \mathbf{v}_\varepsilon - \partial_y(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \end{cases}$$

$$\mathbf{v}_\varepsilon(x, \pm\varepsilon, t) = 0$$

$$\boxed{\|\mathbf{u}_\varepsilon\|_{L^2_x(L^2_y)} + \|\mathbf{v}_\varepsilon\|_{L^2_x(L^2_y)} \;\leq\; C_0\; e^{\frac{t}{\varepsilon}\|M'\|_\infty}}$$

Proof : energy estimates on Linearized Euler's Equations

A dimensionless model

Scaling

$$\mathbf{u}_\varepsilon(x,y,t) = u_\varepsilon(x,\frac{y}{\varepsilon},t), \qquad \mathbf{v}_\varepsilon(x,y,t) = \boxed{\varepsilon}\, v_\varepsilon(x,\frac{y}{\varepsilon},t)$$

Scaled model

$$(\mathcal{P})_\varepsilon \quad \begin{cases} (\partial_t + M\partial_x)^2 u_\varepsilon - \partial_x(\partial_x u_\varepsilon + \partial_y v_\varepsilon) = 0 \\ \boxed{\varepsilon^2}(\partial_t + M\partial_x)^2 v_\varepsilon - \partial_y(\partial_x u_\varepsilon + \partial_y v_\varepsilon) = 0 \end{cases}$$

A dimensionless model

Passage to the limit

$$u_\varepsilon \longrightarrow u, \qquad v_\varepsilon \longrightarrow v$$

Formal limit model

$$(\mathcal{P}) \quad \begin{cases} (\partial_t + M\partial_x)^2 u - \partial_x(\partial_x u + \partial_y v) = 0 \\ \partial_x u + \partial_y v = d(x,t) \end{cases}$$

A dimensionless model

Passage to the limit

$$u_\varepsilon \longrightarrow u, \qquad v_\varepsilon \longrightarrow v$$

Formal limit model

$$(\mathcal{P}) \quad \left\{ \begin{array}{l} (\partial_t + M\partial_x)^2 u \ - \partial_x d \ = 0 \\ \\ \partial_x u + \partial_y v = d(x,t) \end{array} \right.$$

The limit model

$$
\boxed{(\mathcal{P}) \quad \begin{cases} (\partial_t + M\partial_x)^2 u - \partial_x d = 0 \\[2ex] \partial_x u + \partial_y v = d(x,t) \end{cases}}
$$

Introducing $E(f)(x,t) := \dfrac{1}{2}\displaystyle\int_{-1}^{1} f(x,y,t)\,dy$, since $v(\pm 1) = 0$

$$
\boxed{\partial_x u + \partial_y v = d(x,t) \quad \Longrightarrow \quad d(x,t) = E\big(\partial_x u\big)}
$$

Since E (mean in y) and ∂_x commute, we obtain

$$
\boxed{(\mathcal{P}) \qquad (\partial_t + M\partial_x)^2\, u - \partial_x^2\big[E(u)\big] = 0}
$$

The quasi 1D model

$$\boxed{(\mathscr{P}) \qquad (\partial_t + M\partial_x)^2\, u - \partial_x^2 \left[E(u)\right] = 0}$$

A quasi-1D model, non local in y

When M is constant, M and E commute :

- One advected 1D wave equation

$$(\partial_t + M\partial_x)^2 \left[E(u)\right] - \partial_x^2 \left[E(u)\right] = 0$$

- Decoupled 1D transport equations

$$(\partial_t + M\partial_x)^2\, u = \partial_x^2 \left[E(u)\right]$$

Main questions relative to this model

For a general Mach profile, is the evolution problem $(\mathcal{P})$ well-posed ?

If not, what are the conditions on the Mach profile for the problem to be well-posed ?

Outline for the rest of the talk

1 Reduction to the spectral analysis of $\mathbf{A}(\mathbb{M})$

Well-posedness $\longleftrightarrow$ spectrum $\subset \mathbb{R}$

2 General structure of the spectrum of $\mathbf{A}(\mathbb{M})$

Non real spectrum is made of eigenvalues

3 Results on the absence of nonreal eigenvalues

Stable Mach profiles

4 Results of existence of nonreal eigenvalues

Unstable Mach profiles

Towards the well-posedness analysis

$$u(x,y,t) \quad \xrightarrow{\mathcal{F}_x} \quad \mathbf{u}(k,y,t)$$

$$\boxed{\mathbf{U}(\mathrm{k,y,t}) = \Big(\ \mathbf{u}(\mathrm{k,y,t}), \ \big[(\partial_t + ikM)\mathbf{u}\big](k,y,t) \Big)^t}$$

First order evolution problem: $\mathrm{d}_t\mathbf{U} + ik\mathbf{A}(M)\mathbf{U} = 0$

where $\mathbf{A}(\mathrm{M})$ is the operator in $\mathrm{L}^2(-1,1)^2$

$$\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ & \\ E & M \end{pmatrix}$$

Towards the well-posedness analysis

As the operator $\mathbf{A}(\mathrm{M})$ is bounded, we can write

$$\widehat{U}(k,t) = e^{-ik\mathbf{A}(M)t}\,\widehat{U}_0(k)$$

The problem is to get uniform bounds in k .

As $\mathbf{A}(\mathrm{M})$ is non normal, general theorems from semi-group theory do not apply.

Theorem : if $\sigma\big(\mathbf{A}(M)\big) \not\subset \mathbb{R}$, $(\mathcal{P})$ is strongly ill-posed

Conjecture : if $\sigma\big(\mathbf{A}(M)\big) \subset \mathbb{R}$, $(\mathcal{P})$ is well-posed (*)

(*) has been proven in some cases (see later)

General properties of $\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ E & M \end{pmatrix}$

Let $S(u,v) = (v,u)$, i. e $S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ then one has

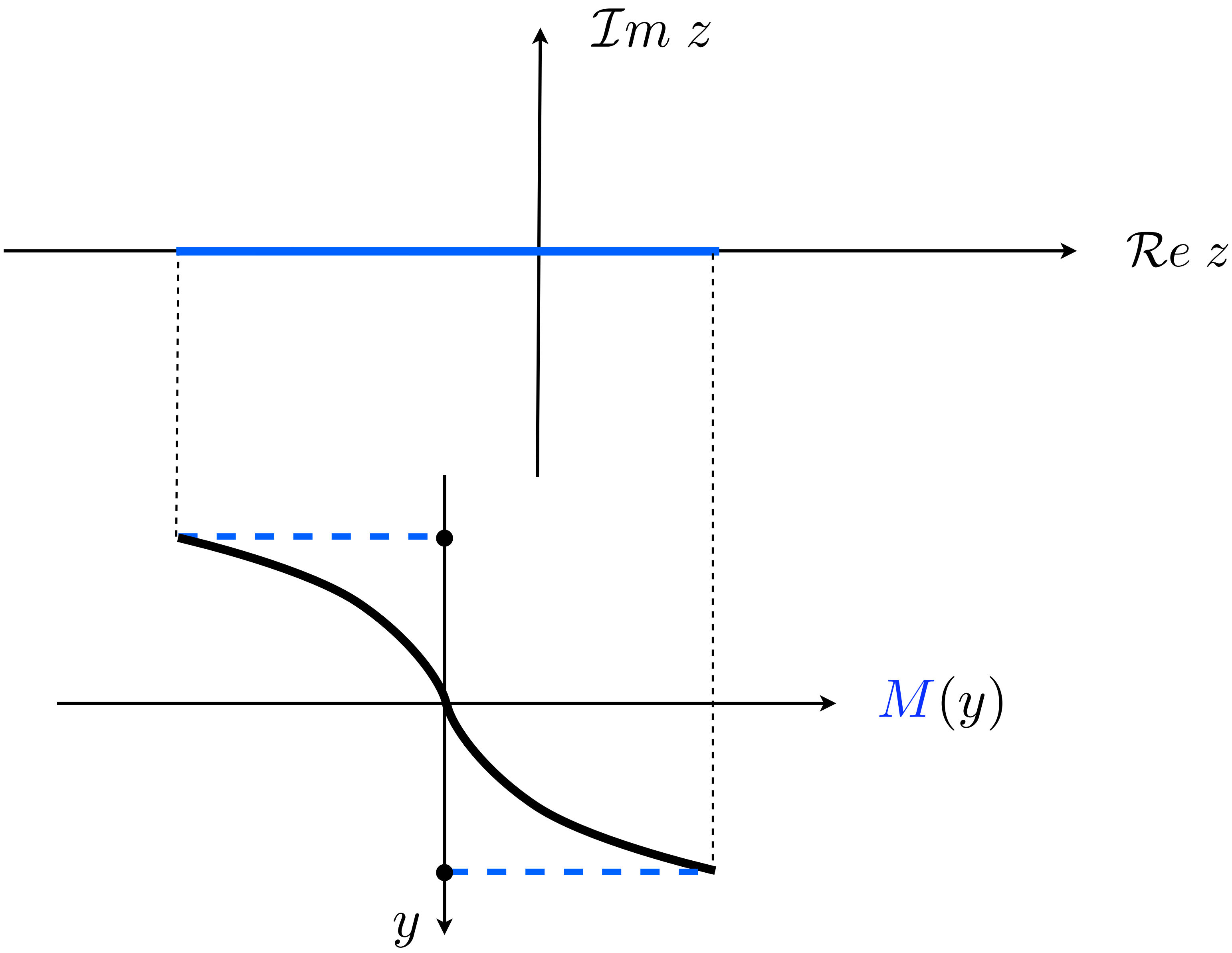
$$\mathbf{A}(M)^* = S \circ \mathbf{A}(M) \circ S$$

The spectrum of $\mathbf{A}(M)$ is symmetric w.r.t. the real axis.

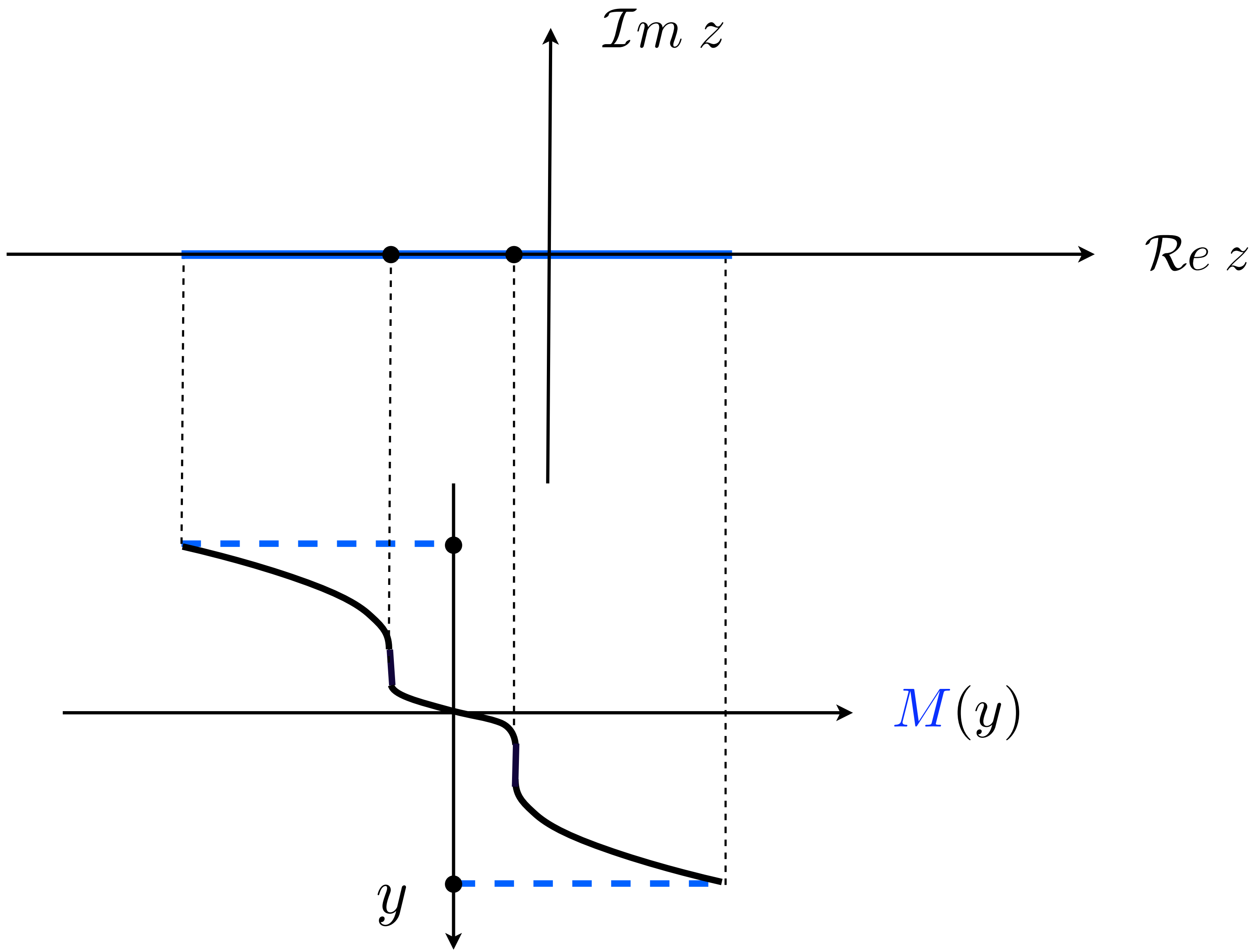
The operator $\mathbf{A}(\mathrm{M})$ is a compact perturbation of

$$\mathbf{A}_0(M) = \begin{pmatrix} M & I \\ \\ 0 & M \end{pmatrix}$$

Structure of the spectrum of $\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ E & M \end{pmatrix}$



Structure of the spectrum of $\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ E & M \end{pmatrix}$



Eigenvalues of $\mathbf{A}(\mathrm{M})$ (1)

Lemma : A number $\lambda \in \mathbb{C} \setminus \mathbf{Im}\, M$ is an eigenvalue of $\mathbf{A}(\mathrm{M})$ if and only if:

$$(\mathcal{E}) \qquad F_M(\lambda) = 1, \quad \text{where } F_M(\lambda) := \frac{1}{2}\int_{-1}^{1} \frac{dy}{\big(\lambda - M(y)\big)^2}$$

This eigenvalue is simple associated with

$$\mathbf{U}_\lambda := (\mathbf{u}_\lambda, \mathbf{v}_\lambda) = \Big(\frac{1}{(\lambda - M)^2}, \frac{1}{(\lambda - M)}\Big)$$

