A new complex spectrum associated to invisibility in waveguides

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Time-harmonic scattering in waveguide

The acoustic waveguide: $\Omega = \mathbb{R} \times (0,1)$, $k = \omega/c$, $e^{-i\omega t}$

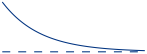
$$\frac{\partial u}{\partial \nu} = 0$$

$$\Delta u + k^2 u = 0 \qquad 1$$

$$\frac{\partial u}{\partial \nu} = 0$$

- A finite number of propagative modes for $k > n\pi$:
  $u_n^{\pm}(x,y) = \cos(n\pi y)e^{\pm i\beta_n x} \quad \beta_n = \sqrt{k^2 - n^2\pi^2}$

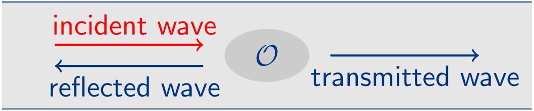
- An infinity of evanescent modes for $k < n\pi$:
  $u_n^{\pm}(x,y) = \cos(n\pi y)e^{\mp\gamma_n x} \quad \gamma_n = \sqrt{n^2\pi^2 - k^2}$



($+/-$ correspond to right/left going modes)

Time-harmonic scattering in waveguide

$\mathcal{O} \subset \Omega$
$1 + \rho \geq 0$
$\mathrm{supp}(\rho) \subset \mathcal{O}$



- The total field $u = u_{inc} + u_{sca}$ satisfies the equations

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

- The incident wave is a superposition of propagative modes:

$$u_{inc} = \sum_{n=0}^{N_P} a_n u_n^+$$

- The scattered field u_{sca} is outgoing:

$$x \to \pm\infty \quad u_{sca} = \sum_{n=0}^{\infty} b_n^{\pm} u_n^{\pm} \sim \sum_{n=0}^{N_P} b_n^{\pm} u_n^{\pm}$$

Scattering problem and trapped modes

By Fredholm analytic theory:

Theorem

The scattering problem is well-posed except maybe for a countable sequence of $k \in \mathscr{T}$ at which trapped modes exist.

Definition

A trapped mode of the perturbed waveguide is a solution $u \neq 0$ of

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

such that $u \in L^2(\Omega)$.

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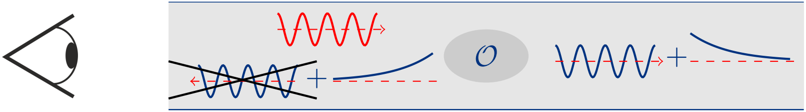
- There is a huge literature on trapped modes: Davies, Evans, Exner, Levitin, McIver, Nazarov, Vassiliev, ...
- Existence of trapped modes is proved in specific configurations (for instance symmetric with respect to the horizontal mid-axis) (Evans, Levitin and Vassiliev, JFM, 1994)

At particular frequencies k, it may occur that, for some u_{inc},

$$x \to -\infty \quad u_{sca} \to 0$$

It means that the obstacle $\mathcal{O}$ produces no reflection. It is invisible for an observer located far at the left-hand side.



Invisibility notions

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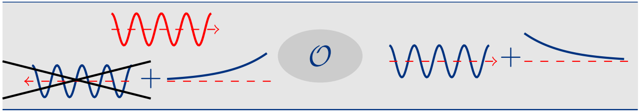
$k \in \mathscr{K}$



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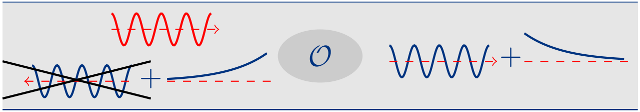
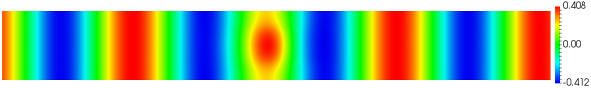


OBJECTIVE

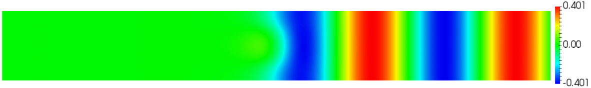
Find a way to compute directly $\mathscr{K}$ by solving an eigenvalue problem, instead of sweeping in k.

Invisibility notions

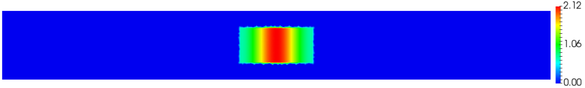
Incident field $u_{inc} = e^{ikx}$



Total field u



Scattered field u_{sca}



Perturbation ρ

Invisibility notions

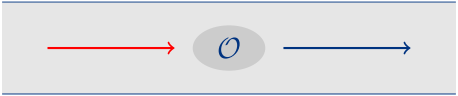
The total field u satisfies homogeneous equations:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

but this is not an eigenvalue problem because $u \notin L^2(\Omega)$.

