Angles of Gaussian primes, Luminy, May 2017

Zeev Rudnick, TAU, joint with Ezra Waxman

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Primes of the form $a^2 + b^2$

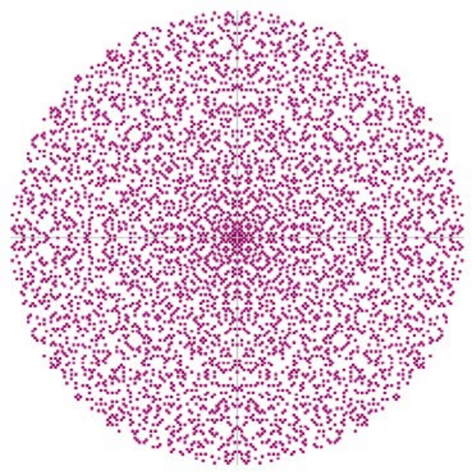
Pierre de Fermat: An odd prime is expressible as $p = a^2 + b^2$ if and only if $p = 1 \bmod 4$
(letter to Mersenne, December 25, 1640). Proof given by Euler (1752-55).

In that case, a+ib is a prime in the Gaussian integers $\mathbf{Z}[i]$, $i = \sqrt{-1}$.

The representation is unique if we assume a>b>0.

We can then find a unique angle $\theta_p \in \left[0, \frac{\pi}{4}\right)$ such that $a + ib = \sqrt{p}e^{i\theta_p}$

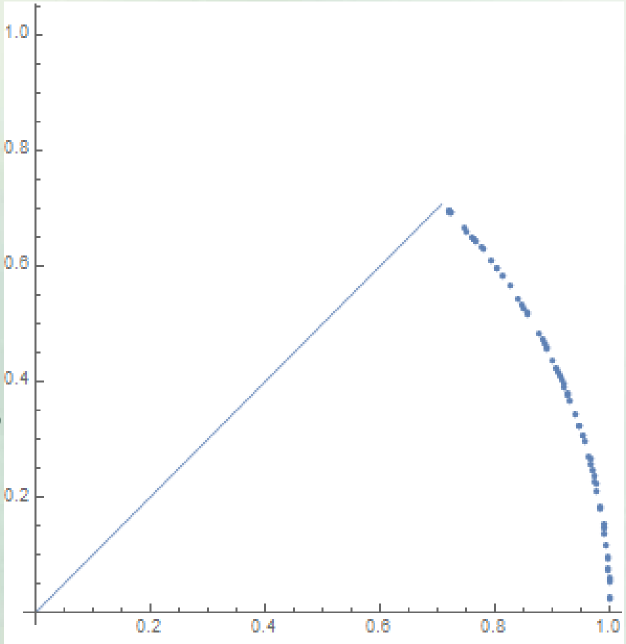
Goal: understand the distribution of these Gaussian primes in the plane.



The angular distribution of Gaussian primes

<u>Hecke (1918):</u> The angles of Gaussian primes are uniformly distributed: For fixed $0 \le \alpha < \beta < \pi/4$

$$\lim_{x\to\infty} \frac{\#\{p = 1 \bmod 4, \quad p \le x: \quad \theta_p \in [\alpha,\beta]\}}{\#\{p = 1 \bmod 4, \quad p \le x\}} = \frac{\beta-\alpha}{\pi/4}$$



Angular distribution $(a+ib)/\sqrt{p}$ of the 67 primes 1000<p<2000, p=1 mod 4, a>b>0

Question: Are the Gaussian angles “random”? i.e. do the first N Gaussian angles have the same statistics as N random points in $\left[0,\frac{\pi}{4}\right)$?

“Random Points” – picked independently and uniformly in $\left[0,\frac{\pi}{4}\right)$

Deviation from randomness: Maximal gap

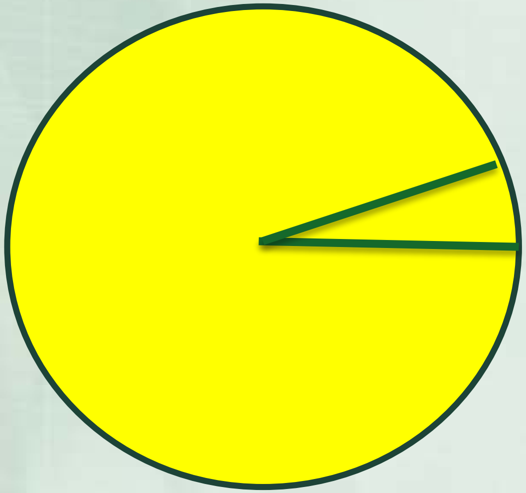
Question: Are the Gaussian angles “random”? i.e. do the first N Gaussian angles have the same statistics as N random points in $\left[0,\frac{\pi}{4}\right)$?