Remark : we can also write $F_M(\lambda) := E\big((\lambda - M)^{-2}\big)$

$$\boxed{\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ E & M \end{pmatrix}}$$

$$\mathbf{A}(\mathrm{M})\ \mathrm{U} = \lambda\, U \quad \Longleftrightarrow \quad \begin{cases} M\ u + v = \lambda\ u \\ E(u) + M\ v = \lambda\ v \end{cases}$$

$$\boxed{\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ E & M \end{pmatrix}}$$

$$\mathbf{A}(\mathrm{M})\ \mathrm{U} = \lambda\ U \iff \begin{cases} u = v\,/\,(\lambda - M) \\ v = E(u)/(\lambda - M) \end{cases}$$

$$\implies \quad u = \frac{E(u)}{(\lambda - M)^2} \quad \text{and} \quad v = \frac{E(u)}{(\lambda - M)}$$

$$\implies \quad E(u)\left[E((\lambda - M)^{-2}) - 1 \right] = 0$$

$$\boxed{\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ E & M \end{pmatrix}}$$

$$\mathbf{A}(\mathrm{M})\ \mathrm{U} = \lambda\ U \quad \Longleftrightarrow \quad \begin{cases} u = v \,/\, (\lambda - M) \\ E(u) + M\ v = \lambda\ v \end{cases}$$

$$\Longrightarrow \quad u = \frac{E(u)}{(\lambda - M)^2} \quad \text{and} \quad v = \frac{E(u)}{(\lambda - M)}$$

$$U \neq 0 \implies E(u) \neq 0 \quad \text{and} \quad E\left((\lambda - M)^{-2}\right) = 1$$

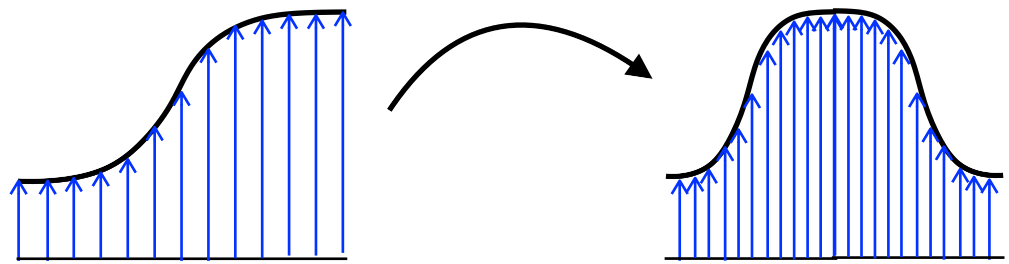
Invariance properties

The function $F_M(\lambda) = \dfrac{1}{2}\displaystyle\int_{-1}^{1} \frac{dy}{(\lambda - M(y))^2}$ is invariant by

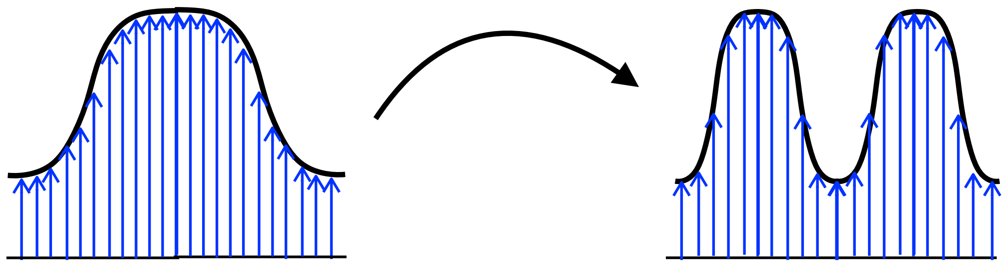
Rearrangement : $M \longrightarrow M_g := M \circ g$

$\boxed{\begin{array}{c} g : [-1,1] \mapsto [-1,1] \\ \text{measure preserving bij.} \end{array}}$

Symmetrization :



Periodization :



The stability / instability of a profile is preserved by such transformations

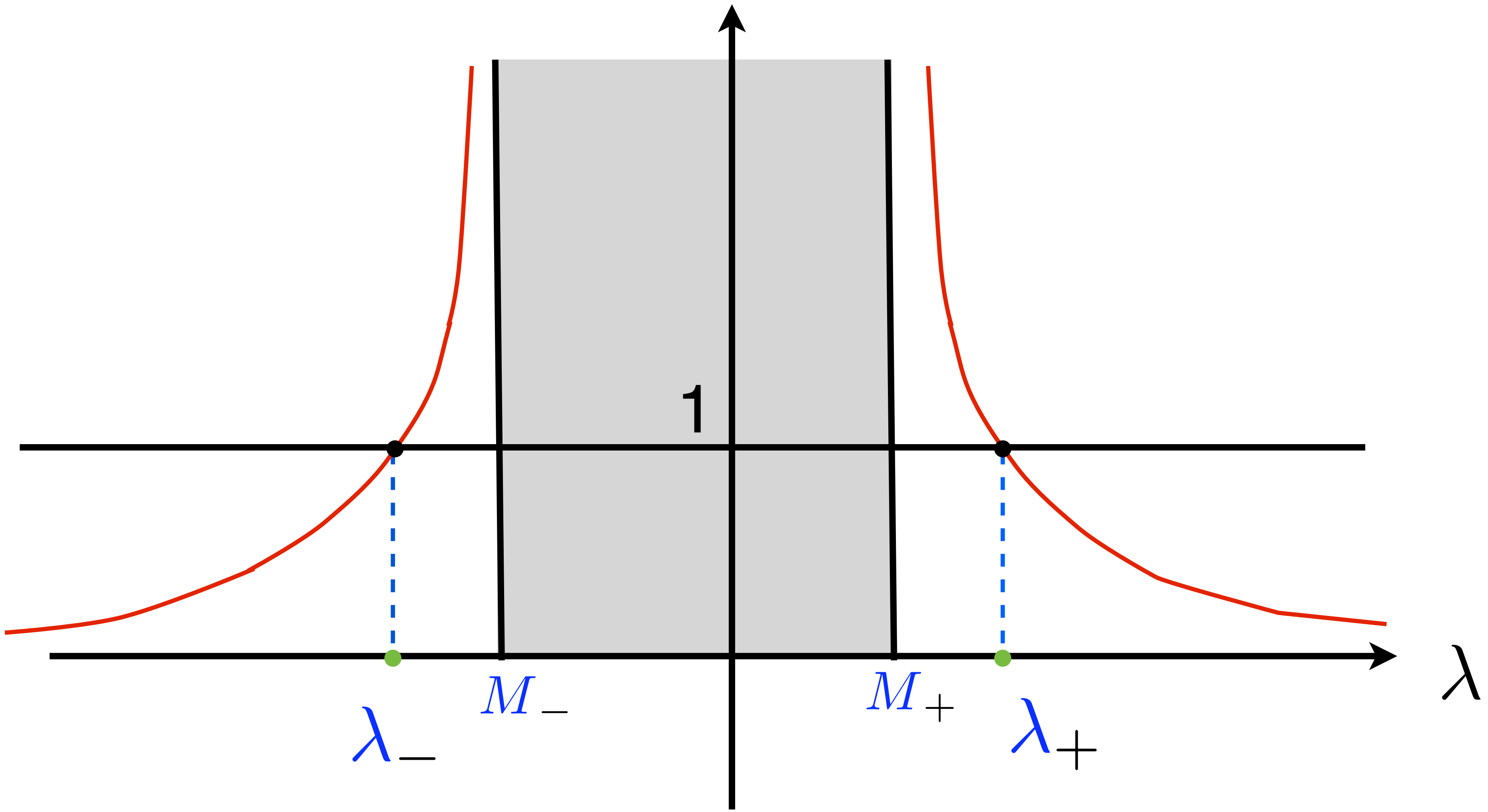
Eigenvalues of $\mathbf{A}(\mathrm{M})$ (2)

The study of real eigenvalues is easier because $F_M(\lambda)$ is real-valued along the real axis

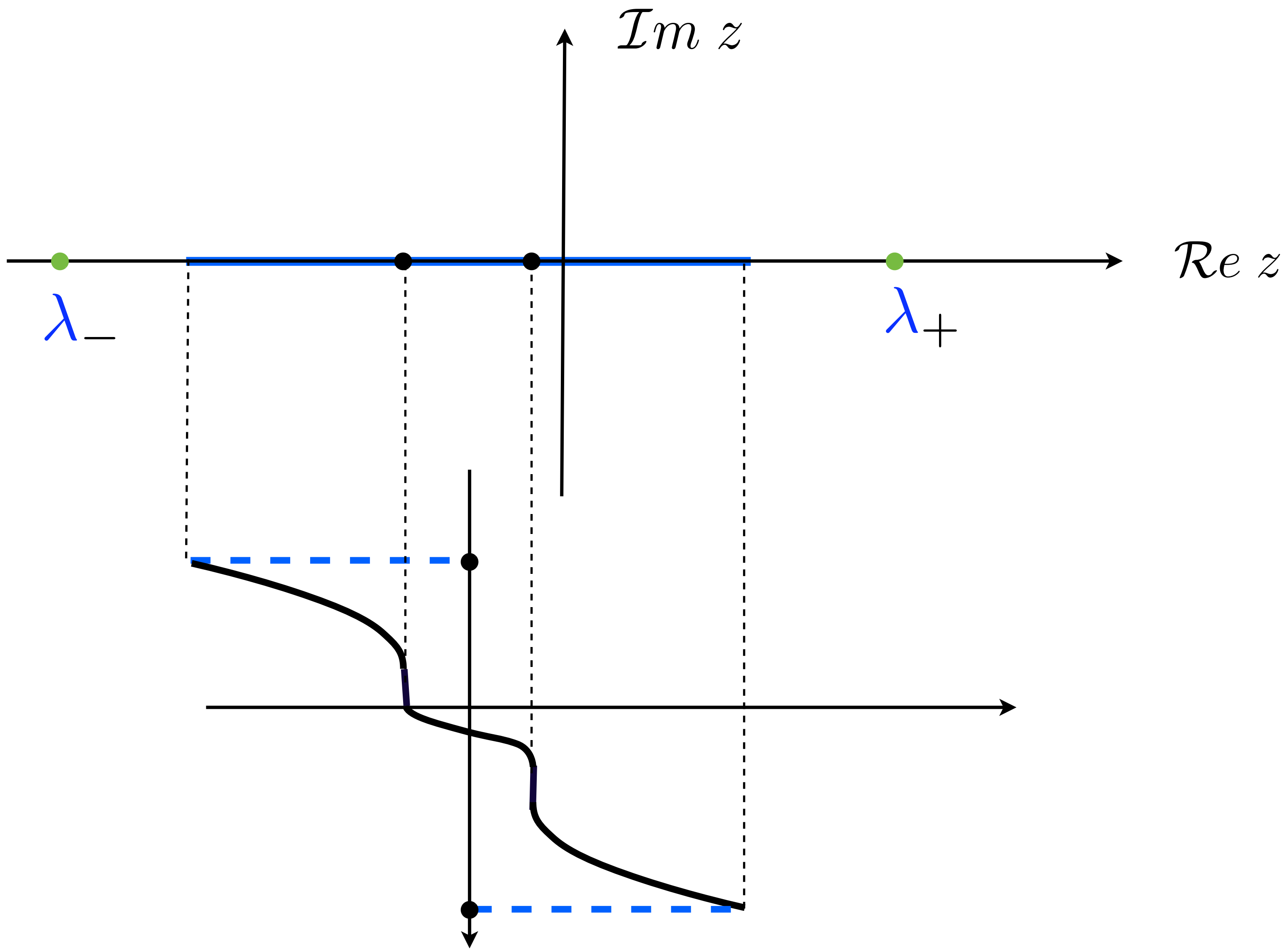
Lemma : The operator $\mathbf{A}(\mathrm{M})$ has exactly two real eigenvalues outside the interval $[M_-, M_+]$

$$\lambda_- < M_- < M_+ < \lambda_+$$

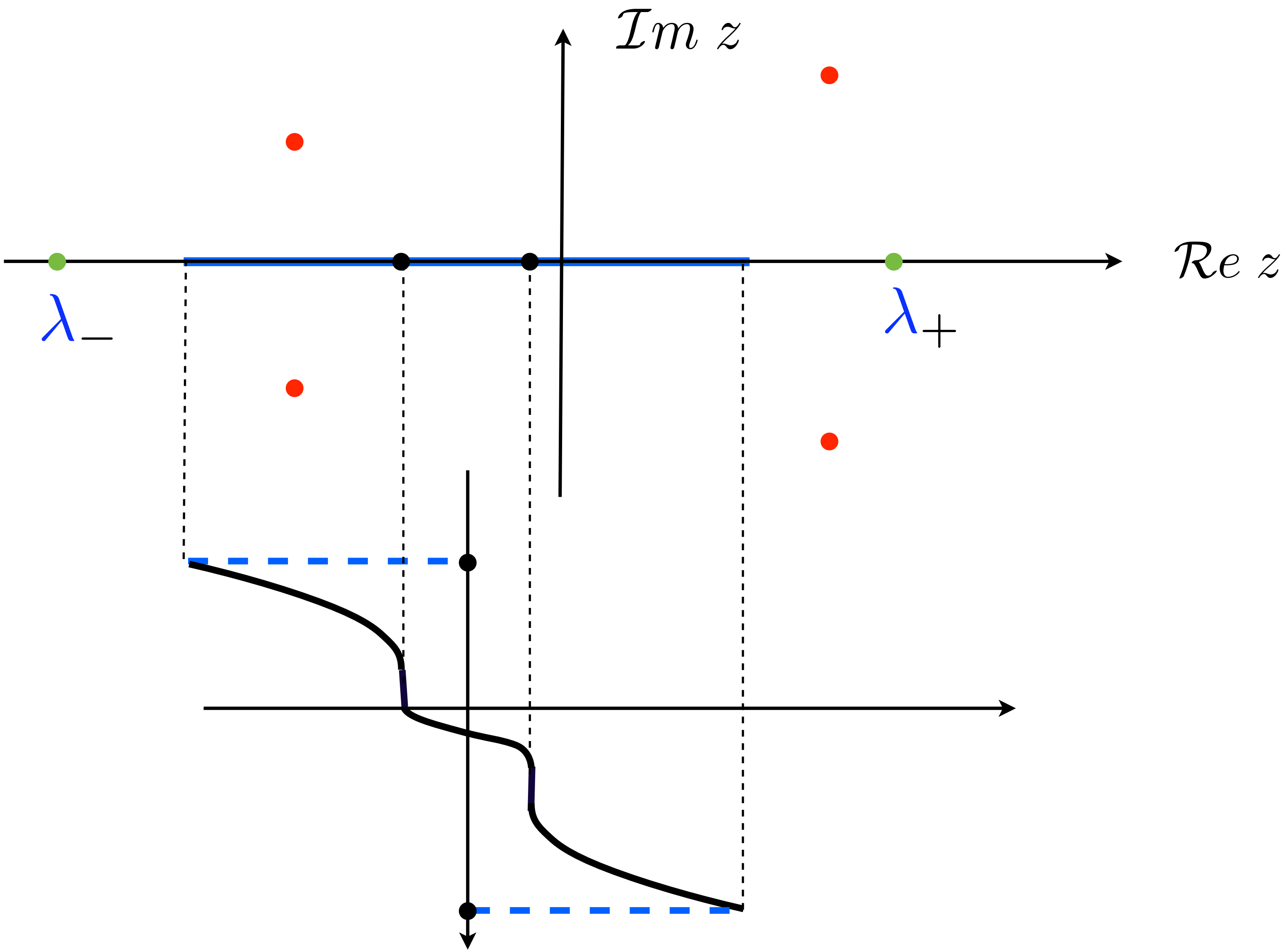
$$F_M(\lambda) = \frac{1}{2}\int_{-1}^{1} \frac{dy}{\left(\lambda - M(y)\right)^2}$$



Back to the spectrum of $\mathbf{A}(\mathrm{M})$



Back to the spectrum of $\mathbf{A}(\mathrm{M})$



Definition of a stable profile

Definition : a Mach profile $\mathbb{M}$ is unstable if

$$\boxed{(\mathcal{E}) \text{ has non real solutions}}$$

and stable if not.