A simple and important remark

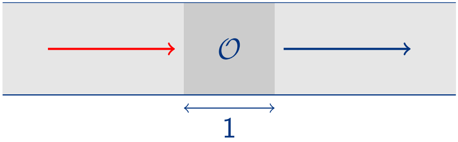
For $k \in \mathscr{K}$, the total field is ingoing at the left-hand side of $\mathcal{O}$ and outgoing at the right-hand side of $\mathcal{O}$.



The idea is to use an analytic dilation (and numerically PMLs), with complex conjugate parameters at both sides of the obstacle, so that the transformed u will belong to $L^2(\Omega)$.

A simple example

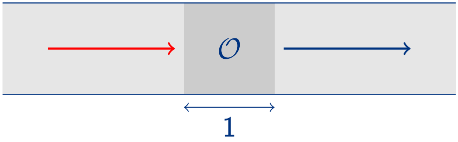
Suppose $\mathcal{O} = (-0.5, 0.5) \times (0, 1)$ and ρ is a constant in $\mathcal{O}$.



Then $\mathscr{K}$ can be computed explicitly (no mode coupling):

$$\mathscr{K} = \bigcup_{n \geq 0} \mathscr{K}^n$$

A simple example



Then $\mathscr{K}$ can be computed explicitly (no mode coupling):

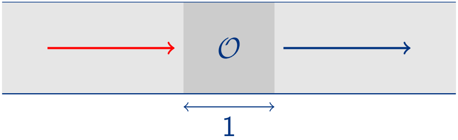
$$\mathscr{K} = \bigcup_{n \geq 0} \mathscr{K}^n$$

with

$$\mathscr{K}^n = \left\{ \sqrt{\frac{j^2 + n^2}{1 + \rho}}\pi; j \in \mathbb{N} \right\}$$

They is always ($\forall \rho \neq 0$) an infinite countable sequence of values of k such that $\mathcal{O}$ produces no reflection.

A simple example



The study of $\mathscr{K}^0$, with a spectral point of view, has been done in:

H. Hernandez-Coronado, D. Krejcirik and P. Siegl,
Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem,
Physics Letters A (2011).

Our approach allows to extend some of their results to higher dimensions.

Outline

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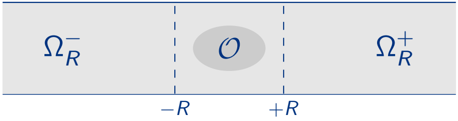
A main tool: Perfectly Matched Layers

(or complex stretching)



In order to use Perfectly Matched Layers to solve the scattering problem, we start by splitting the waveguide into three parts:

$\Omega_R = \Omega \cap \{|x| < R\}$, $\Omega_R^+ = \Omega \cap \{x > R\}$ and $\Omega_R^- = \Omega \cap \{x < -R\}$.

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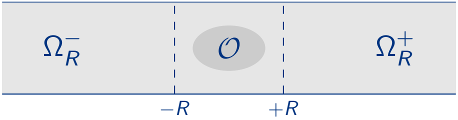

Reformulation of the scattering problem:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega_R) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega \cap \{|x| < R\})$$

$$u = u^+ \text{ and } \frac{\partial u}{\partial x} = \frac{\partial u^+}{\partial x} \quad (x = R)$$

$$u - u_{inc} = u^- \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \frac{\partial u^-}{\partial x} \quad (x = -R)$$

with $u^\pm = \sum b_n^\pm u_n^\pm$ in $\Omega_R^\pm$.

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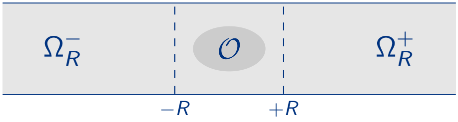

Formulation with a stretching in $\Omega_R^\pm$:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega_R) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega \cap \{|x| < R\})$$

$$u = u_\alpha^+ \text{ and } \frac{\partial u}{\partial x} = \alpha \frac{\partial u_\alpha^+}{\partial x} \quad (x = R)$$

$$u - u_{inc} = u_\alpha^- \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \alpha \frac{\partial u_\alpha^-}{\partial x} \quad (x = -R)$$

with $u_\alpha^\pm(x,y) = u^\pm\left(\pm R + \dfrac{x \mp R}{\alpha}, y\right)$ for $(x,y) \in \Omega_R^\pm$.

