One-dimensional degenerate elliptic operators on L_p-spaces with complex coefficients

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Outline

Aim

Consider a divergence-form sectorial operator

$$A = -\sum_{k,l=1}^{d} \partial_k \, c_{kl} \, \partial_l$$

in $L_2(\mathbb{R}^d)$ with $c_{kl} \in W^{1,\infty}(\mathbb{R}^d)$ complex valued.
Let S be the semigroup generated by A on $L_2(\mathbb{R}^d)$.

Problem. Does S extend consistently to a contraction C_0-semigroup on $L_p(\mathbb{R}^d)$?

Let $-A_p$ be the generator in $L_p(\mathbb{R}^d)$.
Clearly $C_c^\infty(\mathbb{R}^d) \subset D(A_p)$.

Problem. Is $C_c^\infty(\mathbb{R}^d)$ a core for A_p?

Setting

For all $\theta \in [0, \frac{\pi}{2})$ define

$$\Sigma_\theta = \{r\, e^{i\varphi} : r \in [0, \infty) \text{ and } \varphi \in [-\theta, \theta]\}.$$

For all $k, l \in \{1, \ldots, d\}$ let $c_{kl} \in W^{1,\infty}(\mathbb{R}^d)$.
Suppose that there exists a $\theta \in [0, \frac{\pi}{2})$ such that

$$\sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \in \Sigma_\theta$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$.

Operator

Define the form $\mathfrak{a}_0 \colon W^{1,2}(\mathbb{R}^d) \times W^{1,2}(\mathbb{R}^d) \to \mathbb{C}$ by

$$\mathfrak{a}_0(u,v) = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} c_{kl}\, (\partial_l u)\, \overline{\partial_k v}.$$

Then $\mathfrak{a}_0$ is sectorial and closable. Let $\mathfrak{a} = \overline{\mathfrak{a}_0}$ be the closure of $\mathfrak{a}_0$.

Let A be the operator associated with $\mathfrak{a}$.
Then A is m-sectorial. Hence $-A$ is the generator of a holomorphic semigroup S on $L_2(\mathbb{R}^d)$ which is contractive on the sector $\Sigma_{\frac{\pi}{2}-\theta}$.
Obviously $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and

$$Au = -\sum_{k,l=1}^{d} \partial_k\, c_{kl}\, \partial_l u$$

for all $u \in C_c^\infty(\mathbb{R}^d)$.

Extension to L_p, strongly elliptic case

If A is strongly elliptic, that is, there exists a $\mu > 0$ such that

$$\mathrm{Re} \sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k \, \overline{\xi_l} \geq \mu \, |\xi|^2$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, then the semigroup S on $L_2(\mathbb{R}^d)$ extends consistently to a C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.
In general $S^{(p)}$ is not a contraction semigroup.
Let $-A_p$ be the generator of $S^{(p)}$.
Obviously $C_c^\infty(\mathbb{R}^d) \subset D(A_p)$ and

$$A_p u = -\sum_{k,l=1}^{d} \partial_k \, c_{kl} \, \partial_l u$$

for all $u \in C_c^\infty(\mathbb{R}^d)$.

Theorem. The space $C_c^\infty(\mathbb{R}^d)$ is a core for A_p.

Extension to L_p, real valued coefficients

(Without strong ellipticity.)
Suppose the coefficients are real valued.
Then the Beurling–Deny theorem implies that S is sub-Markovian.

With duality S extends consistently to a contraction C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.
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Smooth real valued coefficients

Theorem (Wong-Dzung). Suppose the $c_{kl} \in C^2(\mathbb{R}^d)$ are real valued and the matrix (c_{kl}) is symmetric.
Then $C_c^\infty(\mathbb{R}^d)$ is a core for A_p for all $p \in [1, \infty)$.

Theorem (Ouhabaz). The same is valid on L_2 if the $c_{kl} \in W^{2,\infty}(\mathbb{R}^d)$ are real valued and the matrix (c_{kl}) is symmetric.

Real coefficient on $[0, 1]$

Theorem (Campiti–Metafune–Pallara).
Let $c\colon [0, 1] \to [0, \infty)$ be a real valued Lipschitz continuous function with $c(0) = c(1) = 0$ and $c(x) > 0$ for all $x \in (0, 1)$.
Consider divergence form operator $A_p = -\frac{d}{dx}\, c\, \frac{d}{dx}$ on $L_p(0, 1)$, where $p \in [1, \infty)$.
They presented a characterisation when $C_c^\infty(0, 1)$ is a core for A_p.