Stability results

They have been obtained with the following process

1. The profile M is approximated by a piecewise linear continuous profile M_h such that

$$\| M_h - M \|_{L^\infty} \to 0, \quad h \to 0$$

2. One analyzes the equation $(\mathcal{E})$ for M_h

Key point : the function $\mathrm{F}_{M_h}(\lambda)$ is a rational fraction

3. One concludes using perturbation theory for eigenvalue problems (Kato)

3. One concludes using perturbation theory for eigenvalue problems (Kato)

$$\mathbf{A}(M) - \mathbf{A}(M_h) = \begin{pmatrix} M - M_h & 0 \\ 0 & M - M_h \end{pmatrix}$$

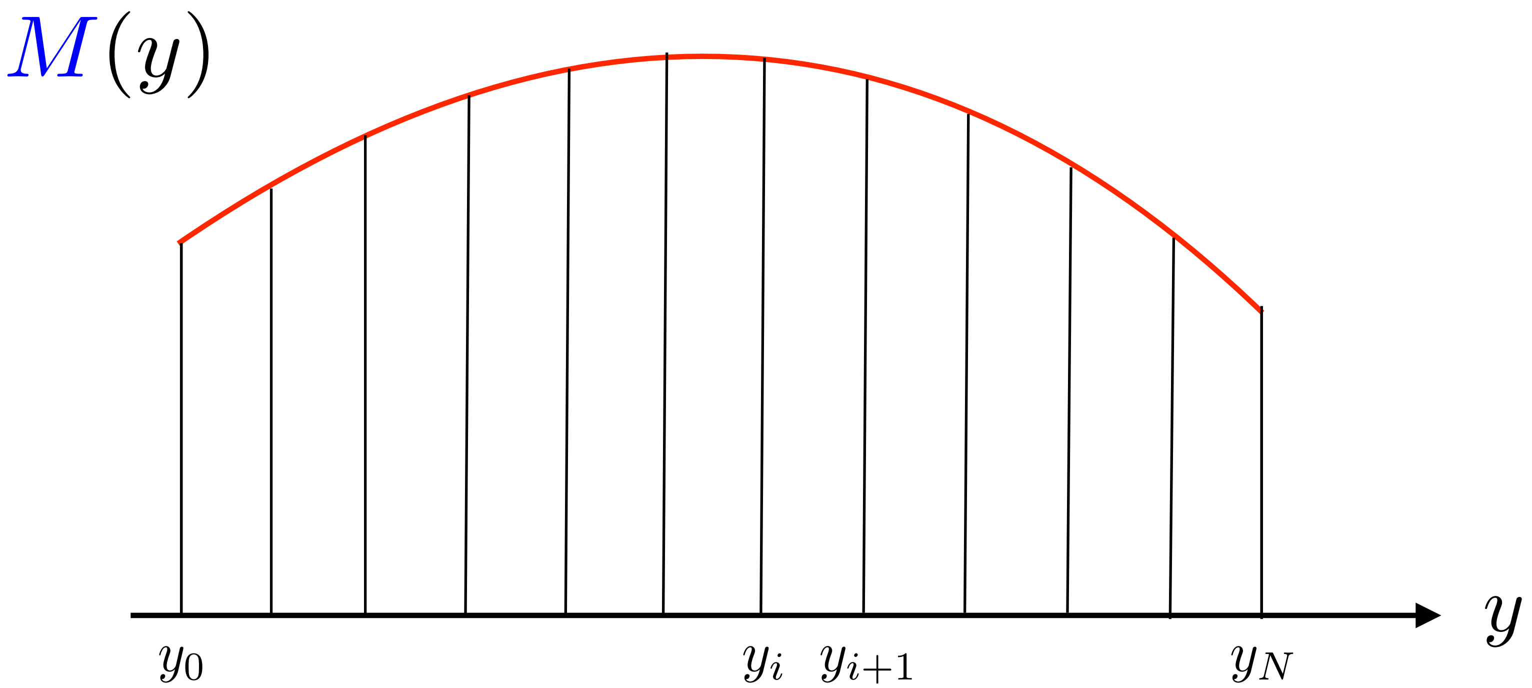
$$\Longrightarrow \quad \|\mathbf{A}(M) - \mathbf{A}(M_h)\| = \|M - M_h\|_{L^\infty}$$

(Kato) $\Longrightarrow \quad d\big[\sigma\big(\mathbf{A}(M)\big), \sigma\big(\mathbf{A}(M_h)\big)\big] \leq \|\mathbf{A}(M) - \mathbf{A}(M_h)\|$

Assuming that $\sigma\big(\mathbf{A}(M_h)\big) \subset \mathbb{R}$ for any $h > 0$, then

$$d\big[\sigma\big(\mathbf{A}(M)\big), \mathbb{R}\big)\big] \leq \|M - M_h\|_{L^\infty} \quad \underset{h \to 0}{\Longrightarrow} \quad \sigma\big(\mathbf{A}(M)\big) \subset \mathbb{R}$$

1. The profile M is approximated by a piecewise linear continuous profile M_h



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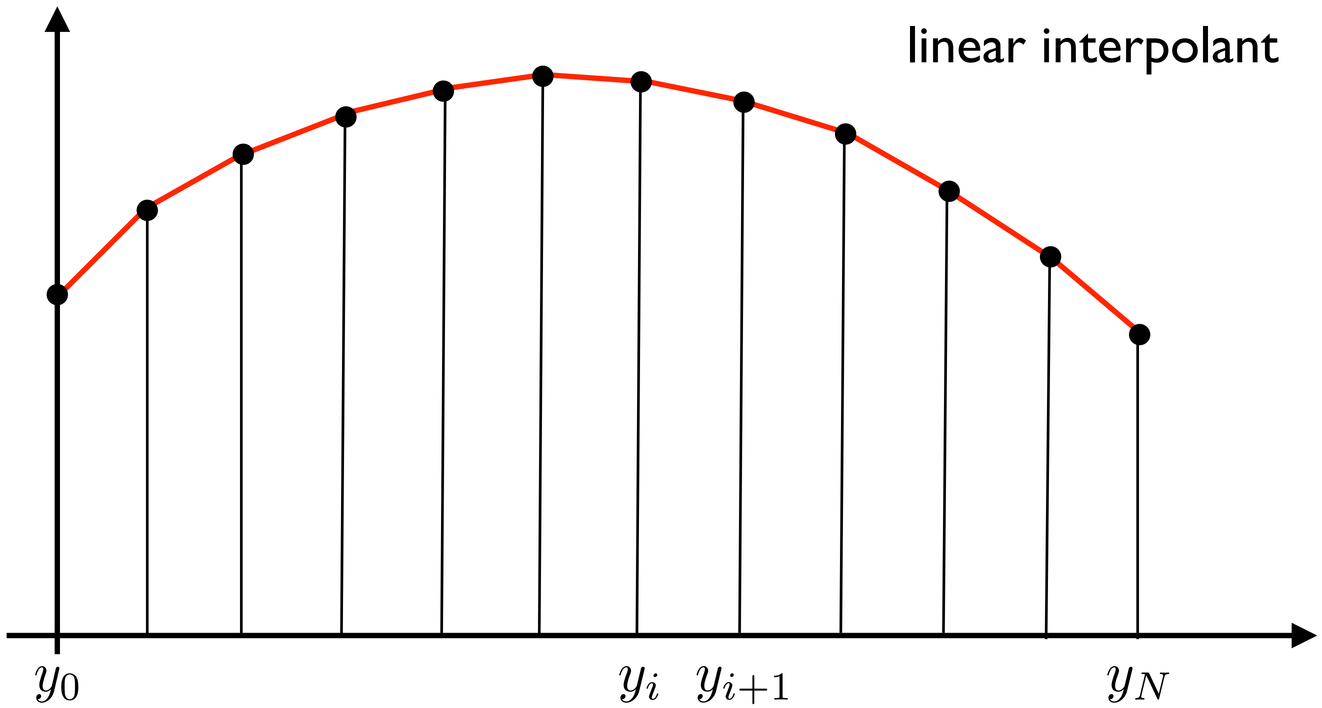
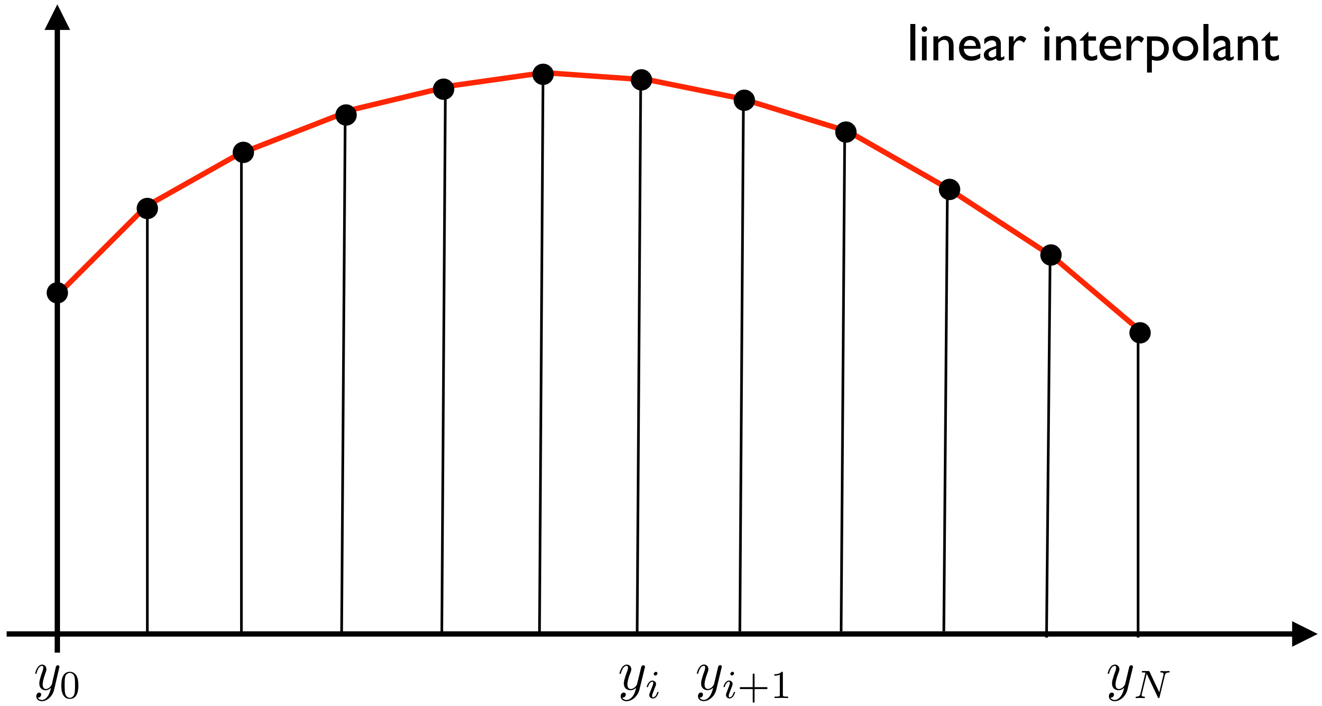


2. One analyzes the equation ($\mathcal{E}$) for M_h



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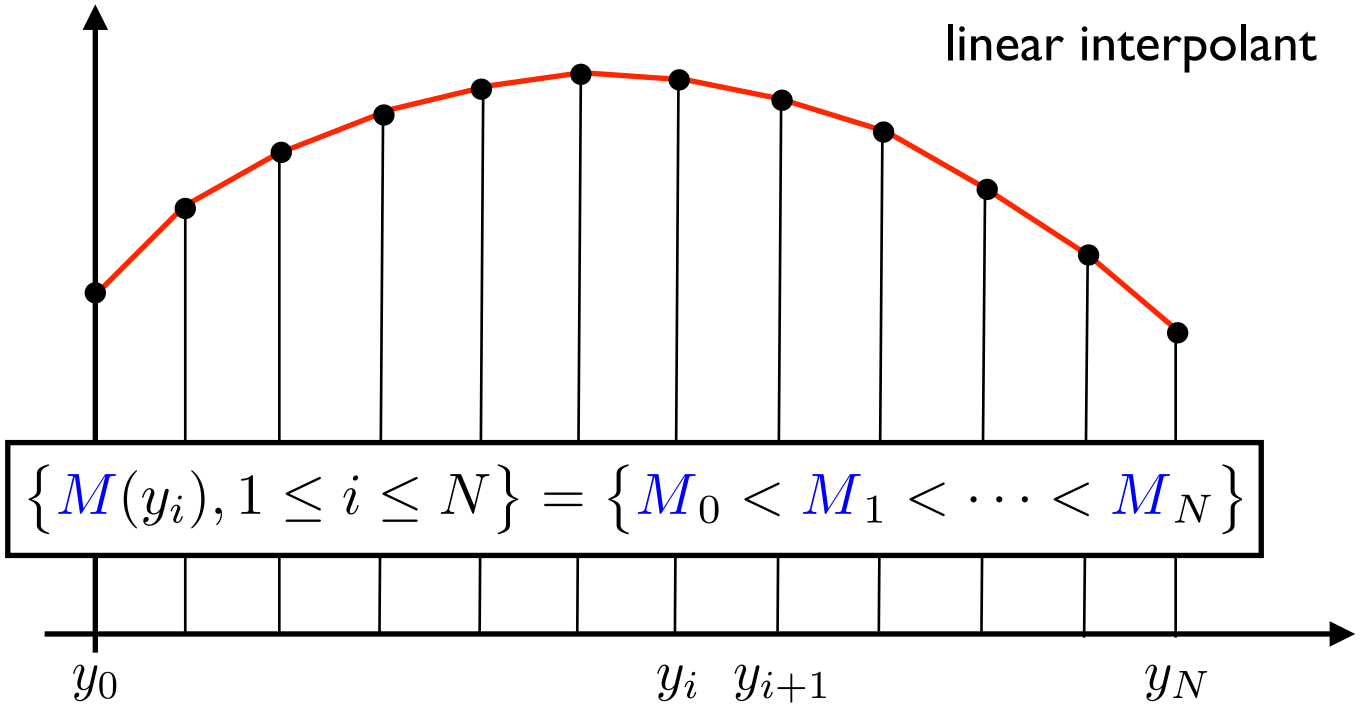