“Random Points” – picked independently and uniformly in $\left[0,\frac{\pi}{4}\right)$

Claim: The **maximal gap** between the first N angles is $> \frac{1}{\sqrt{N\log N}}$

Compare: The maximal gap between N **random** angles is $(\log N)/N$ almost surely
- which is much smaller.

Claim: The arc $(0,1/\sqrt{X})$ does not contain any angle of a prime p<X.
.
Proof: if $p=a^2+b^2\le X,\quad 0<b<a$ has angle θ_p close to zero then



$$\theta_p \sim \tan\theta_p = \frac{b}{a} \ge \frac{1}{a} \ge \frac{1}{\sqrt{a^2+b^2}} \ge \frac{1}{\sqrt{X}}$$

Deviation from randomness: The minimal gap

The **minimal** gap between angles: For N random, independent uniform $\theta_1, \ldots, \theta_N \in [0, \frac{\pi}{4})$

$$\min\{|\theta_i - \theta_j|: \quad i \neq j \leq N\} \approx \frac{1}{N^2} \qquad \text{almost} \quad \text{surely}$$

Note that the average gap is 1/N

For the Gaussian angles, we have “repulsion”: The minimal distance between the first N angles is $\approx \frac{1}{N \log N}$

$$\min\{|\theta_p - \theta_q|: p \neq q \leq X\} \geq \frac{1}{X} \approx \frac{1}{N \log N}, \qquad N = \#\{p \leq X : p = 1 \bmod 4\}$$

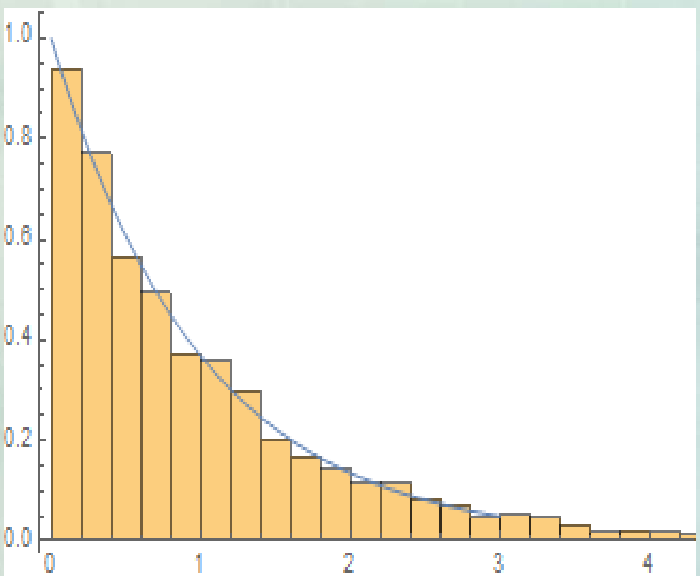
Level spacing distribution -numerics

Let $E_1 < E_2 < \cdots < E_N$ be the **reordering** of the first N angles $\{\theta_p\}$

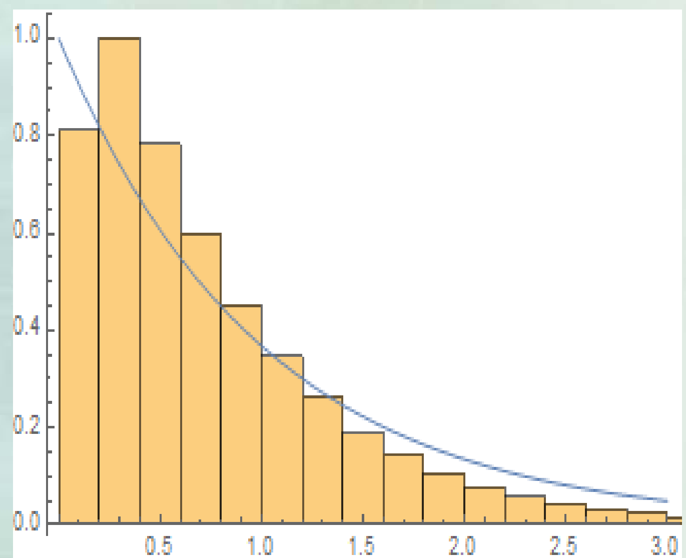
P(s) :=limiting distribution of the normalized gaps δ_n between adjacent levels

$$\delta_n := \frac{E_{n+1} - E_n}{\textbf{mean spacing}}$$

$$\frac{1}{N}\#\{n \le N : \delta_n < x\} \xrightarrow[N\to\infty]{} \int_0^x P(s)\,ds$$