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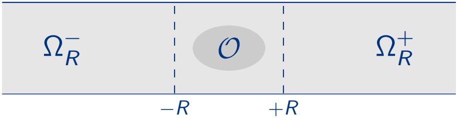
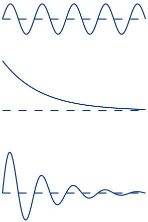


The magic idea of PMLs: using $\alpha \in \mathbb{C}$ such that $u_\alpha^\pm \in L^2(\Omega_R^\pm)$.

Indeed, if $\alpha = e^{-i\theta}$ with $0 < \theta < \pi/2$, propagative modes become evanescent:

$$
\begin{aligned}
u^+(x,y) = \;& \sum_{n=0}^{N_P} a_n \cos(n\pi y) e^{i\sqrt{k^2-n^2\pi^2}(x-R)} \\
+ \;& \sum_{n>N_P} a_n \cos(n\pi y) e^{-\sqrt{n^2\pi^2-k^2}(x-R)} \\
u_\alpha^+(x,y) = \;& \sum_{n\geq 0} a_n \cos(n\pi y) e^{-\sqrt{\frac{n^2\pi^2-k^2}{\alpha^2}}(x-R)}
\end{aligned}
$$



where $\Re e\sqrt{z} > 0$ for $z \in \mathbb{C}\setminus\mathbb{R}^-$.

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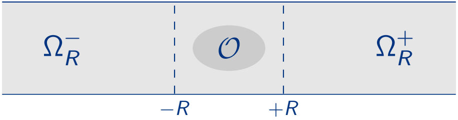

Final PML formulation:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega_R) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega \cap \partial\Omega_R)$$

$$\Delta_\theta u_\alpha^\pm + k^2 {u_\alpha}^\pm = 0 \quad (\Omega_R^\pm) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega \cap \{\pm x > R\})$$

$$u = u_\alpha^+ \text{ and } \frac{\partial u}{\partial x} = \alpha \frac{\partial u_\alpha^+}{\partial x} \quad (x = R)$$

$$u - u_{inc} = u_\alpha^- \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \alpha \frac{\partial u_\alpha^-}{\partial x} \quad (x = -R)$$

where $\Delta_\theta = e^{-2i\theta} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and $u_\alpha^\pm \in L^2(\Omega_R^\pm)$.

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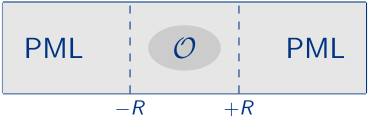

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where $\Delta_\theta = e^{-2i\theta}\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and $u_\alpha^\pm \in L^2(\Omega_R^\pm)$.

Outline

The spectral problem for trapped modes

Let us consider the following unbounded operator of $L^2(\Omega)$:

$$D(A) = \{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \qquad Au = -\frac{1}{1+\rho}\Delta u$$

The trapped modes ($k \in \mathscr{T}$) correspond to real eigenvalues k^2 of A.

THE SPECTRAL PROBLEM FOR TRAPPED MODES

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The trapped modes ($k \in \mathscr{T}$) correspond to real eigenvalues k^2 of A.

SPECTRAL FEATURES OF A

- A is a positive self-adjoint operator.
- $\sigma(A) = \sigma_{ess}(A) = \mathbb{R}^+$ and $\sigma_{disc}(A) = \emptyset$
- Trapped modes are embedded eigenvalues of A !

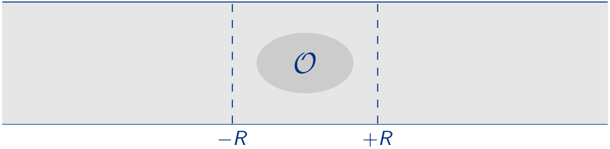
Solution: introduce an analytic dilation (Aguilar, Balslev, Combes, Simon... 70)

The spectral problem for trapped modes

Let us consider now the following unbounded operator:

$$D(A_\alpha) = \{u \in L^2(\Omega); A_\alpha u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$$

$$A_\alpha u = -\frac{1}{1+\rho(x)} \left(\alpha(x) \frac{\partial}{\partial x} \left(\alpha(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)$$



where $\alpha(x) = e^{-i\theta}$ $\alpha(x) = 1$ $\alpha(x) = e^{-i\theta}$

THE SPECTRAL PROBLEM FOR TRAPPED MODES

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$$
\begin{aligned}
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A_\alpha u &= -\frac{1}{1+\rho(x)}\left(\alpha(x)\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}
$$