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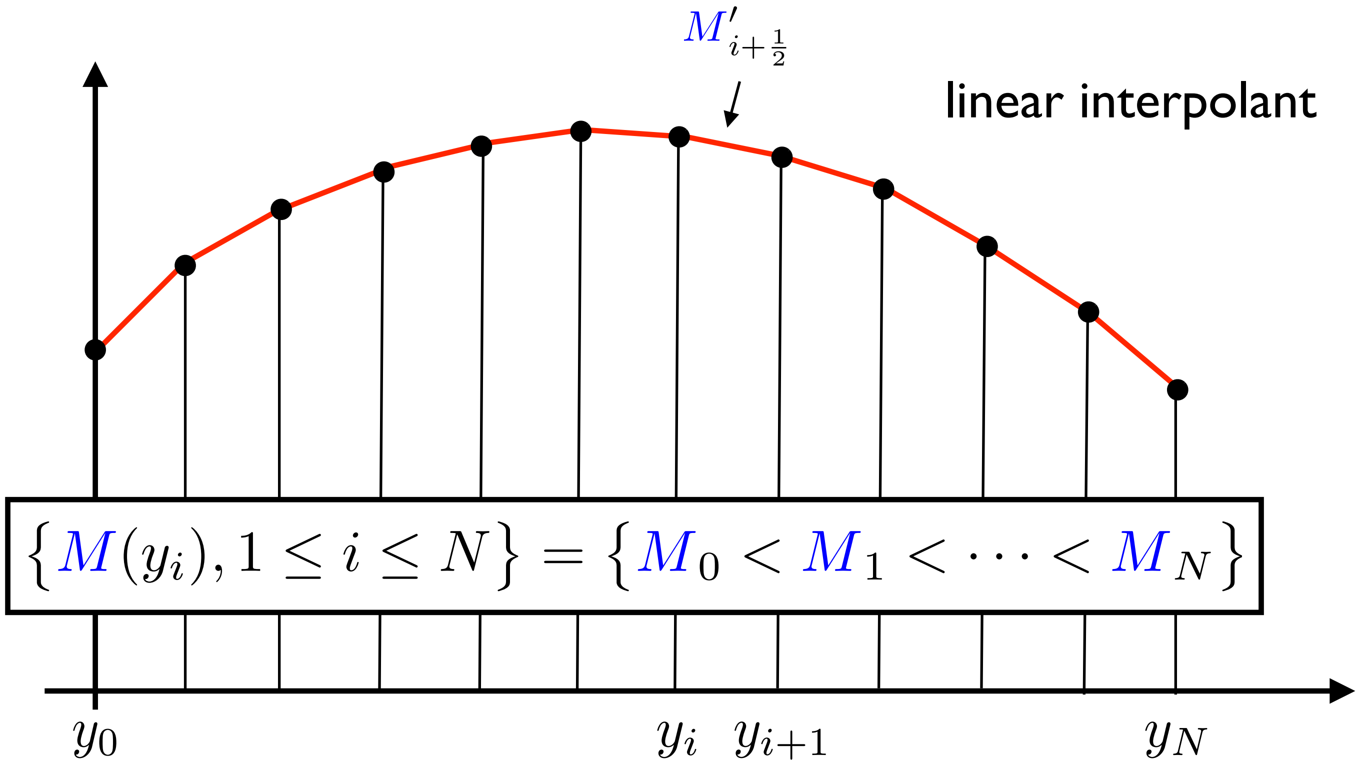

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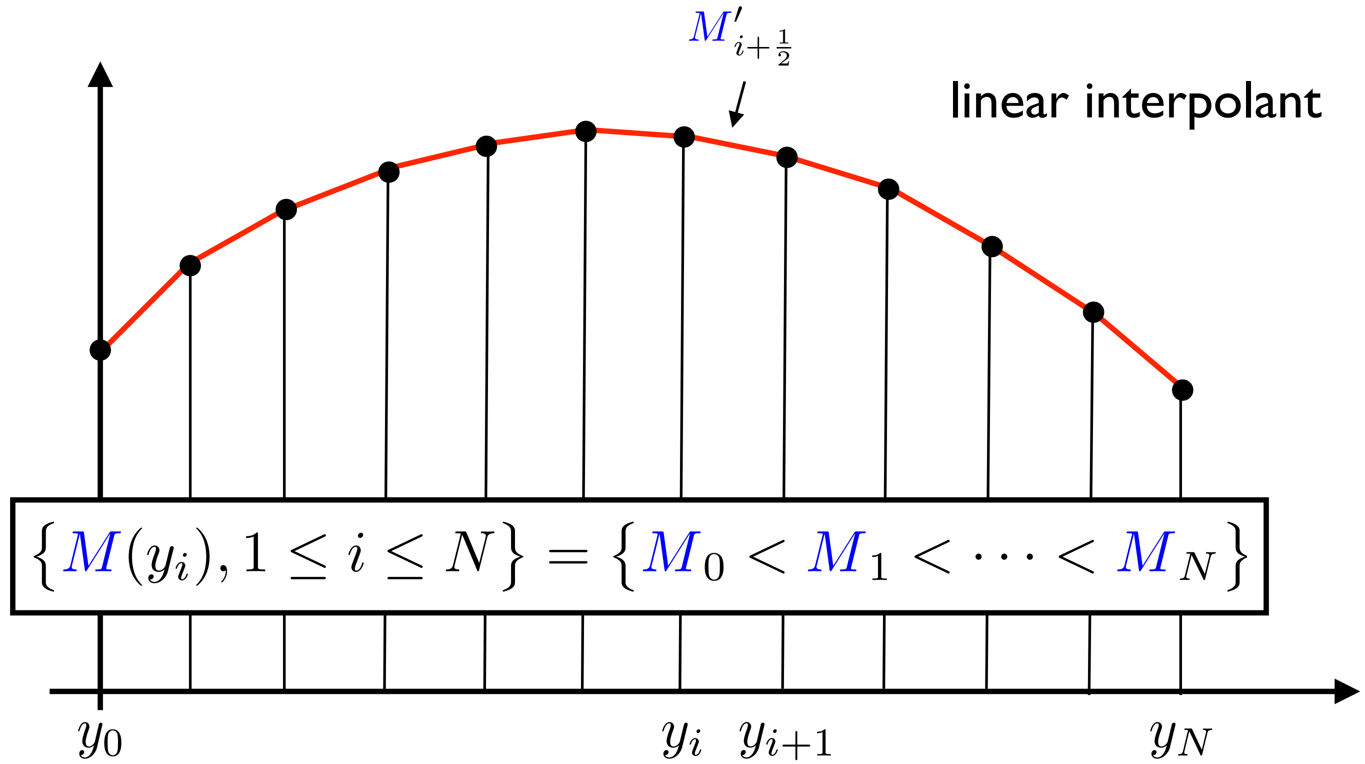

$$\boxed{\int_{y_j}^{y_{j+1}} \frac{dy}{\left(\lambda - M_h(y)\right)^2} = \frac{1}{M'_{j+\frac{1}{2}}} \int_{y_j}^{y_{j+1}} \frac{M'_h(y)\, dy}{\left(\lambda - M_h(y)\right)^2}}$$

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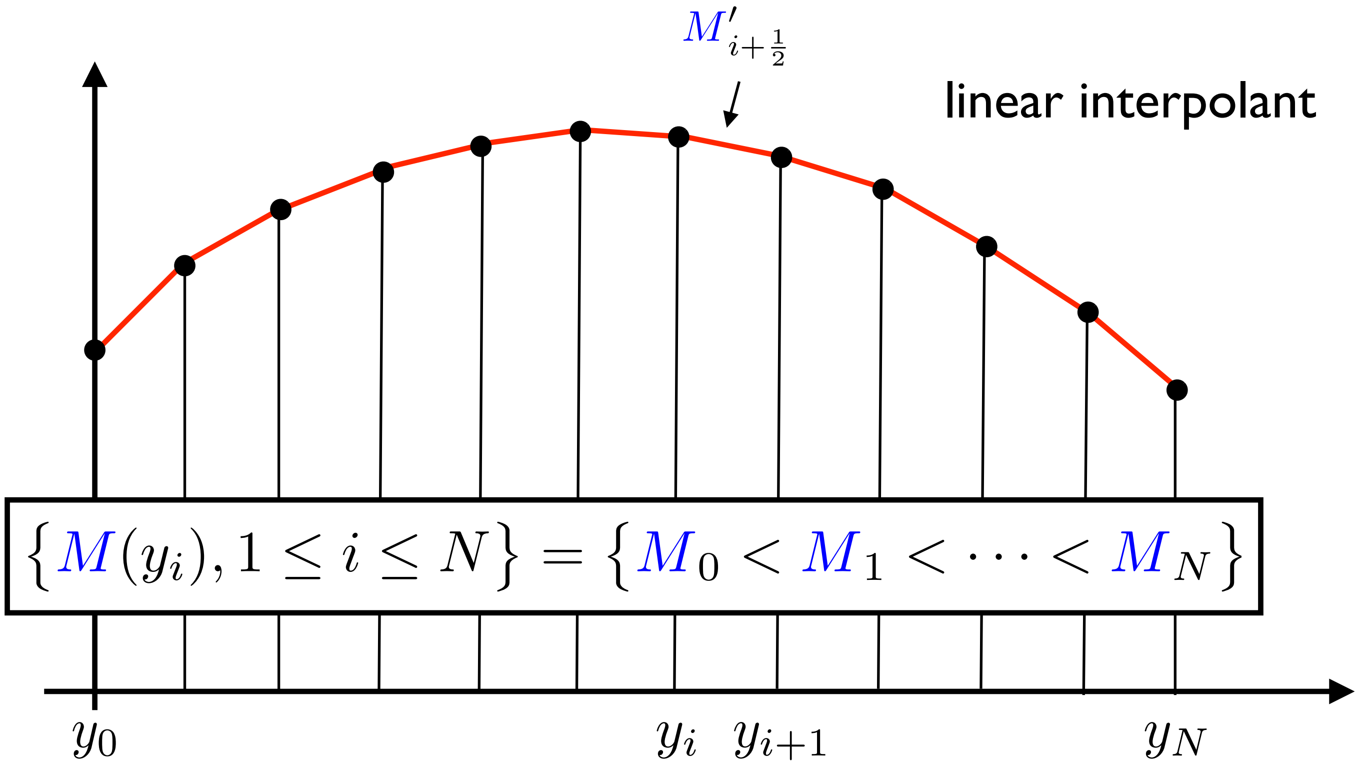

$$\int_{y_j}^{y_{j+1}} \frac{dy}{\left(\lambda - M_h(y)\right)^2} = \frac{1}{M'_{j+\frac{1}{2}}} \int_{y_j}^{y_{j+1}} \frac{M'_h(y)\,dy}{\left(\lambda - M_h(y)\right)^2} = -\frac{1}{M'_{j+\frac{1}{2}}} \left[\frac{1}{\lambda - M_h}\right]_{y_j}^{y_{j+1}}$$

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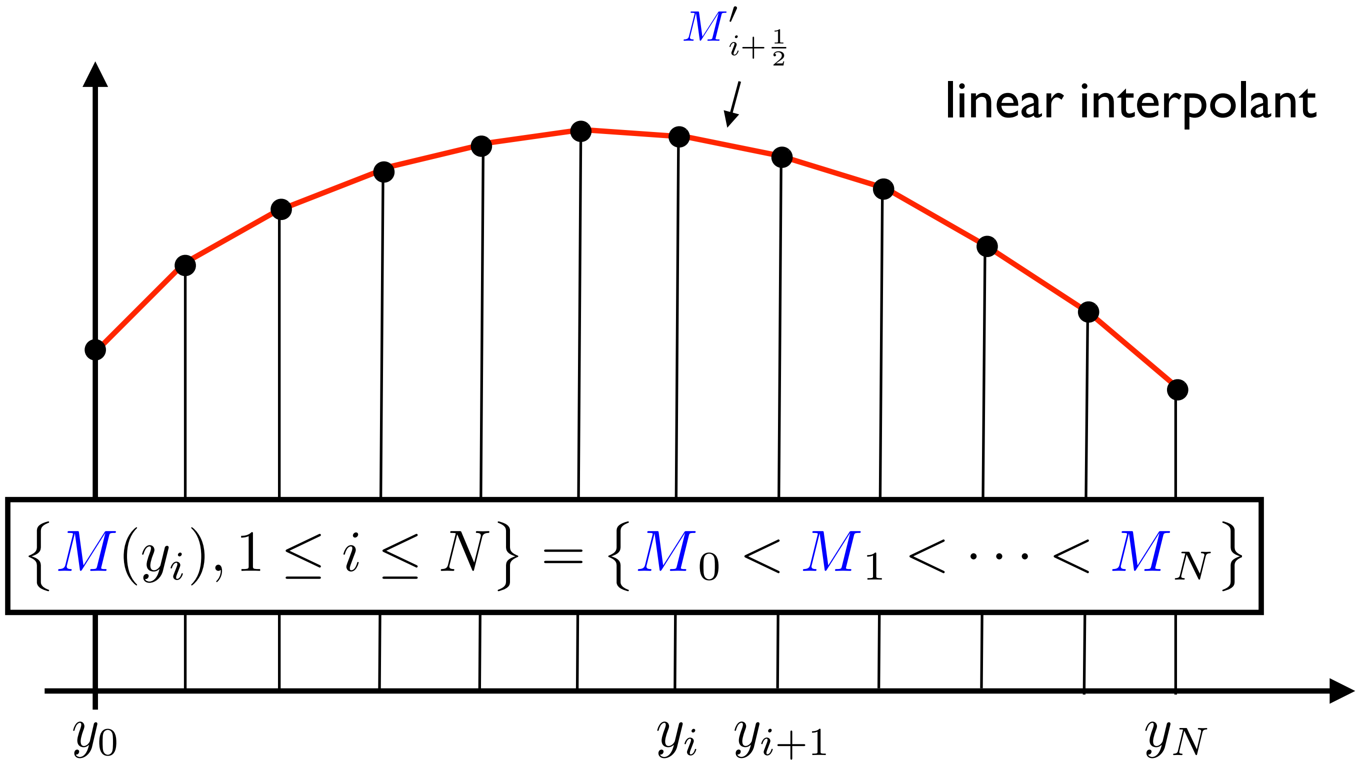

$$F_{M_h}(\lambda) := \int_{-1}^{1} \frac{dy}{\left(\lambda - M_h(y)\right)^2} = \sum_{j=0}^{N} \frac{\gamma_j}{\lambda - M_j}$$

$$\boxed{M_j = M(x_i) \qquad \gamma_j = \frac{1}{2}\,\frac{M'_{i+\frac{1}{2}} - M'_{i-\frac{1}{2}}}{M'_{i+\frac{1}{2}}\,M'_{i-\frac{1}{2}}}}$$