Spacings of 5000 random points $P(s) = \exp(-s)$



Spacings between the 39174 angles θ_p $p < 1{,}000{,}000$

Small scale distribution of Gaussian angles

Hecke (1918): The angles of Gaussian primes are uniformly distributed: For **fixed** $0 \le \alpha < \beta < \pi/4$

$$\#\left\{p \le x, \quad p = 1 \bmod 4: \quad \theta_p \in [\alpha, \beta]\right\} \sim \frac{\beta - \alpha}{\pi/4} \cdot \#\{p \le x, \quad p = 1 \bmod 4\}, \qquad x \to \infty$$

We look for prime angles in "short" (**shrinking**) arcs.
To have a good chance to find them, we need the length of the arc to be a bit bigger than

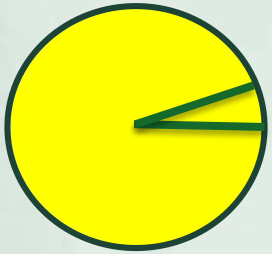
$$\beta - \alpha \gg \frac{1}{\#\{p \le x, p = 1 \bmod 4\}} \approx \frac{\log x}{x}$$

Moreover, we can ask if uniform distribution persists on shrinking arcs.

Assuming GRH, uniform distribution holds for **every** arc of length $\beta - \alpha \gg x^{-\frac{1}{2} + o(1)}$

Unconditionally, this holds with 1/2 replaced by 12/37=0.324… (Kubilius 1952, …, Maknys 1977), Harman & Lewis (2001) 0.381 (existence of angles, without equidistribution).

Note: GRH gives sharp result, since we saw that the arc $(0, 1/\sqrt{X})$ does not contain any angle of a prime p<X.



Almost all short arcs contain an angle

Theorem (ZR & Waxman / Parzanchevski and Sarnak, 2017):

Assuming GRH, **almost all** arcs of length $\frac{(\log x)^3}{x}$ contain an angle $\theta_p, \quad p \leq X$.

Unconditionally, can get arcs of length $x^{-(\frac{1}{2}+\delta)}$ for a suitable $\delta > 0$ by using a zero-density theorem.

This is achieved by giving a bound on the **variance** of the number of angles in short arcs.

The number variance

Counting angles in a small arc: Divide $[0,\frac{\pi}{4}]$ into K small arcs and ask how many of the N prime angles fall into each:

$$N := \#\{p \le X,\ p = 1 \bmod 4\} \sim \frac{1}{2}\frac{X}{\log X}$$

$$\mathcal{N}_{K,N}(\theta) := \#\left\{ p \le x : \quad \theta_{\mathfrak{p}} \in [\theta, \theta + \frac{\pi/4}{K}] \right\}$$

<u>**Expected value**</u>

$$\mathbb{E}(\mathcal{N}_{K,N}) := \frac{1}{\pi/4}\int_0^{\pi/4} \mathcal{N}_{K,N}(\theta)\,d\theta = \frac{N}{K}$$

<u>**Variance:**</u>

"Thm": Assume GRH. Then

$$\mathrm{Var}(\mathcal{N}_{K,N}) := \frac{1}{\pi/4}\int_0^{\pi/4} \left| \mathcal{N}_{K,N}(\theta) - \frac{N}{K} \right|^2 d\theta \ll \frac{N}{K}(\log K)^2$$

Assuming GRH, <u>**almost all**</u> arcs of length $\frac{1}{K}$ contain an angle θ_p, $\quad p \le K(\log K)^3$.

Compare: For N random points, $\mathrm{Var}(\mathcal{N}_{K,N}^{\text{random}}) \sim \dfrac{N}{K}$

Asymptotic for the variance ?