Specific aim

- What happens on the full real line $\mathbb{R}$, so dimension one?
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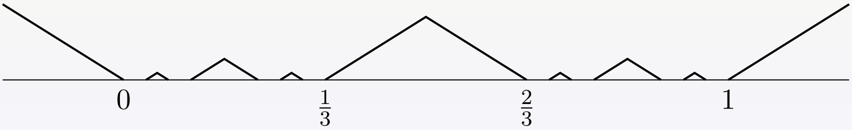
Remark. A $W^{1,\infty}(\mathbb{R})$-function with values in a sector can have many zeros, even without being zero on any nontrivial interval.
An example is $x \mapsto d(x, K) \wedge 1$, where K is the Cantor set.



Set-up

Let $c \in W^{1,\infty}(\mathbb{R})$, $\theta \in [0, \frac{\pi}{2})$ and suppose that $c(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}$. Define the sectorial form $\mathfrak{a}_0 \colon W^{1,2}(\mathbb{R}) \times W^{1,2}(\mathbb{R}) \to \mathbb{C}$ by

$$\mathfrak{a}_0(u,v) = \int_{\mathbb{R}} c\, u' \, \overline{v'}.$$

Then $\mathfrak{a}_0$ closable. Let $\mathfrak{a} = \overline{\mathfrak{a}_0}$ be the closure.

Let A be the operator associated with $\mathfrak{a}$ on $L_2(\mathbb{R})$. So if $u, f \in L_2(\mathbb{R})$, then

$$\Big(u \in D(A) \text{ and } Au = f\Big) \Leftrightarrow \Big(u \in D(\mathfrak{a}) \text{ and } \forall_{v \in D(\mathfrak{a})} [\mathfrak{a}(u,v) = (f,v)_{L_2(\mathbb{R})}]\Big).$$

Let S be the semigroup on $L_2(\mathbb{R})$ generated by $-A$.

Extension to L_p

Recall $c(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}$ and $\mathfrak{a}_0(u,v) = \int_{\mathbb{R}} c\, u'\, \overline{v'}$.

Proposition (Do–tE). Let $p \in [1,\infty)$ and suppose that $\left|1 - \frac{2}{p}\right| \leq \cos\theta$. Then S extends consistently to a contraction C_0-semigroup on $L_p(\mathbb{R})$.

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Remark. Actually $c \in W^{1,\infty}(\mathbb{R})$ is not needed in the above proposition, merely $c \in L_\infty(\mathbb{R})$ and then A is the operator associated with $\mathfrak{a}_0$.

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Theorem. Suppose θ is minimal. Let $p \in [1,\infty)$. Then S extends consistently to a contraction C_0-semigroup on $L_p(\mathbb{R})$ if and only if $\left|1 - \frac{2}{p}\right| \leq \cos\theta$.

Observation. If $-A_p$ is the generator of the C_0-semigroup on $L_p(\mathbb{R})$, then $C_c^\infty(\mathbb{R}) \subset D(A_p)$ and $A_p u = -\frac{d}{dx}\, c\, \frac{d}{dx} u$ for all $u \in C_c^\infty(\mathbb{R})$.

Notation

Recall $c \in W^{1,\infty}(\mathbb{R})$, $\theta \in [0, \frac{\pi}{2})$ and $c(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}$.

Define

$$\mathcal{P} = [\operatorname{Re} c > 0] \quad \text{and} \quad \mathcal{N} = [\operatorname{Re} c = 0].$$

Let $\{I_k : k \in K\}$ be the set of connected components of $\mathcal{P}$.
Write $I_k = (a_k, b_k)$ for all $k \in K$, with $a_k, b_k \in [-\infty, \infty]$. Let

$$E = \{a_k, b_k : k \in K\} \cap \mathbb{R}$$

be the set of all finite endpoints.
For all $k \in K$ define

$$m_k = \begin{cases} \dfrac{a_k + b_k}{2} & \text{if } a_k \in \mathbb{R} \text{ and } b_k \in \mathbb{R}, \\ a_k + 1 & \text{if } a_k \in \mathbb{R} \text{ and } b_k = \infty, \\ b_k - 1 & \text{if } a_k = -\infty \text{ and } b_k \in \mathbb{R}, \\ 0 & \text{if } a_k = -\infty \text{ and } b_k = \infty. \end{cases}$$

Main result

Define the function $Z\colon \mathbb{R} \to \mathbb{R}$ by

$$Z(x) = \begin{cases} \int_x^{m_k} \frac{1}{\operatorname{Re} c} & \text{if } x \in I_k \text{ and } k \in K, \\ \infty & \text{if } x \in \mathcal{N}. \end{cases}$$

Theorem. Let $p \in [1, \infty)$ and assume that $\left|1 - \frac{2}{p}\right| \leq \cos\theta$.
Let $-A_p$ be the generator of the contraction C_0-semigroup on $L_p(\mathbb{R})$ which is consistent with S.
Then $C_c^\infty(\mathbb{R})$ is a core for A_p if and only if $Z|_{(x-\delta,x+\delta)} \notin L_q(x-\delta, x+\delta)$ for all $x \in E$ and $\delta > 0$, where q is the dual exponent of p.