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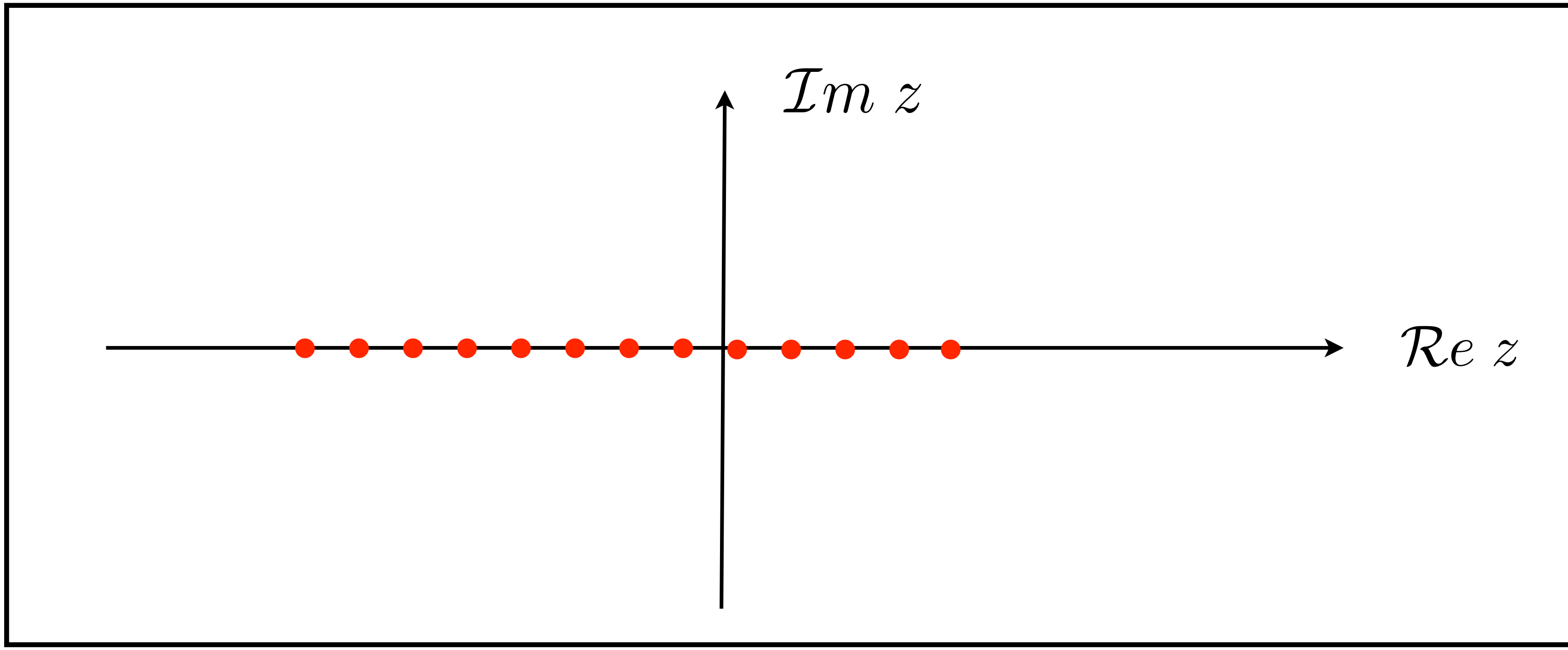
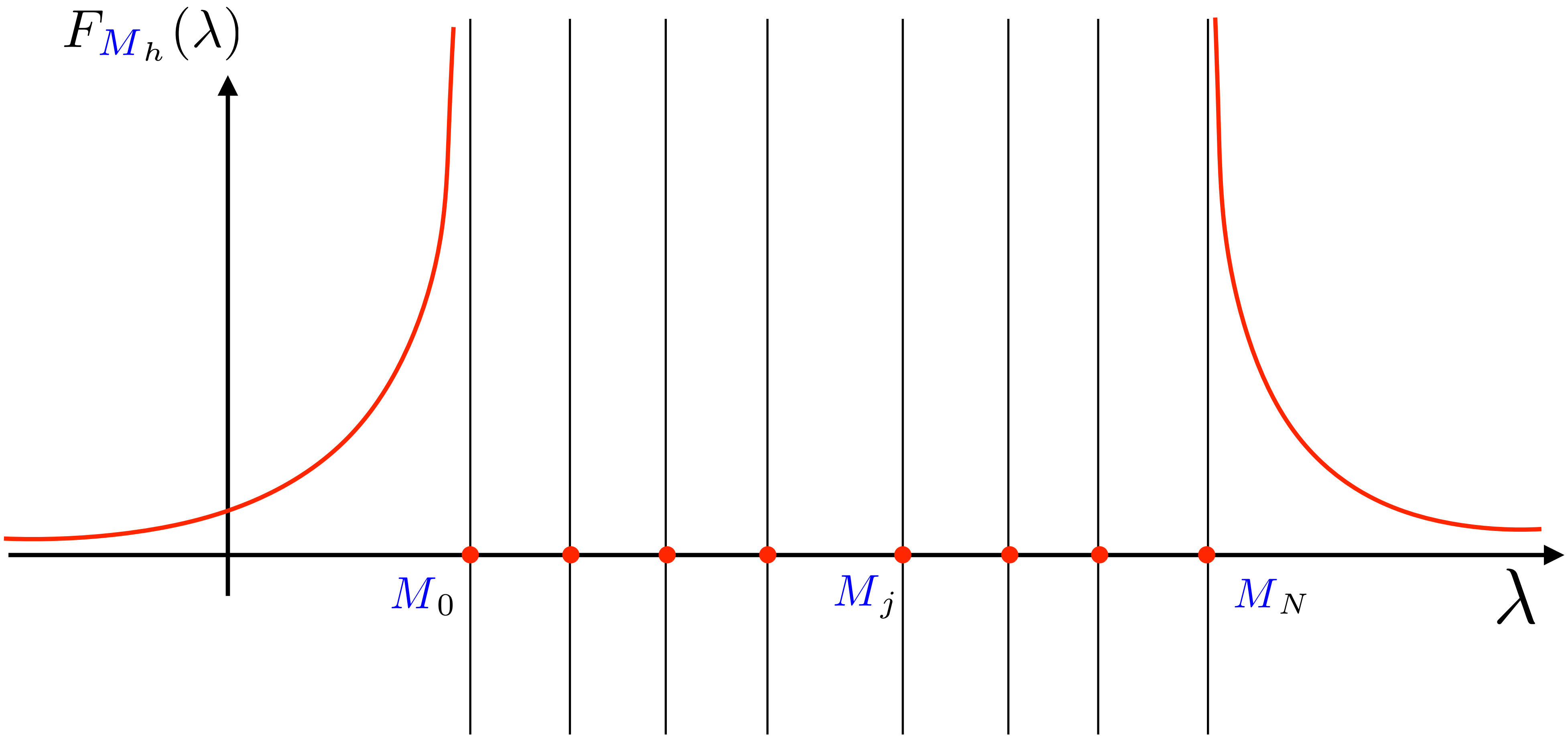


$$F_{M_h}(\lambda) := \int_{-1}^{1} \frac{dy}{\big(\lambda - M_h(y)\big)^2} = \sum_{j=0}^{N} \frac{\gamma_j}{\lambda - M_j}$$

$\boxed{F_{M_h}(\lambda) = 1 \quad \text{polynomial equation of degree } N+1}$

2. One analyzes the equation ($\mathcal{E}$) for M_h



If one shows that the equation $F_{M_h}(\lambda) = 2$ has
has one real root in $N-1$ intervals, all roots are real

$\Longrightarrow$ the profile M_h is stable

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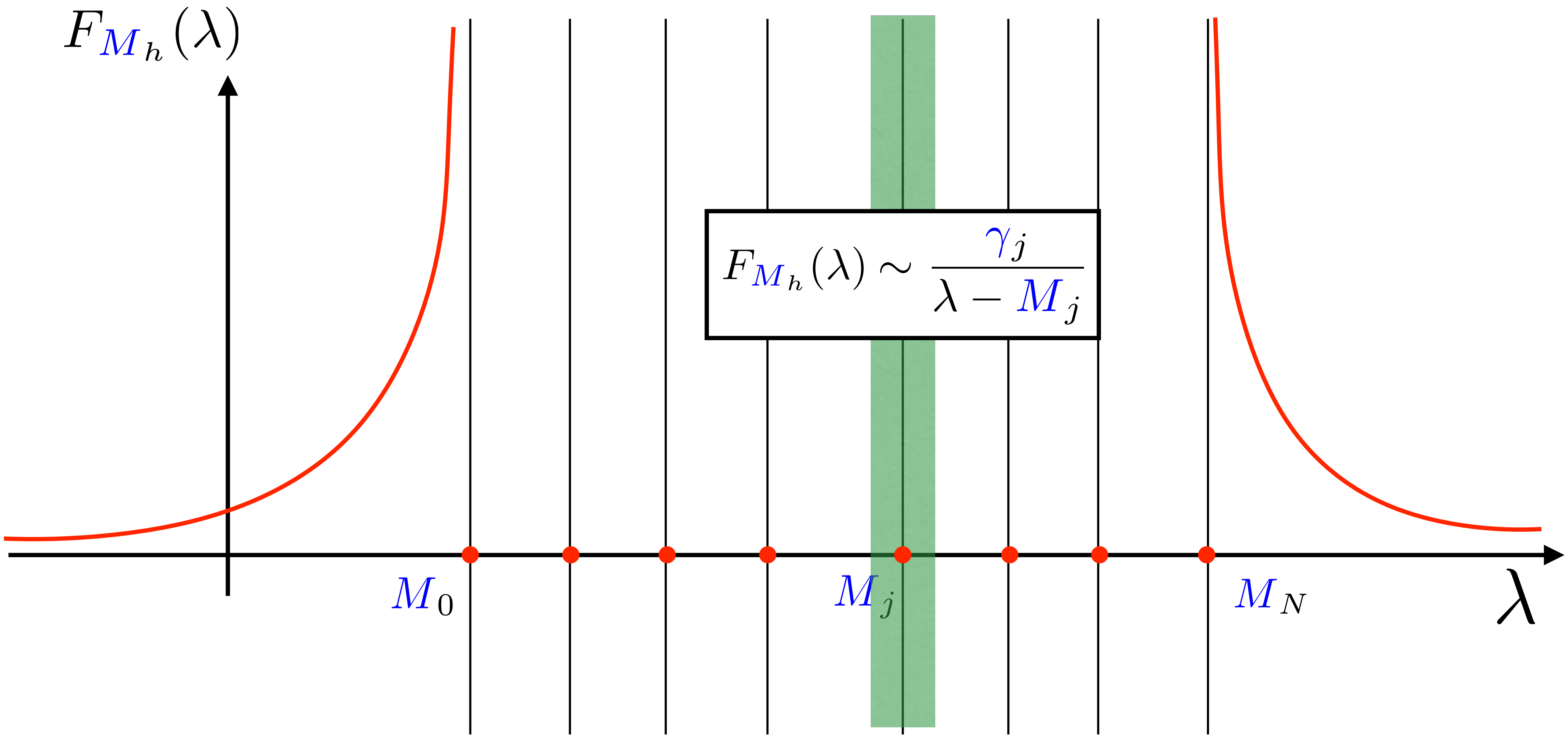

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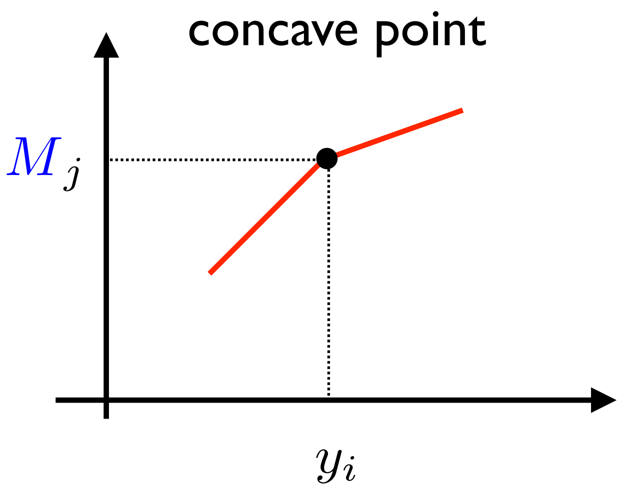
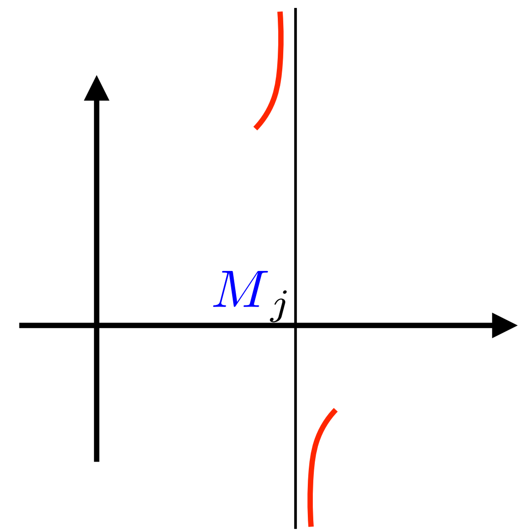
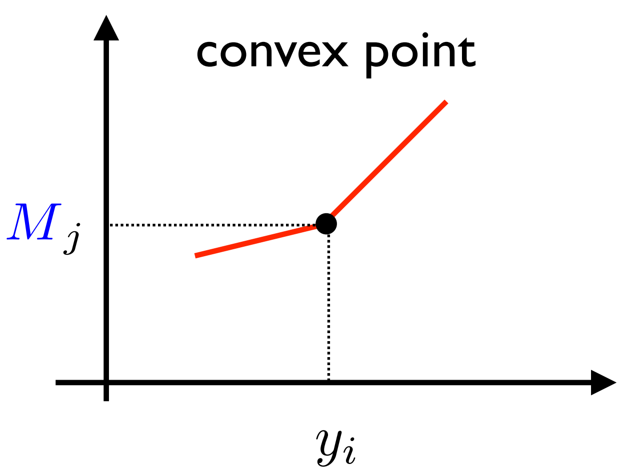


$$\gamma_j = \frac{1}{2}\,\frac{M'_{i+\frac{1}{2}} - M'_{i-\frac{1}{2}}}{M'_{i+\frac{1}{2}}\,M'_{i-\frac{1}{2}}} < 0$$