Conjecture: $$\mathrm{Var}(\mathcal{N}_{K,N}) \sim \frac{N}{K}\min(1, 2\frac{\log K}{\log N})$$

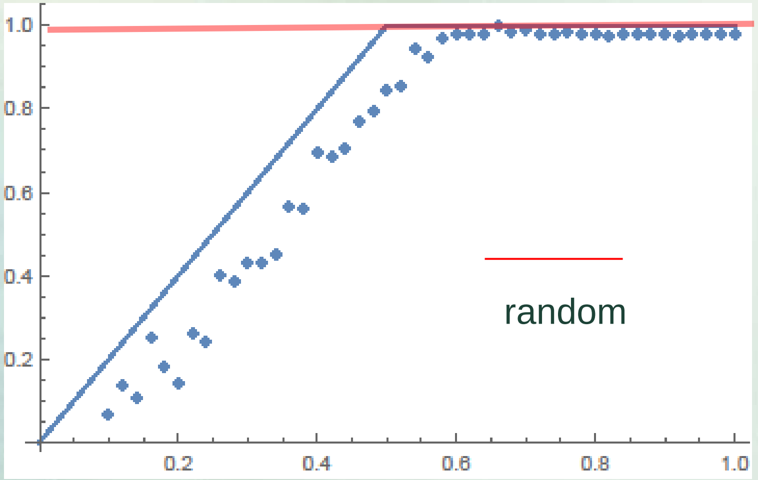
Compare: For N random points,

$$\mathrm{Var}(\mathcal{N}_{K,N}^{\text{random}}) \sim \frac{N}{K}$$

Motivation for conjecture:

a) A **random matrix model**: Express variance through zeros of a certain family of Hecke L-functions, then replace these zeros by eigenphases of a suitable ensemble of random matrices.

b) A **function field analogue**



$$\frac{\mathrm{Var}(\mathcal{N}_{K,N})}{N/K} \quad \text{vs} \quad \log K / \log \mathrm{N}$$

Data: 35241 angles of the Gaussian primes $10^8 < p < 2 * 10^8$

A function field analogue

$\mathbf{F}_q[T]$= polynomials $f(T) = a_0+a_1T+a_2T^2+....+a_dT^d$,   with coefficients a_i ε$\mathbf{F}_q$

<u>analogues</u>:

integers $\mathbf{Z}$ ↔ polynomials $\mathbf{F}_q[T]$

primes p ↔ irreducible polynomial P(T) (“prime”)

<u>positive</u> integer n>0 ↔ <u>monic</u> polynomial $P(T)=T^d+$….

In both cases we have the Fundamental Theorem of Arithmetic – unique factorization into primes (prime polynomials).

Prime Number Theorem ↔ Prime Polynomial Theorem

Sums of two squares $p = a^2 + b^2$ ↔ polynomials $P(T) = A^2 + TB^2$

Gaussian integers ↔ $\mathbf{F}_q[\sqrt{-T}]$

$\boxed{\text{Advantage of } \mathbf{F}_q[T] \text{ : Can take } q\to\infty}$

Analogue of Gaussian integers

$\mathbf{F}_q[\sqrt{-T}]$ Euclidean domain, equipped with Galois conjugation $\sigma(f)(S) := f(-S)$
and norm: $\mathrm{Norm}(f) := f \cdot \sigma(f) \in \mathbf{F}_q\,[T]$

analogue of the unit circle $S^1 = \{z \in \mathbf{C} : \bar{z}\, z = 1\}$

$$\mathbb{S}^1 := \{\ f \in \mathbb{F}_q[[\sqrt{-T}\,]] : f(0) = 1,\ \mathrm{Norm}(\ f\ )=1\,\}$$

Direction of Gaussian polynomial

$$U(f) := \sqrt{f / \sigma(f)} \in \mathbb{S}^1 \quad \Longleftrightarrow \quad \alpha / \bar{\alpha} = e^{2\sqrt{-1}\,\theta} \in S^1, \qquad \alpha = |\alpha|\, e^{\sqrt{-1}\,\theta} \in \mathbb{C}$$

Sectors/arcs on the unit circle

$$\mathrm{Sect}(u;k) := \{\, v \in \mathbb{S}^1 : \|u - v\| \le \frac{1}{q^k}\,\}$$

$$\|u - v\| \le \frac{1}{q^k} \quad \Leftrightarrow \quad u = v \mod (\sqrt{-T}\,)^k$$

K:=Number of distinct sectors Sect(u;k) $\qquad K = q^{\kappa}, \qquad \kappa = \lfloor k / 2 \rfloor$

Sums of two squares in $\mathbf{F}_q[T]$

A monic irreducible $P(T) \in \mathbf{F}_q[T]$, coprime to T, is of the form $P(T) = A(T)^2 + TB(T)^2$ if and only if $P(0)$ is a square in $\mathbf{F}_q$