Corollary. If c is real valued then $C_c^\infty(\mathbb{R})$ is a core for A_1.

Consequence

Corollary. Let $p \in (1, \infty) \setminus \{2\}$, let $c \in W^{2-\frac{1}{p},\infty}(\mathbb{R})$ and suppose that $c(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}$, where $\theta = \arccos \left|1 - \frac{2}{p}\right|$.
Then $C_c^\infty(\mathbb{R})$ is a core for A_p.

If $p = 2$, then require $\theta \in [0, \frac{\pi}{2})$, $c \in W^{2-\frac{1}{2},\infty}(\mathbb{R})$ and $c(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}$.

Some ingredients of the proof

For all $u \in L_{1,\mathrm{loc}}(\mathbb{R})$ with $u|_{\mathcal{P}} \in W^{1,1}_{\mathrm{loc}}(\mathcal{P})$ define $Du \colon \mathbb{R} \to \mathbb{C}$ by

$$(Du)(x) = \begin{cases} u'(x) & \text{if } x \in \mathcal{P}, \\ 0 & \text{if } x \in \mathcal{N}. \end{cases}$$

Theorem. Let $p \in [1, \infty)$ and suppose that $\left|1 - \frac{2}{p}\right| \leq \cos\theta$. Then

$$D(A_p) = \{u \in L_p(\mathbb{R}) \cap W^{1,p}_{\mathrm{loc}}(\mathcal{P}) : c\,Du \in W^{1,p}_{\mathrm{loc}}(\mathcal{P}),\ D(c\,Du) \in L_p(\mathbb{R}),$$

$$\lim_{x \downarrow a} (c\,Du)(x) = 0 \text{ for all } a \in E_l \text{ and}$$

$$\lim_{x \uparrow b} (c\,Du)(x) = 0 \text{ for all } b \in E_r\}.$$

If $u \in D(A_p)$, then $A_p u = -D(c\,Du)$.

Some ingredients of the proof (2)

Lemma. If $(a_k, b_k) = I_k$ and $u \in D(A_p)$, then $u\,\mathbb{1}_{(a_k,\infty)} \in D(A_p)$ and $u\,\mathbb{1}_{(-\infty,a_k)} \in D(A_p)$.

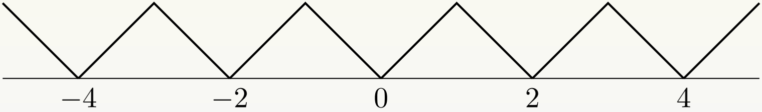
Lemma. If $(a_k, b_k) = I_k$, $a_k \in \mathbb{R}$ and $Z|_{(a_k,m_k)} \in L_q(a_k, m_k)$, then $D(A_p) \subset C[a_k, m_k]$.

Example (1)

Let $\kappa \in (1, \infty)$.
Define $c \colon \mathbb{R} \to [0, \infty)$ by

$$c(x) = \Big(d(x, 2\mathbb{Z}) \Big)^{\kappa}.$$

Then $c \in W^{1,\infty}(\mathbb{R})$.



Then

$$Z(x) = (\kappa - 1)^{-1} \Big(x^{-(\kappa - 1)} - 1 \Big)$$

for all $x \in (0, 1)$.
Let $p \in (1, \infty)$.
It follows from the main theorem that $C_c^{\infty}(\mathbb{R})$ is a core for A_p if and only if $\kappa \geq 2 - \frac{1}{p}$.

Example (2)

Fix $\lambda \in [0,1)$. Let $K \subset [0,1]$ be the generalized Cantor set with $|K| = \lambda$. So $K = \bigcap_{n=0}^{\infty} K_n$, with $K_0 = [0,1]$ and for any $n \in \mathbb{N}_0$ construct K_{n+1} by removing the central open interval of length $(1-\lambda)\, 3^{-(n+1)}$ from each of the 2^n intervals of K_n.

Fix $\kappa \in (1,\infty)$ and define $c \colon \mathbb{R} \to [0,\infty)$ by

$$c(x) = \Big(d(x,K) \wedge 1 \Big)^{\kappa}.$$

Let $p \in (1,\infty)$ and let q be the dual exponent of p.

Then $C_c^\infty(\mathbb{R})$ is a core for A_p if and only if $\lambda > 0$ or $2 \geq 3^{1-(\kappa-1)q}$.

References

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One-dimensional degenerate elliptic operators on L_p-spaces with complex coefficients.
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