SPECTRAL FEATURES OF A_α

- A_α is a non self-adjoint operator.
- $\sigma_{ess}(A_\alpha) = \cup_{n\geq 0}\{n^2\pi^2 + e^{-2i\theta}t^2; t \in \mathbb{R}\}$ *(Weyl sequences)*
- $\sigma(A_\alpha)\backslash\sigma_{ess}(A_\alpha) = \sigma_{disc}(A_\alpha)$
- $\sigma_{disc}(A_\alpha) \subset \{z \in \mathbb{C}; -2\theta < \arg(z) \leq 0\}$

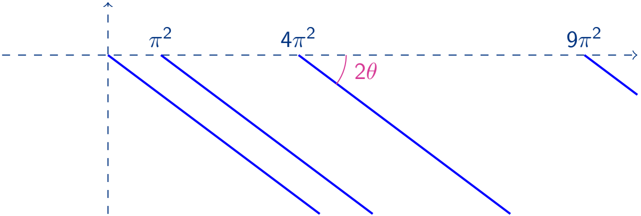
Proof of the second item:

$$\begin{aligned}\sigma_{ess}(A_\alpha) &= \sigma_{ess}(-\Delta_\theta) \\ &= \bigcup_{n\geq 0} \sigma_{ess}(-\Delta_\theta^{(n)}) \\ &= \bigcup_{n\geq 0} \{n^2\pi^2 + e^{-2i\theta} t^2 ; t \in \mathbb{R}\}\end{aligned}$$

$$\Delta_\theta = e^{-2i\theta}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Delta_\theta^{(n)} = e^{-2i\theta}\frac{\partial^2}{\partial x^2} + n^2\pi^2$$

Essential spectrum of A_α:



Proof of the third item: $\sigma(A_\alpha)\backslash\sigma_{ess}(A_\alpha) = \sigma_{disc}(A_\alpha)$

For $z \in \mathbb{C}$ and $u \in H^1(\Omega)$:

$$\left(\frac{1+\rho}{\alpha}(A_\alpha - z)u, u\right)_{L^2(\Omega)} = \int_\Omega \alpha \left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{\alpha}\left|\frac{\partial u}{\partial y}\right|^2 - \frac{z(1+\rho)}{\alpha}|u|^2$$

Since $\Re e(\alpha(x)) = \Re e\left(\frac{1}{\alpha(x)}\right) \geq \cos\theta > 0$, we get for $z = -t^2 < 0$:

$$\Re e\left(\frac{1+\rho}{\alpha}(A_\alpha - z)u, u\right)_{L^2(\Omega)} \geq C\|u\|^2_{H^1(\Omega)}$$

so that $z = -t^2 \notin \sigma(A_\alpha)$.

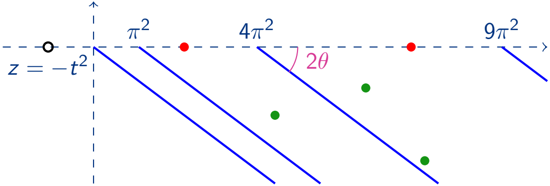
The spectral problem for trapped modes

Proof of the third item: $\sigma(A_\alpha)\backslash\sigma_{ess}(A_\alpha) = \sigma_{disc}(A_\alpha)$

The result follows because:

1. $U = \mathbb{C}\backslash\sigma_{ess}(A_\alpha)$ is a connected set.
2. There is a point $z \in U$ such that $A_\alpha - z$ is invertible.

(See D.E. Edmunds and W.D. Evans, Spectral theory and differential operators, 1987.)

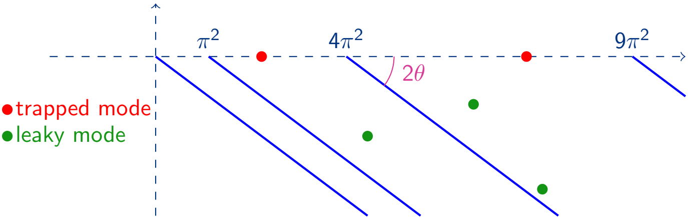


The spectral problem for trapped modes

Discrete spectrum of A_α

- Trapped modes are discrete real eigenvalues of A_α !
- Complex discrete eigenvalues correspond to leaky modes (or complex resonances), which are exponentially growing at infinity.

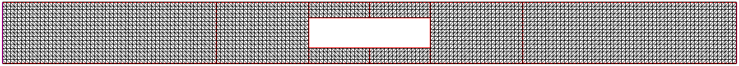
Spectrum of A_α:



Numerical illustration

The numerical results have been obtained by Lucas Chesnel with FreeFem++.