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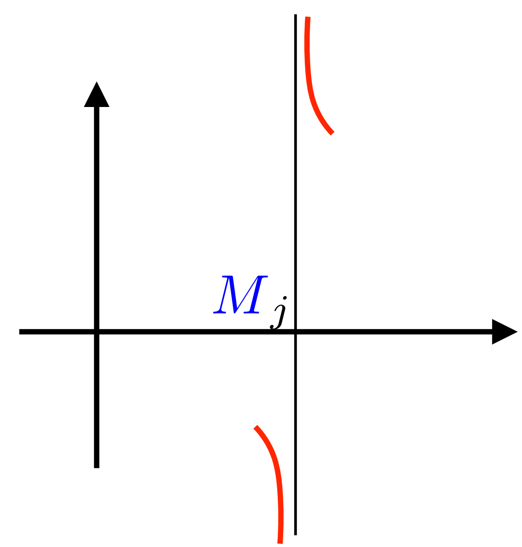
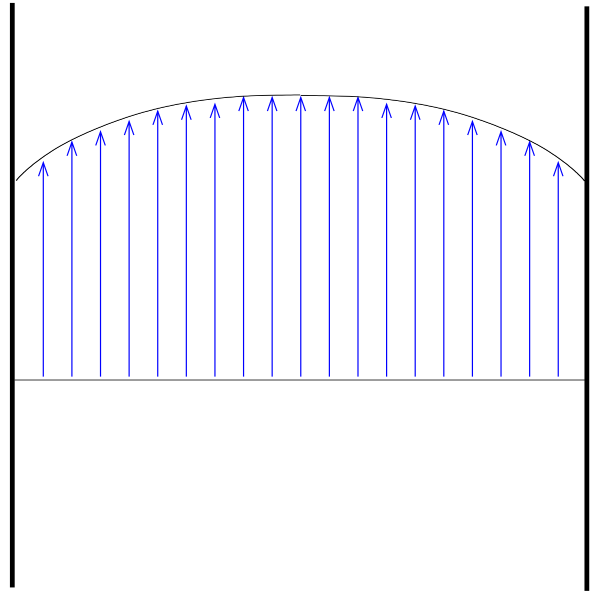
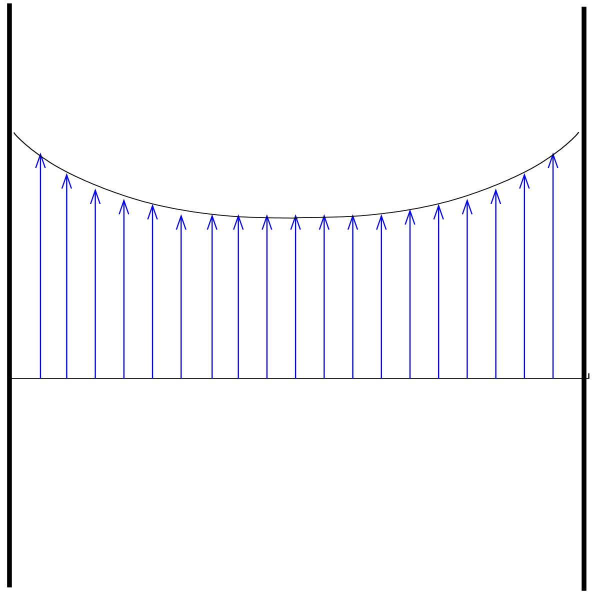


Stability results

Theorem : the profile M is stable in the following 3 cases

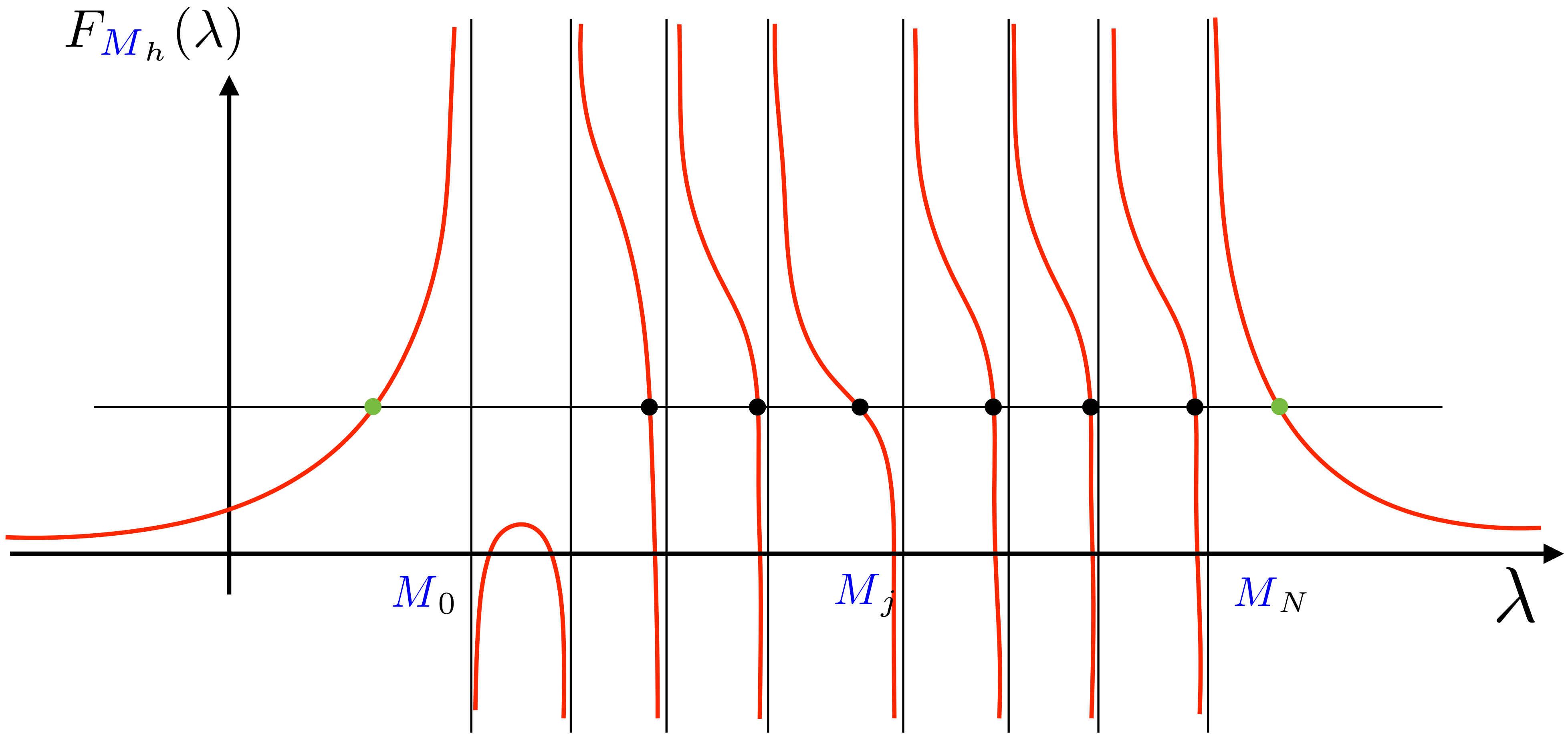
1. M is convex or concave in $[-1,1]$



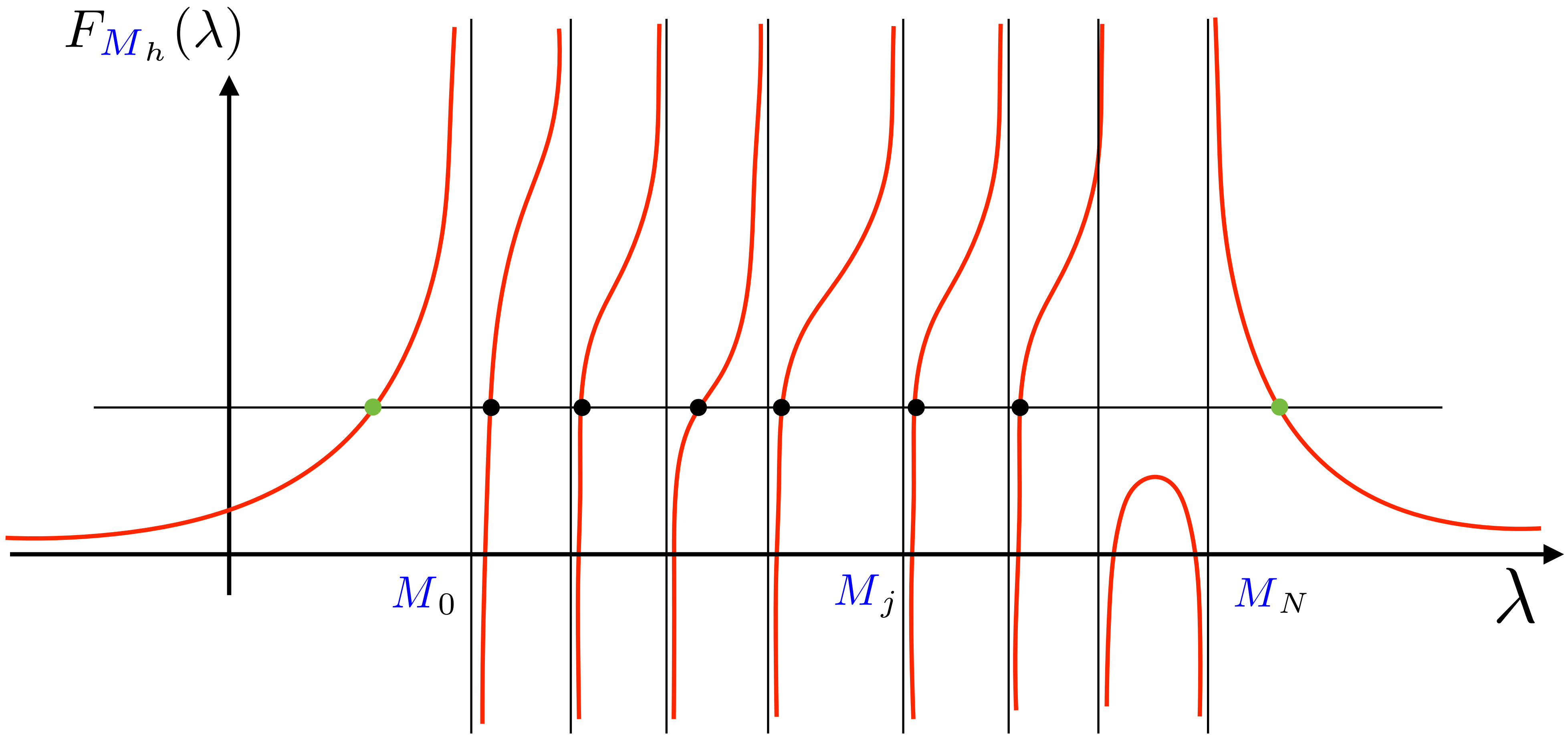


corresponds to the Rayleigh criterion in the incompressible case

The convex case



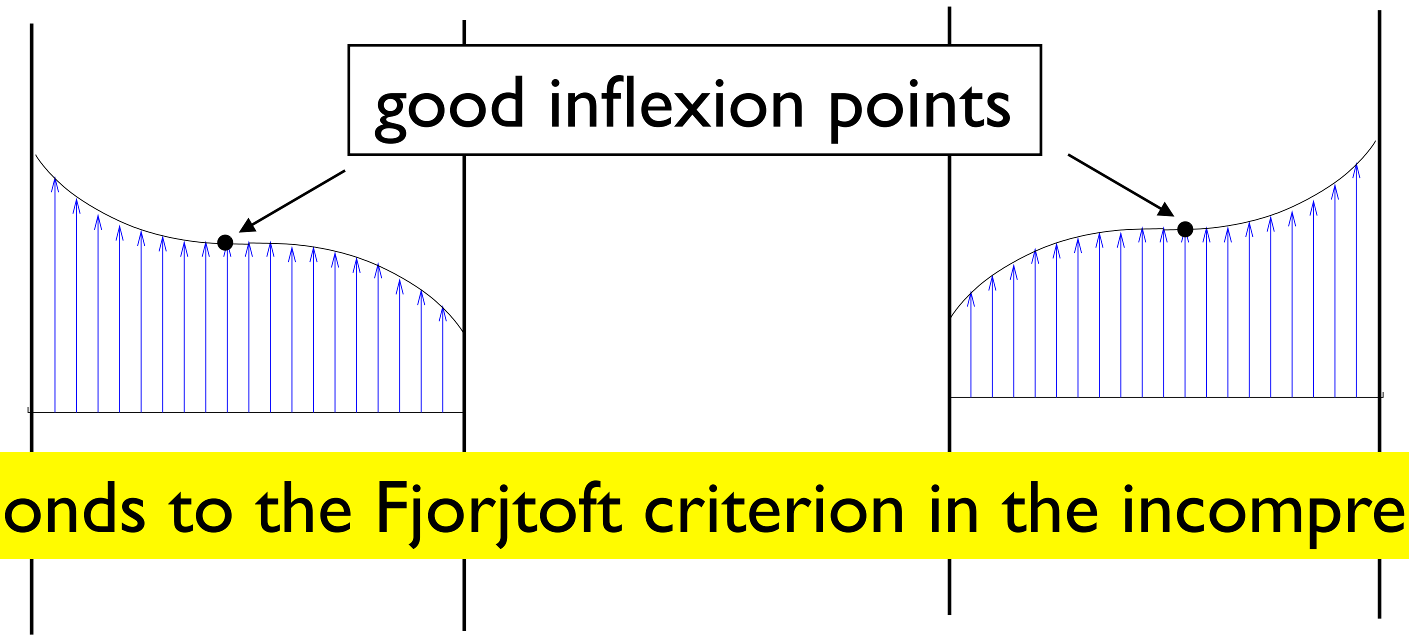
The concave case



Stability results

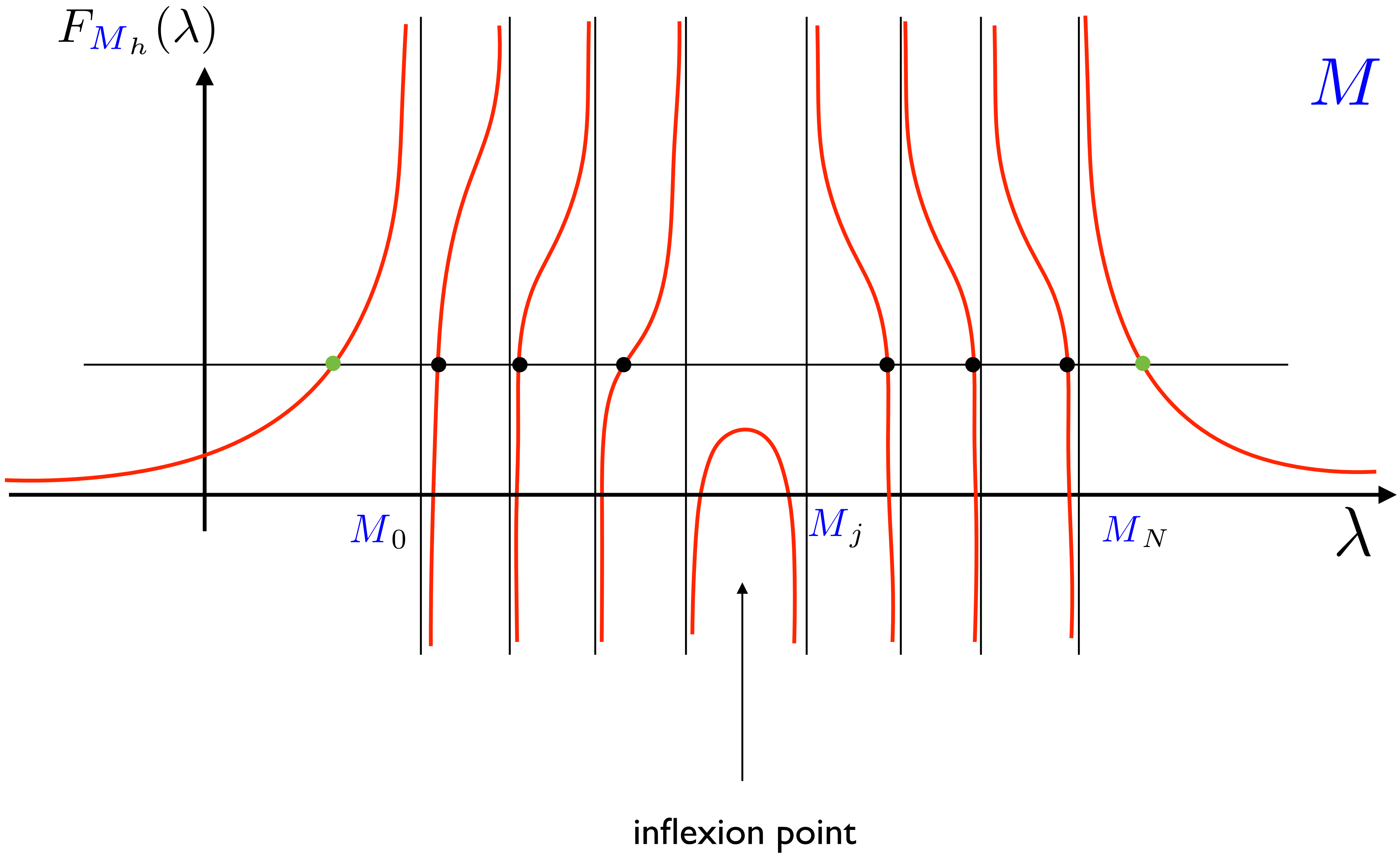
Theorem : the profile M is stable in the following 3 cases

