# A singular family of J-selfadjoint Schrödinger operators

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# What we know

# Isotonic harmonic oscillator

## Separation of variables rotationally symmetric harmonic oscillator in $\mathbb{R}^d$

$\Re(\gamma) > 0$ and $m \in \mathbb{C}$ eventually $\Re(m) > -1$

$$L_{m,\gamma} = -\partial_r^2 + \frac{m^2 - \frac{1}{4}}{r^2} + \frac{\gamma^2}{4} r^2 \qquad \text{in } C_0^\infty]0,\infty[ \subset L^2[0,\infty[$$

- Complex harmonic oscillator: $m = \pm\frac{1}{2}$
- Complex Calogero operator: limiting case $\gamma \to 0$

# J-selfadjoint extensions

$L_{m,\min,\gamma}$: minimal operator
$L_{m,\max,\gamma}$: maximal operator

As $r \to 0$ we always have asymptotics

$$f \in \mathrm{D}(L_{m,\min,\gamma}) \qquad \Rightarrow \qquad f = \begin{cases} o(r^{\frac{3}{2}}) & m \neq 0 \\ o(r^{\frac{3}{2}} \ln(r)) & m = 0 \end{cases}$$

Lemma.

$$\dim \left[ \frac{\mathrm{D}(L_{m,\max,\gamma})}{\mathrm{D}(L_{m,\min,\gamma})} \right] = \begin{cases} 0 & |\Re(m)| \geq 1 \\ 2 & |\Re(m)| < 1 \end{cases}$$

$\Rightarrow$ Case $|\Re(m)| \geq 1$

$$L_{m,\min,\gamma} = L_{m,\max,\gamma} = L^*_{\overline{m},\min,\overline{\gamma}}$$

J-selfadjoint for $Jf = \bar{f}$

# J-selfadjoint extensions

Case $|\Re(m)| < 1$ $\quad\quad \eta(r) = \begin{cases} 1 & r < 1 \\ 0 & r > 2 \end{cases}$

$w_+(r) = \eta(r) r^{\frac{1}{2}+m}$ $\quad\quad w_-(r) = \eta(r) \begin{cases} r^{\frac{1}{2}-m} & m \neq 0 \\ r^{\frac{1}{2}} \ln(r) & m = 0 \end{cases}$

Proposition 1.

For any closed

$$L_{m,\min,\gamma} \subset S_{m,\gamma} \subset L_{m,\max,\gamma}$$

there exists $\alpha_\pm \in \mathbb{C} \setminus \{0\}$ such that

$$\mathrm{D}(S_{m,\gamma}) = \mathrm{D}(L_{m,\min,\gamma}) + \mathbb{C}[\alpha_+ w_+ + \alpha_- w_-]$$

$$\mathrm{D}(S^*_{m,\gamma}) = \mathrm{D}(L_{\overline{m},\min,\overline{\gamma}}) + \mathbb{C}\overline{[\alpha_+ w_+ + \alpha_- w_-]}$$

$$(1-\eta)\mathrm{D}(S_{m,\gamma}) \subset H^2(\mathbb{R}) \cap \widehat{H^2(\mathbb{R})}$$

# Semi-homogeneous extension

## Unitary dilations $e^{iD\tau}\phi(r) = e^{\tau/2}\phi(e^{\tau} r)$

We have

$$e^{iD\tau} L_{m,\gamma} e^{-iD\tau} = e^{-2\tau} L_{m,e^{2\tau}\gamma}.$$

Also true for min and max operators.

And also true for the J-selfadjoint family

$$H_{m,\gamma} : \qquad \mathrm{D}(H_{m,\gamma}) = \mathrm{D}(L_{m,\min,\gamma}) + \mathbb{C} w_+.$$

This family is holomorphic for $\Re(m) > -1$ and $\Re(\gamma) > 0$.