Here the scatterer is a non-penetrable rectangular obstacle in the middle of the waveguide:



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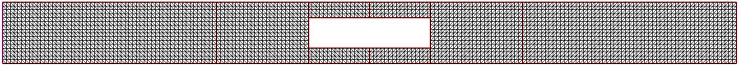
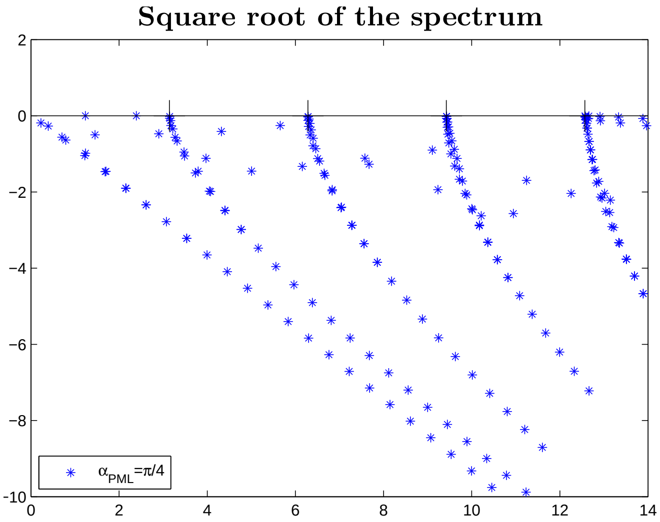


In the next slides, we represent the square-root of the spectrum, which corresponds to k values.

Numerical illustration



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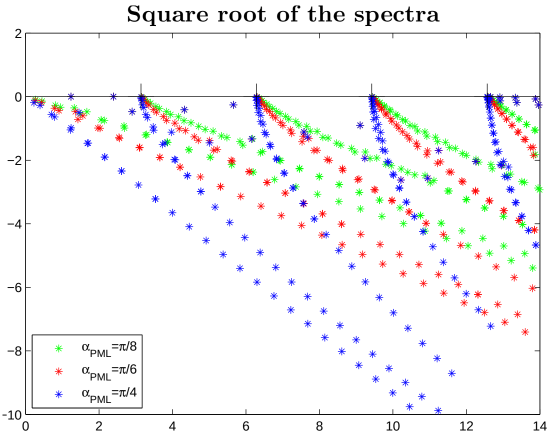
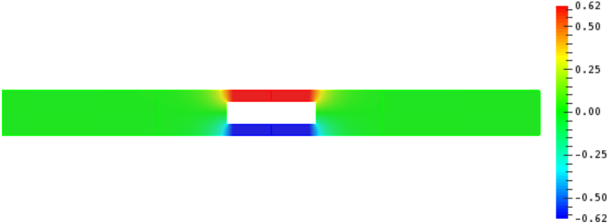
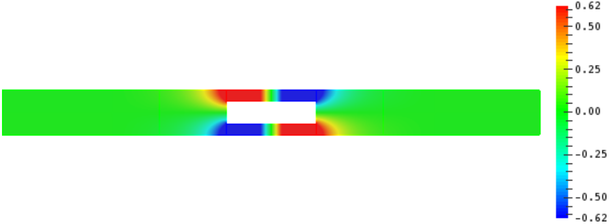


Numerical illustration

There are two trapped modes:





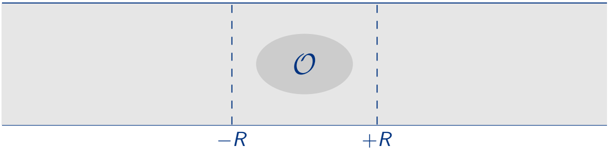
Outline

A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

Let us consider now the following unbounded operator :

$$
\begin{aligned}
D(A_{\tilde{\alpha}}) &= \{u \in L^2(\Omega); A_{\tilde{\alpha}} u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \\
A_{\tilde{\alpha}} u &= -\frac{1}{1+\rho}\left(\tilde{\alpha}(x)\frac{\partial}{\partial x}\left(\tilde{\alpha}(x)\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}
$$



$\tilde{\alpha}(x) = e^{i\theta}$ $\qquad$ $\tilde{\alpha}(x) = 1$ $\qquad$ $\tilde{\alpha}(x) = e^{-i\theta}$

A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

Let us consider now the following unbounded operator :