1. M is convex or concave in $[-1,1]$
2. M is decreasing and convex - concave
3. M is increasing and concave - convex



corresponds to the Fjorjtoft criterion in the incompressible case

The concave-convex case



Instability results (I)

The stability property of a profile in unstable in L^∞-norm

Let M be continuous and $\{y_j\}$ be a regular mesh of $[-1,1]$ of stepsize $h>0.$

Let M_h be the piecewise constant profile given by

$$M_h(y) = \frac{1}{h}\int_{y_j}^{y_{j+1}} M(y)\,dy, \qquad y \in [y_j, y_{j+1}]$$

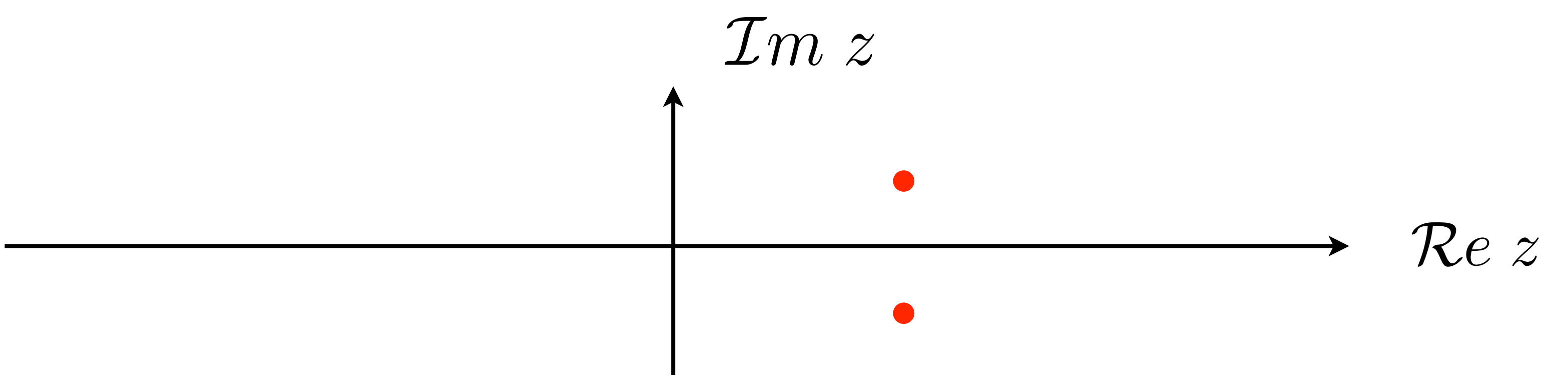
Then, for h small enough, M_h is unstable.

Instability results (2)

Note that if M is unstable, $M + C$ is unstable too.

One can always impose $M(0) = 0$ for instance

If M is unstable, $\widetilde{M}$ is unstable for $\|\widetilde{M} - M\|_{L^\infty}$ small enough



Instability results (2)

Instability results : hard for general smooth profiles

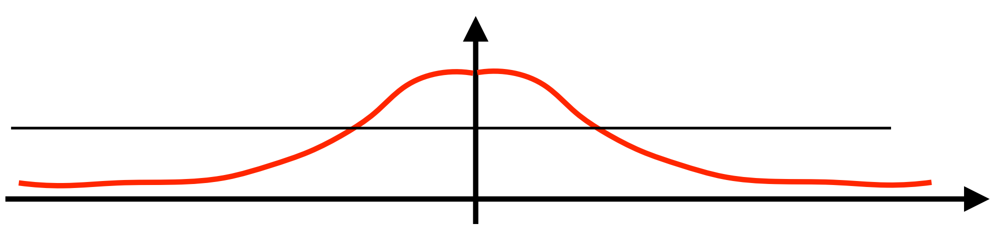
However, it is possible to obtain several results in the case of odd profiles, for which one has

$$\forall\ \nu \in \mathbb{R}, \qquad F_M(i\nu) = \int_0^1 \frac{M(y)^2 + \nu^2}{\left(M(y)^2 + \nu^2\right)^2}\, dy$$

which leads to looking for purely imaginary roots of ($\mathcal{E}$).

$$\lim_{\nu \to \pm\infty} F_M(i\nu) = 0$$

$\Longrightarrow$ sufficient instability condition $\limsup_{\nu \to 0} F_M(i\nu) > 1$



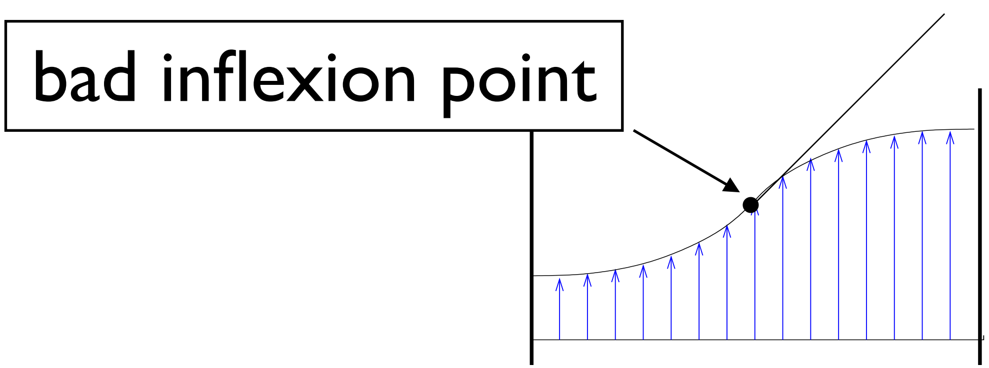
Instability results (2)

Theorem : Assume that M is odd, of class C^2 and

$$\int_{-1}^{1} \frac{M'(0)^2\, y^2 - M(y)^2}{y^2\; M(y)^2}\, dy > 1 + M'(0)^2$$

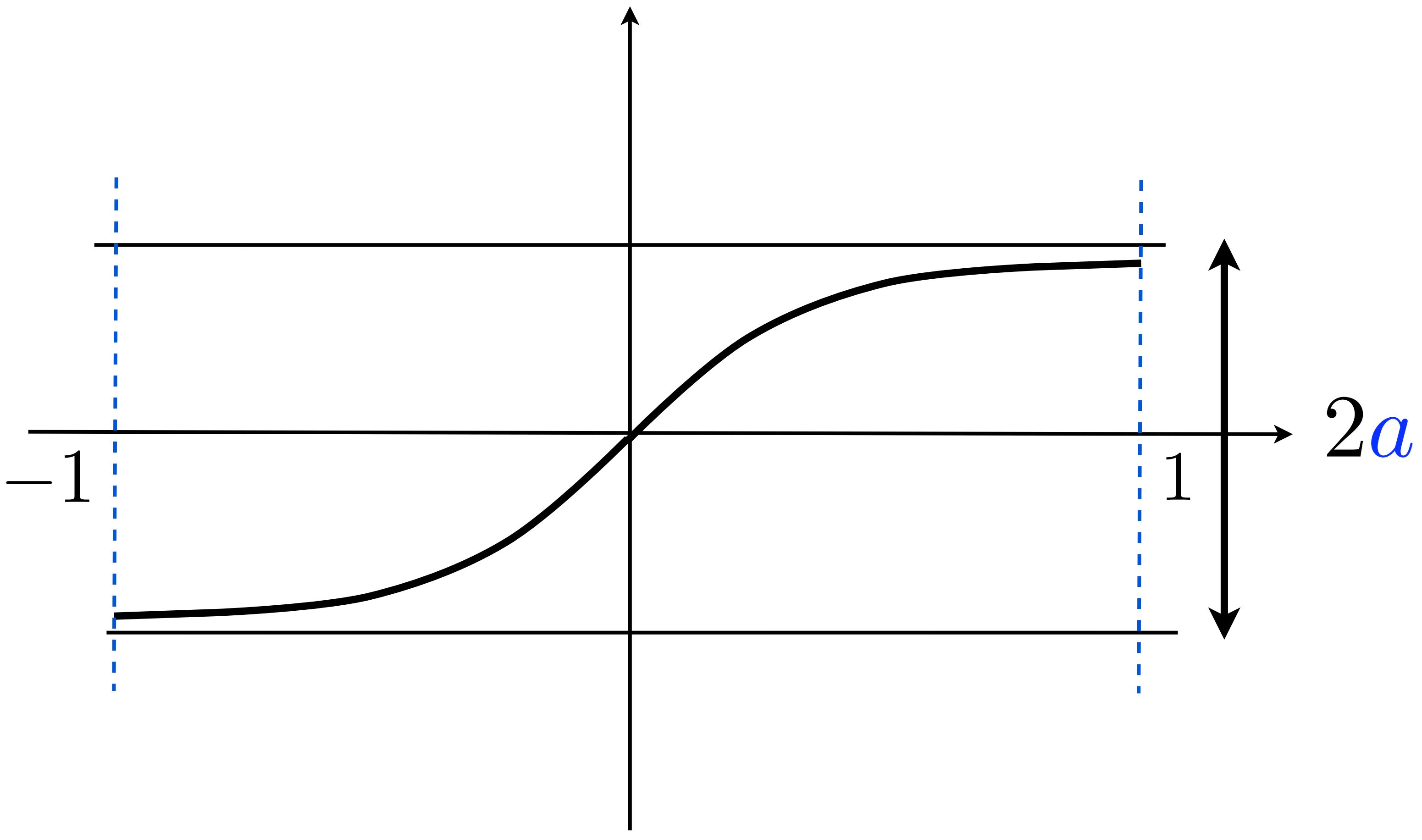
the profile M is unstable.

If moreover, M is increasing and concave for $y > 0$ the condition is also necessary.



Instability results (I)

Application : $M(y) = a\tanh(\alpha\, y), \quad a > 0, \quad \alpha > 0.$



Instability results (I)

Application : $M(y) = a\tanh(\alpha\, y), \quad a > 0, \quad \alpha > 0.$

Let α^* the unique solution of

$$\alpha \tanh \alpha = 1 \quad (\alpha^* \simeq 1.1996)$$

The profile M is unstable if and only if (*)

$$\boxed{\alpha > \alpha^* \quad \text{and} \quad a < \left[1 - \alpha \tanh \alpha\right]^{\frac{1}{2}}}$$

(*) $\qquad \alpha > \alpha^* \implies \alpha \tanh \alpha < 1.$

A by-product : hydrodynamic instabilities

Theorem : if M is unstable, $(\mathcal{P})_\varepsilon$ is unstable, i. e.

$$(\mathcal{P})_\varepsilon \begin{cases} (\partial_t + M\partial_x)^2 u_\varepsilon - \partial_x(\partial_x u_\varepsilon + \partial_y v_\varepsilon) = 0 \\ \varepsilon^2\,(\partial_t + M\partial_x)^2 v_\varepsilon - \partial_y(\partial_x u_\varepsilon + \partial_y v_\varepsilon) = 0 \end{cases}$$

$$\|u_\varepsilon\|_{L^2_x(L^2_y)} + \|v_\varepsilon\|_{L^2_x(L^2_y)} \geq C(u_0, u_1)\, e^{\alpha \frac{t}{\varepsilon}}$$

These are new results for hydrodynamic instabilities in compressible fluids, proven by perturbation theory

Computation of discrete spectra

With finite dimension approximation spaces $V_h \subset L^2(-1,1)$ one constructs discrete approximations $\mathbf{A}_h(M)$ of $\mathbf{A}(\mathrm{M})$

One computes the spectrum of $\mathbf{A}_h(M)$

$$M\,u + v = \lambda\,u \qquad\qquad E(u) + M\,v = \lambda\,v$$

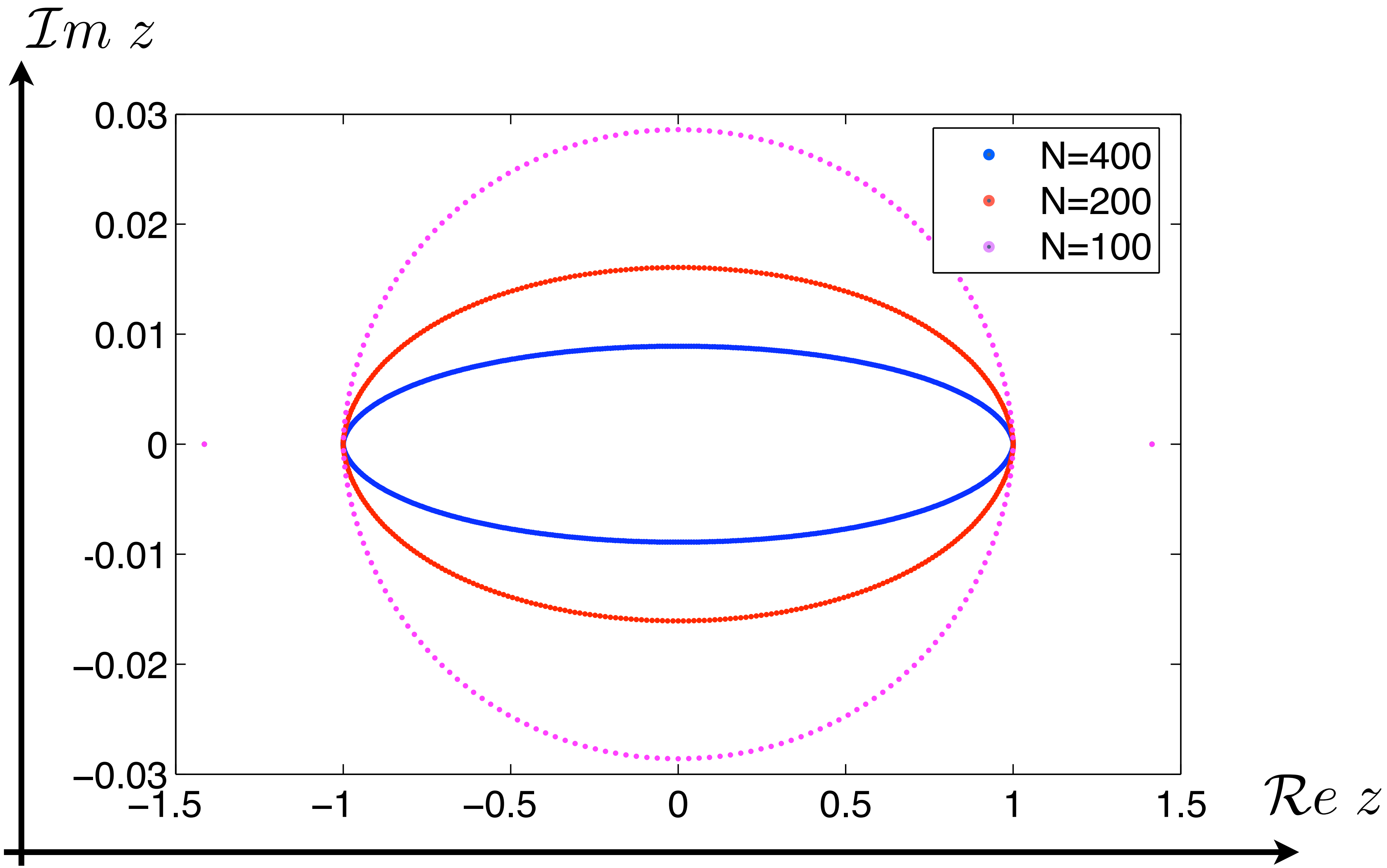
$$\Downarrow$$

Find $(u_h, v_h) \in V_h \times V_h \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that