Equivalently,

$$P(T) = (A(-T) + \sqrt{-T}\,B(-T)) \cdot (A(-T) - \sqrt{-T}\,B(-T))$$

Counting Gaussian prime polynomials in sectors

$$\mathcal{N}_{k,\nu}(u) := \#\{\, P \;\text{ prime},\; \deg P = \nu,\; U(P) \in \mathrm{Sect}(u,k) \,\}$$

mean value $$\frac{1}{K}\sum_u \mathcal{N}_{k,\nu}(u) = \frac{1}{K}\#\{P \text{ prime} : \deg P = \nu\} = \frac{N}{K}$$

$$N := \#\{P \text{ prime} : \deg P = \nu\} \sim \frac{q^\nu}{\nu}$$

Variance in polynomial sectors

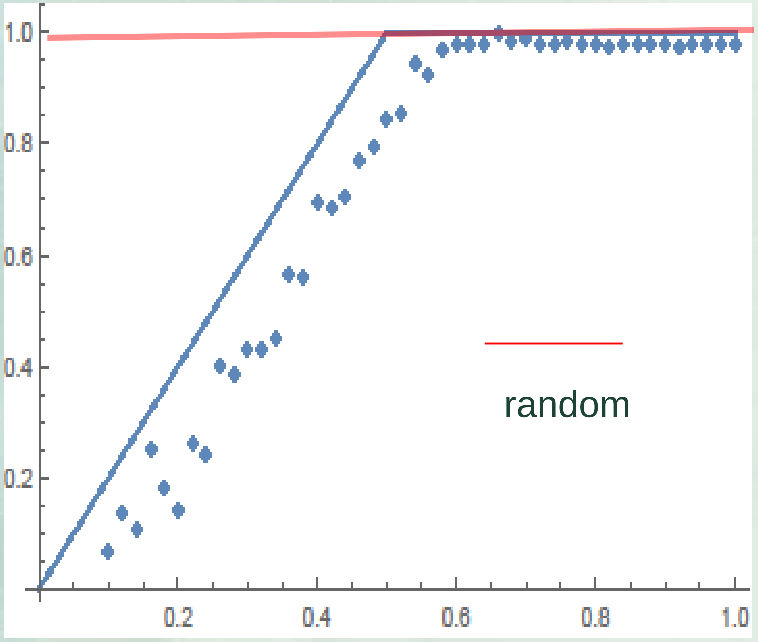
Theorem : As $q \to \infty$, the number variance is

$$\frac{\mathrm{Var}(\mathcal{N}_{\kappa,\nu})}{N/K} \sim \begin{cases} 2\dfrac{\log_q K}{\log_q N} - \dfrac{2}{\log_q N}, & \log_q K \le \dfrac{1}{2}\log_q N \\ 1 + \dfrac{\eta(\log_q N) - 1}{\log_q N}, & \log_q K > \dfrac{1}{2}\log_q N\,. \end{cases}$$

$$\eta(m) = \begin{cases} 1, & m \quad \text{even} \\ 0, & m \quad \text{odd} \end{cases}$$

This matches our conjecture over the integers:

$$\frac{\mathrm{Var}(\mathcal{N}_{K,N})}{N/K} \sim \min\left(2\frac{\log K}{\log N}, 1\right)$$



Picking out directions in sectors

Main tool – "super-even" Dirichlet characters modulo $S^{2\kappa}, S = \sqrt{-T}$

Definition: A Dirichlet character modulo $S^{2\kappa}$ is a homomorphism

$$\chi : \left(\mathbb{F}_q[S]/(S^{2\kappa})\right)^\times \to \mathbb{C}^\times$$

It is "even" if it is trivial on the scalars $\mathbf{F}_q^*$

It is "super even" if in addition it is trivial on the subgroup of even polynomials {f(S)=f(-S) modulo $S^{2\kappa}$}

There are exactly $K = q^\kappa$ super-even characters modulo $S^{2\kappa}$

Key fact: For a polynomial $f = A^2 + TB^2$, the direction $U(f) := \frac{A+\sqrt{-T}B}{A-\sqrt{-T}B} \in \mathbb{S}^1$ lies in the sector $\mathrm{Sect}(u, k)$ if and only if

$$\chi(f) = \chi(u), \qquad \forall \quad \text{super-even} \quad \chi \bmod S^{2\kappa}$$

The L-function for a super-even character

The L-function associated to χ : for Re(s)>1

$$L(s,\chi) := \sum_{f \text{ monic}} \frac{\chi(f)}{\|f\|^s} = \prod_{P \text{ prime}} \left(1 - \frac{\chi(P)}{\|P\|^s}\right)^{-1}$$