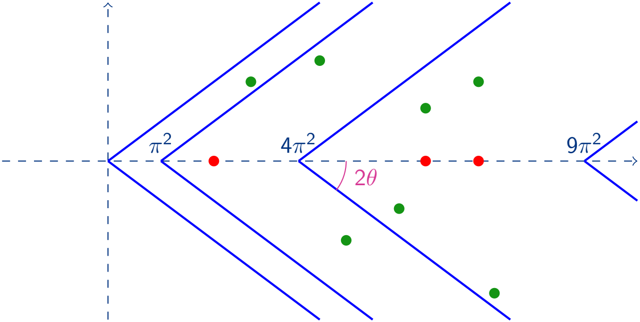
$$\begin{aligned} D(A_{\tilde\alpha}) &= \{u \in L^2(\Omega); A_{\tilde\alpha} u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \\ A_{\tilde\alpha} u &= -\frac{1}{1+\rho}\left(\tilde\alpha(x)\frac{\partial}{\partial x}\left(\tilde\alpha(x)\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u}{\partial y^2}\right) \end{aligned}$$

Spectral features of $A_{\tilde\alpha}$

- $A_{\tilde\alpha}$ is a non self-adjoint operator.
- $\sigma_{ess}(A_{\tilde\alpha}) = \bigcup_{n\geq 0}\{n^2\pi^2 + e^{2i\theta}t^2; t \in \mathbb{R}\} \cup \{n^2\pi^2 + e^{-2i\theta}t^2; t \in \mathbb{R}\}$
- $\sigma_{disc}(A_{\tilde\alpha}) \subset \{z \in \mathbb{C}; -2\theta < \arg(z) < 2\theta\}$
- Conjecture: $\sigma(A_{\tilde\alpha})\backslash\sigma_{ess}(A_{\tilde\alpha}) = \sigma_{disc}(A_{\tilde\alpha})$ if $\rho \neq 0$.

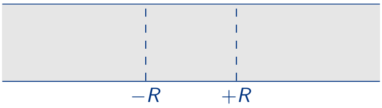
Difficulty: $\mathbb{C}\backslash\sigma_{ess}(A_{\tilde\alpha})$ is not a connected set.

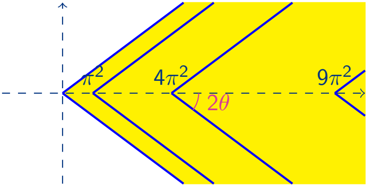
Typical expected spectrum of $A_{\tilde{\alpha}}$:



Pathological cases

In the unperturbed case ($\rho = 0$):





$$\sigma_p(A_{\tilde{\alpha}}) = \{z \in \mathbb{C}; -2\theta < \arg(z) \leq 2\theta\}$$

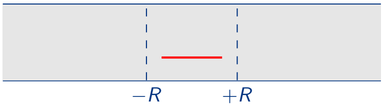
Proof: Use the strechted plane wave as an eigenvector:

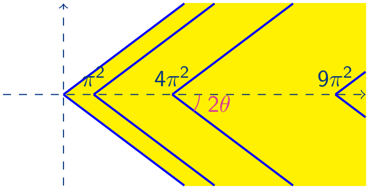
$$A_{\tilde{\alpha}} u = k^2 u$$

for $u(x,y) = \begin{cases} e^{ik(-R+(x+R)e^{-i\theta})} & \text{if} \quad x < -R \\ e^{ikx} & \text{if} \quad -R < x < R \\ e^{ik(R+(x-R)e^{i\theta})} & \text{if} \quad R < x \end{cases}$

Pathological cases

And the same result holds with horizontal cracks !





$$\sigma_p(A_{\tilde{\alpha}}) = \{z \in \mathbb{C}; -2\theta < \arg(z) \leq 2\theta\}$$

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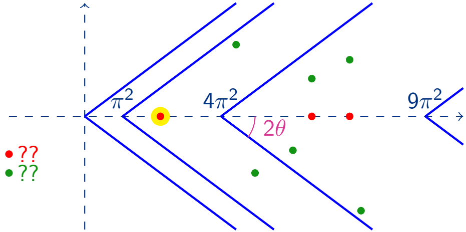
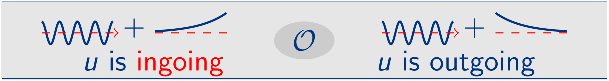
LINK BEWTEEN THE DISCRETE SPECTRUM AND $\mathscr{K}$



For real eigenvalues, the eigenmode is such that

u is ingoing $\mathcal{O}$ u is outgoing

For $k^2 \in \sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R}$, the eigenmode is such that:



There are two cases:

- Either u on the left-hand side contains a propagative part and it is a case of no-reflection: $k \in \mathscr{K}$.
- Either u is evanescent on both sides and k is associated to a trapped mode: $k \in \mathscr{T}$.