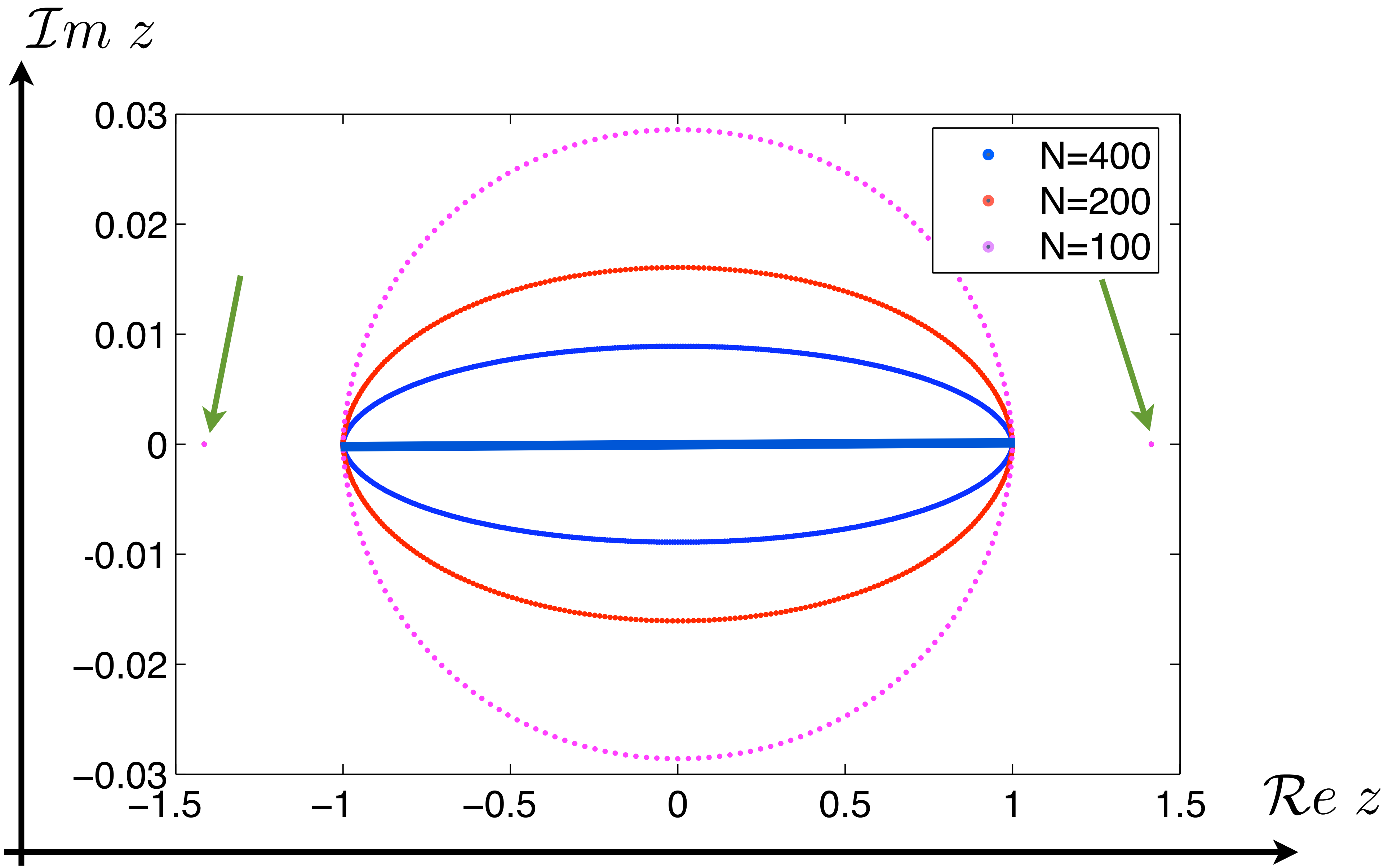
$$\int M\,u_h\,\widetilde{v}_h\,dy + \int v_h\,\widetilde{u}_h\,dy = \lambda \int u_h\,\widetilde{u}_h\,dy \qquad \forall\,\widetilde{u}_h \in V_h$$

$$\frac{1}{2}\int\int u_h(y)\,\widetilde{v}_h(y')\,dydy' + \int M\,v_h\,\widetilde{v}_h\,dy = \lambda \int v_h\,\widetilde{v}_h\,dy \qquad \forall\,\widetilde{v}_h \in V_h$$

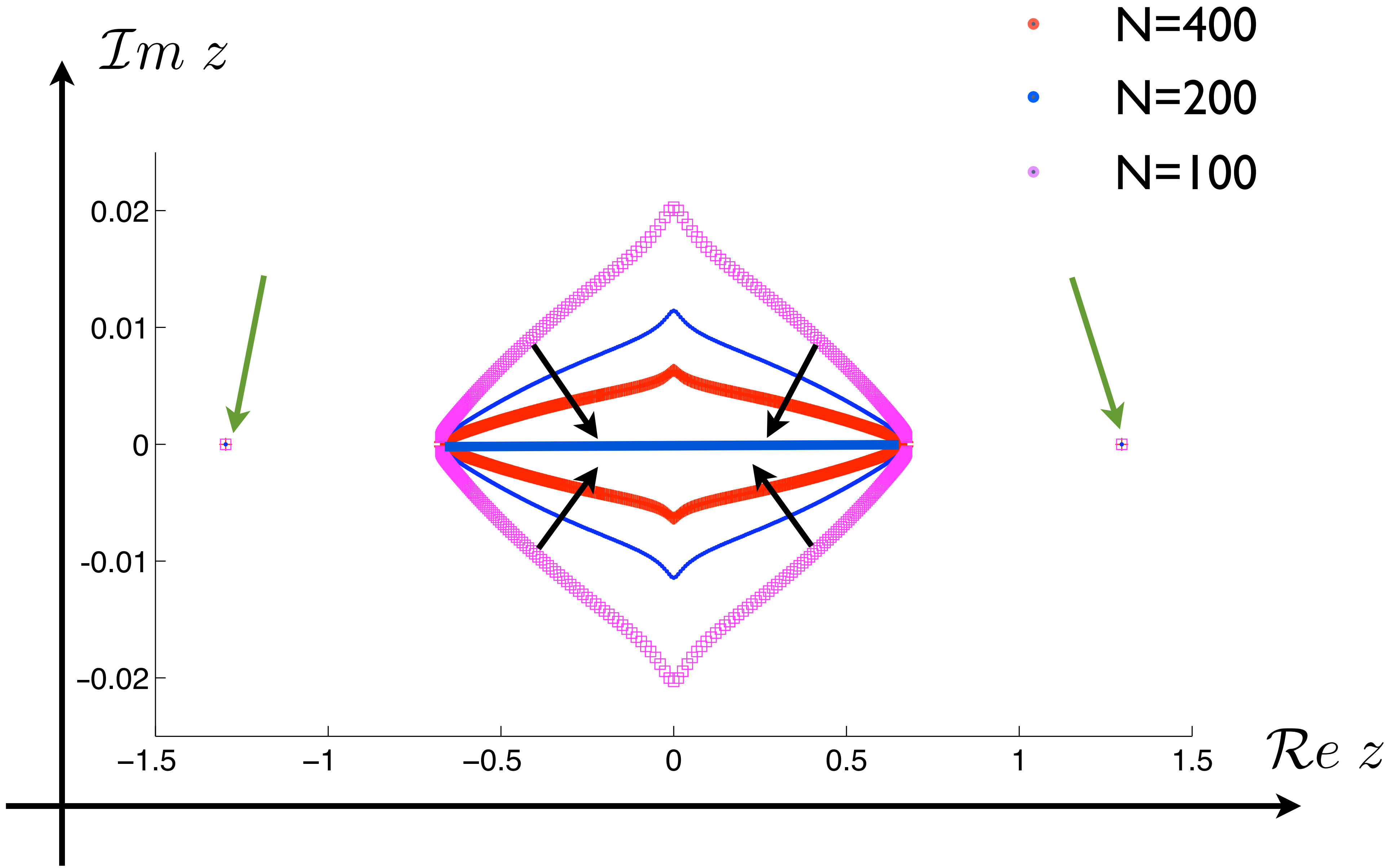
The case of a linear profile $M(y) = y$



The case of a linear profile $M(y) = y$

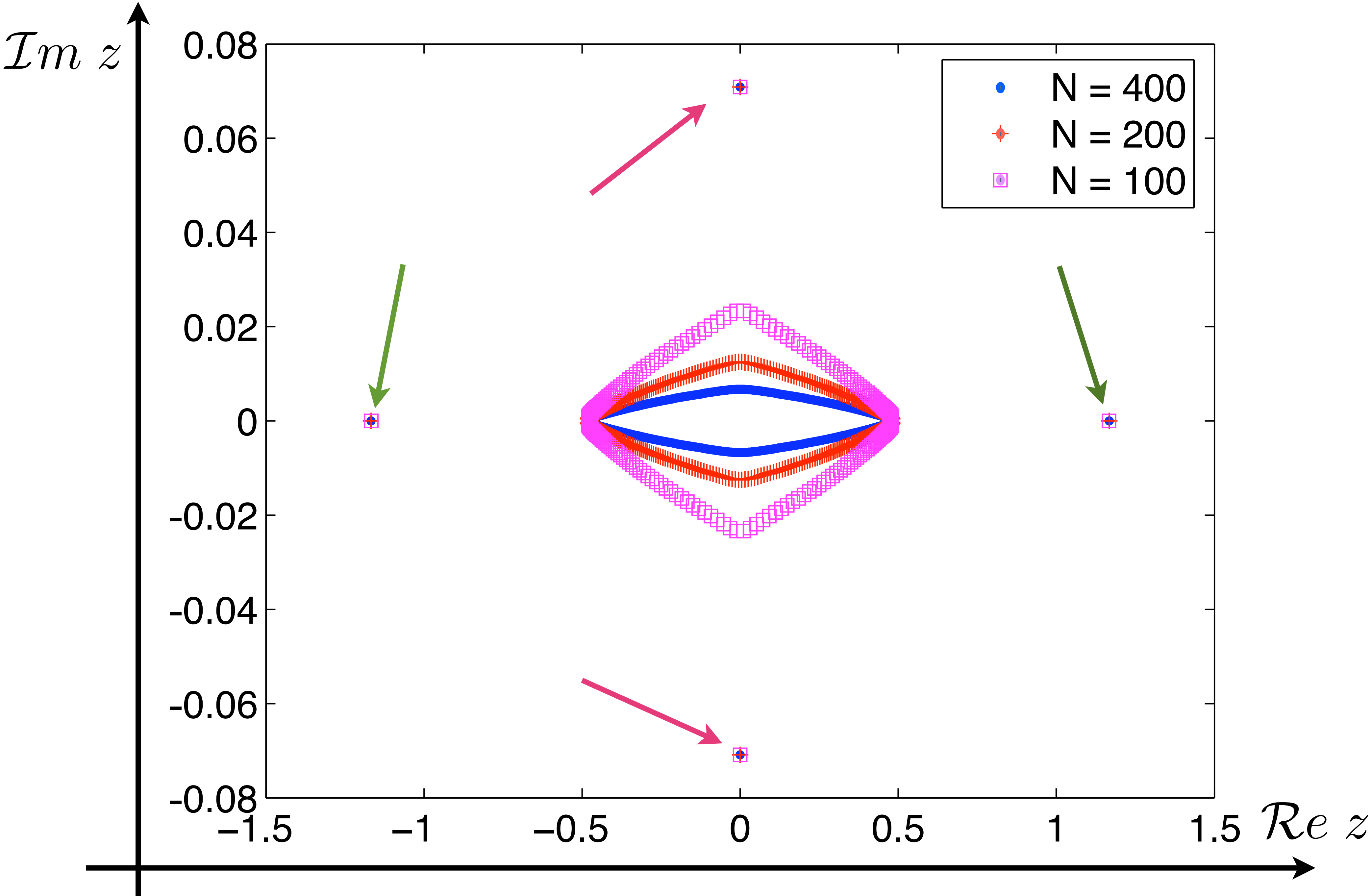


The case of a stable tanh profile



$$M(y) = a \tanh(\alpha\, y)$$

The case of an unstable tanh profile



A well-posedness result

(A) M is stable $\left(\Longleftrightarrow (\ \mathcal{E}\ ) \text{ only has real solutions.} \right)$

(B) $M \in C^{3,\gamma}(-1,1),\ M' \neq 0,\ M'' \neq 0$ in $[-1,1]$

Theorem : Under assumptions (A) and (B), $(\mathcal{P})$ is weakly well-posed : if $(u_0, u_1) \in H^4_x(L^2_y) \times H^3_x(L^2_y)$, there exists a unique solution

$$u \in C^0\big(\mathbb{R}^+; H^1_x(L^2_y)\big) \times C^1\big(\mathbb{R}^+; L^2_x(L^2_y)\big)$$

$$\boxed{\|u(\cdot,t)\|_{H^1_x(L^2_y)} \leq C(M)\,(1+t^3)\,\Big(\|u_0\|_{H^4_x(L^2_y)} + \|u_1\|_{H^3_x(L^2_y)}\Big)}$$

$$\boxed{U(x,t) = U_p(x,t) + U_c(x,t)}$$

U_p is a solution of the generalized wave equation

$$\boxed{\left[(\partial_t - \lambda_+ \partial_x)(\partial_t - \lambda_- \partial_x)\right] U_p = 0}$$

U_c is a continuous superposition on λ of solutions of squared transport equations

$$U_c = \int_{M_-}^{M_+} U_{c,\lambda}\, d\lambda \qquad (\partial_t - \lambda\, \partial_x)^2\, U_{c,\lambda} = 0$$

Thank you for your attention