# Spectrum of $H_{m,\gamma}$

$\Re(m) > -1$

$$A_{b,a} = z\partial_z^2 + (b-z)\partial_z - a \qquad \text{Kummer operator}$$

$$C_{b,\gamma,\tilde{a}} = z\partial_z^2 + (b-\gamma z)\partial_z - \tilde{a} \qquad \text{generalised confluent}$$

$$A_{b,\frac{\tilde{a}}{\gamma}} v(z) = 0 \quad \Rightarrow \quad C_{b,\gamma,\tilde{a}} v(\gamma z) = \gamma A_{b,\frac{\tilde{a}}{\gamma}} v(\gamma z) = 0$$

For $V_{\gamma,\mu} u(z) = \mathrm{e}^{-\frac{\gamma}{2}z} z^{\mu} u(z)$ and $z = \frac{r^2}{2}$

$$V_{\gamma,\frac{m}{2}+\frac{1}{4}} C_{m+1,\gamma,\frac{\gamma(m+1)-\epsilon}{2}} V_{-\gamma,-\frac{m}{2}-\frac{1}{4}} = z\partial_z^2 + \partial_z - \frac{m^2-\frac{1}{4}}{2z} - \frac{\gamma^2}{2} z + \epsilon$$

$$= \partial_r^2 - \frac{m^2-\frac{1}{4}}{r^2} - \frac{\gamma^2}{4} r^2 + \epsilon = -(L_{m,\gamma} - \epsilon)$$

Then $C_{m+1,\gamma,\frac{\gamma(m+1)-\epsilon}{2}} v(z) = 0$ gives $(L_{m,\gamma} - \epsilon)\phi(r) = 0$

$$\phi(r) = \left(\frac{1}{2}\right)^{\frac{m}{2}+\frac{1}{4}} \mathrm{e}^{-\frac{\gamma r^2}{4}} r^{m+\frac{1}{2}} v\left(\frac{\gamma r^2}{2}\right)$$

# Spectrum of $H_{m,\gamma}$

## $\Re(m) > -1$

Take

$$a = \frac{m+1}{2} - \frac{\epsilon}{2\gamma} \qquad \mathbb{M}(a,b,z) = \frac{{}_1F_1(a;b;z)}{\Gamma(b)}$$

$$b = m+1 \qquad U(a,b,z) = \frac{{}_2F_0(a,a+1-b;-;-\frac{1}{z})}{z^a}$$

Get solutions $\Xi$ and $\Theta$ for $(L_{m,\gamma} - \epsilon)\Phi = 0$ such that

$$\mathrm{W}(\Xi,\Theta) = \frac{\sqrt{2}}{\gamma^m \Gamma(a)}.$$

This gives eigenvalues $\varepsilon = \gamma(2n+1+m)$ for $n \in \mathbb{N} \cup \{0\}$ and eigenfunctions

$$\mathrm{e}^{-\frac{\gamma r^2}{4}} r^{m+\frac{1}{2}} L_n^m\left(\frac{\gamma r^2}{2}\right) \in \mathrm{D}(H_{m,\gamma}).$$

# Green's function of $H_{m,\gamma}$

Asymptotes

$r \to \infty$

$$\Xi(r) \sim r^{-\frac{1}{2}-\frac{\epsilon}{\gamma}} \mathrm{e}^{\frac{\gamma r^2}{2}} \qquad \Theta(r) \sim r^{-\frac{1}{2}+\frac{\epsilon}{\gamma}} \mathrm{e}^{-\frac{\gamma r^2}{2}}$$

$r \to 0$

$$\Xi(r) \sim r^{m+\frac{1}{2}} \qquad \Theta(r) \sim \begin{cases} r^{-m+\frac{1}{2}} & m \neq 0,\ \Re(m) \geq 0 \\ r^{\frac{1}{2}} \ln(r) & m = 0 \\ r^{m+\frac{1}{2}} & -1 < \Re(m) < 0 \end{cases}$$

The Green's function is

$$G(r,t) = \frac{\gamma^m \Gamma(a)}{\sqrt{2}} \Xi(\min\{r,t\}) \Theta(\max\{r,t\})$$

Proposition 2.

$$(H_{m,\gamma} - \epsilon)^{-1} \in \mathcal{C}_2(L^2[0,\infty[) \qquad \text{Schatten class}$$

for $\epsilon = -\gamma(m+2)$ and hence all $\epsilon \neq \gamma(2n+1+m)$

# What we don’t know

- Numerical range (some partial enclosures).
- Schatten norm asymptotic of the resolvent.
- Is it a holomorphic family of type (A) or (B) in some range of $m$, $\gamma$?

# References

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- See also Viola's talk.