Theorem

$$\sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R} = \{k^2 \in \mathbb{R}; k \in \mathscr{K} \cup \mathscr{T}\}$$

$\mathcal{P}$ stands for parity and $\mathcal{T}$ for time reversal:

$$\mathcal{P}u(x,y) = u(-x,y) \text{ and } \mathcal{T}u(x,y) = \overline{u(x,y)}$$

$\mathcal{PT}$-SYMMETRY OF $A_{\tilde{\alpha}}$

If the obstacle is symmetric i.e. $\rho(-x,y) = \rho(x,y)$, then the operator $A_{\tilde{\alpha}}$ is $\mathcal{PT}$-symmetric:

$$\mathcal{P}\,\mathcal{T}\,A_{\tilde{\alpha}}\,\mathcal{P}\,\mathcal{T} = A_{\tilde{\alpha}}$$

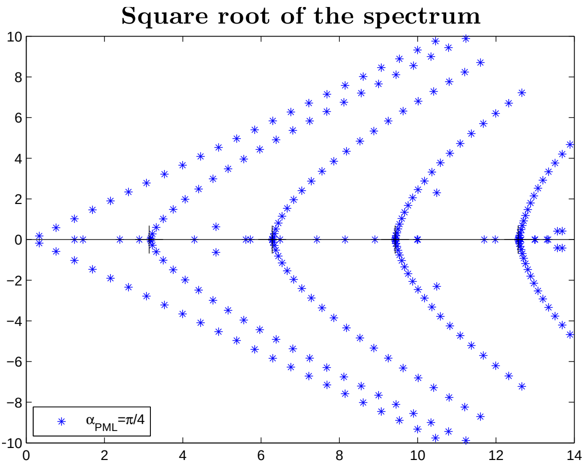
As a consequence $\sigma(A_{\tilde{\alpha}}) = \overline{\sigma(A_{\tilde{\alpha}})}$.

In particular:

$$A_{\tilde{\alpha}}u = \lambda u \Longleftrightarrow A_{\tilde{\alpha}}\tilde{u} = \overline{\lambda}\tilde{u} \quad \text{with} \quad \tilde{u}(x,y) = \overline{u(-x,y)}$$

Numerical illustration

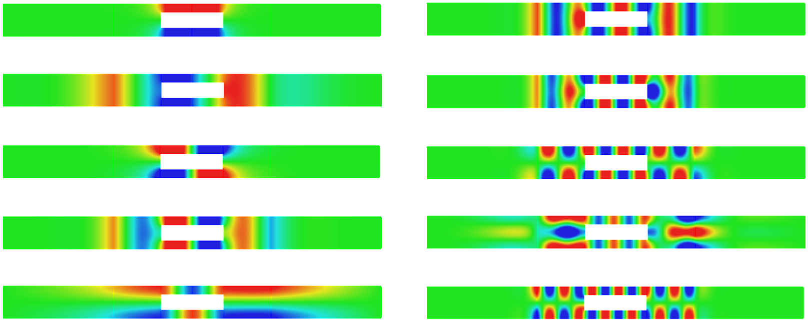
for a rectangular symmetric cavity



- The spectrum is symmetric w.r.t. the real axis ($\mathcal{PT}$-symmetry) .
- There are much more real eigenvalues than for trapped modes.

Numerical illustration

for a rectangular symmetric cavity

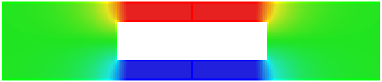


This is a representation of the computed modes for the 10 first real eigenvalues and in the whole computational domain (including PMLs).

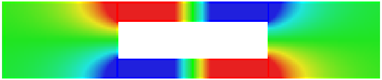
Numerical illustration

for a rectangular symmetric cavity

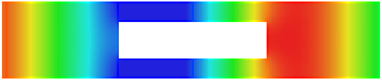
Let us focus on the eigenmodes such that $0 < k < \pi$:



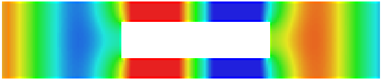
First trapped mode:
$k = 1.2355\cdots$



Second trapped mode:
$k = 2.3897\cdots$



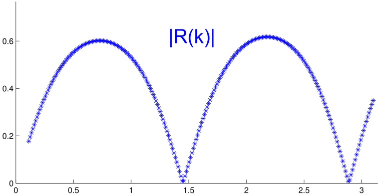
First no-reflection mode:
$k = 1.4513\cdots$

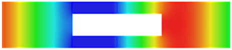


Second no-reflection mode:
$k = 2.8896\cdots$

Numerical illustration

To validate this result, we compute the amplitude of the reflected plane wave for $0 < k < \pi$:



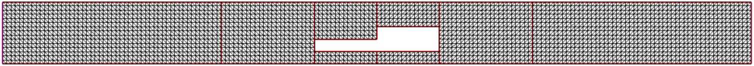


First no-reflection mode:
$k = 1.4513\cdots$

Second no-reflection mode:
$k = 2.8896\cdots$

There is a perfect agreement!

Here the scatterer is a not symetric in x, and neither in y:

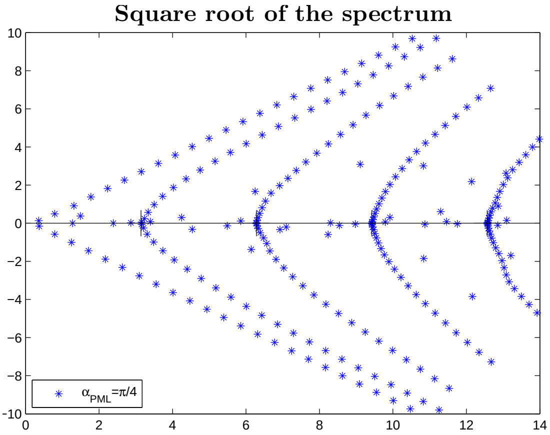


We expect:

- No trapped modes
- No invariance of the spectrum by complex conjugation

Numerical illustration

in a non $\mathcal{PT}$-symmetric case

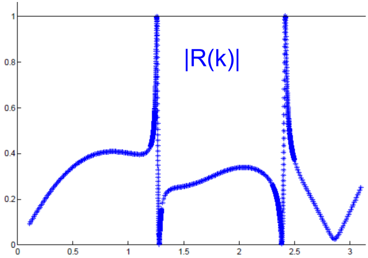


- The spectrum is no longer symmetric w.r.t. the real axis.
- There are several eigenvalues near the real axis.

Numerical illustration

in a non $\mathcal{PT}$-symmetric case

Again results can be validated by computing $R(k)$ for $0 < k < \pi$:



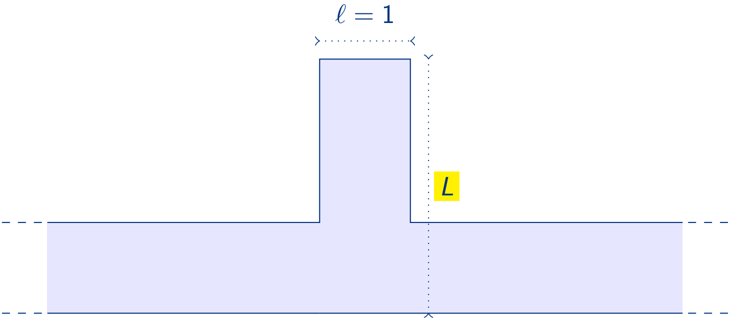
$k = 1.2803 + 0.0003i$ $k = 2.3868 + 0.0004i$ $k = 2.8650 + 0.0241i$

Complex eigenvalues also contain useful information about almost no-reflection.

Numerical illustration of coalescence

$\mathscr{T}$ spectrum in red, $\mathscr{K}$ spectrum in blue

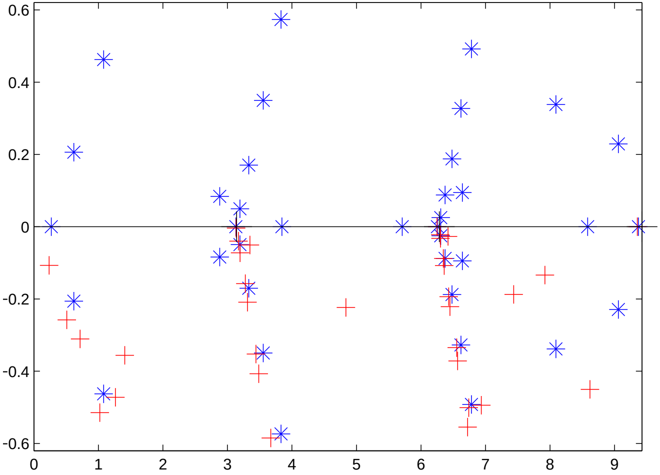
Two series of computations: one with classical PMLs, one with conjugate PMLs. We compute the spectra for a range of L.



Numerical illustration of coalescence

$\mathscr{T}$ spectrum in red, $\mathscr{K}$ spectrum in blue

$L = 1.0267$



Summary and future works

There is still a lot of work to do !

- Prove the conjecture concerning the discrete spectrum of $A_{\tilde{\alpha}}$.
- Prove the existence of real eigenvalues, at least in $\mathcal{PT}$-symmetric cases ($\Leftrightarrow \mathscr{K} \neq \emptyset$).
- Combine this approach with our method for building invisible obstacles.
- ...