Norm of a polynomial: $\|f\| := \#\mathbf{F}_q[S]/(f) = q^{\deg(f)}$ (analogy: for $0 \neq n \in \mathbf{Z}$, $|n| = \#\mathbf{Z}/n\mathbf{Z}$)

If χ is nontrivial ("primitive") character modulo $T^{2\kappa}$ then

- $L(s,\chi)$ is a polynomial in q^{-s} of degree $2\kappa - 1$
- If χ is "even" then there is a trivial zero at s=0
- RH (Weil, 1940's): All non-trivial zeros lie on Re(s)=1/2

$$\Theta_\chi \approx \begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{i\theta_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{i\theta_N} \end{pmatrix}$$

$$L(s,\chi) = (1 - q^{-s}) \cdot \det(I - q^{1/2-s}\,\Theta_\chi)$$

$\Theta(\chi)$ = unitary mxm matrix, m= 2κ-2, called the **"unitarized Frobenius matrix"**

The variance via super-even characters

Theorem: as $q \to \infty$

$$\mathrm{Var}(\mathcal{N}_{k,\nu}) \sim \frac{N}{K} \times \frac{1}{\nu} \times \underset{\chi \text{ super-even mod } S^{2\kappa}}{\mathrm{Average}} \left\{ \left|\mathrm{trace}(\Theta_\chi^\nu)\right|^2 \right\}$$

N.M. Katz (2016): As $q \to \infty$, the unitarized Frobenius classes $\{\Theta_\chi : \chi$ super even mod $S^{2\kappa}\}$ become uniformly distributed in the unitary symplectic group USp(2κ-2)

$$\Rightarrow \quad \lim_{q\to\infty} \underset{\chi \text{ super even mod } S^{2\kappa}}{\mathrm{Average}} \left\{ \left|\mathrm{trace}(\Theta_\chi^\nu)\right|^2 \right\} = \int_{\mathrm{USp}(2\kappa-2)} \left|\mathrm{trace}(U^\nu)\right|^2 dU = \begin{cases} 2\kappa-2, & 2\kappa-2<\nu \\ \nu-1+\eta(\nu), & 1\le\nu\le\kappa-1 \end{cases}$$

$$\Rightarrow \quad \lim_{q\to\infty} \frac{\mathrm{Var}(\mathcal{N}_{k,\nu})}{N/K} \sim \begin{cases} 2\dfrac{\log_q K}{\log_q N}+\ldots, & \log_q K < \dfrac{1}{2}\log_q N \\ 1+\ldots, & \dfrac{1}{2}\log_q N < \log_q K \end{cases}$$

Summary

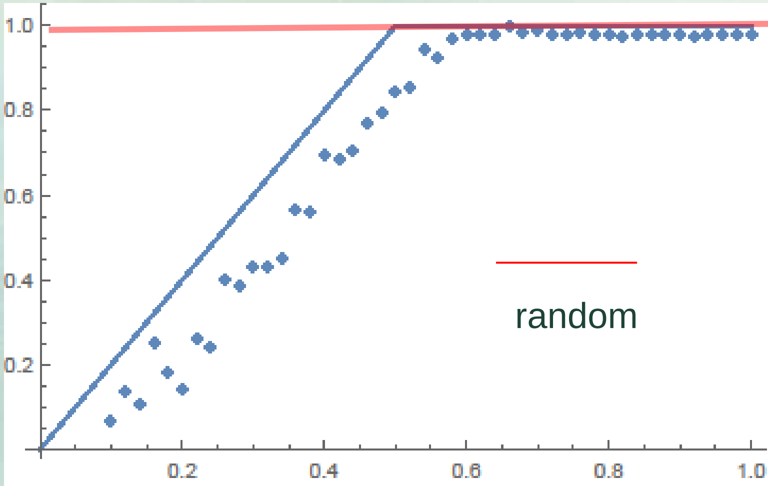
The angles associated to representations of primes as $p = a^2 + b^2$ exhibit randomness on global scale, but deviations on shorter scales.

In particular we predict that the number variance in short arcs exhibits:

- Poissonian statistics for very short arcs,

- Random Matrix Theory statistics for medium-sized arcs

We develop a function field analogue where we **<u>prove</u>** the corresponding statements in the large finite field limit



$$\frac{\mathrm{Var}(\mathcal{N}_{K,N})}{N/K} \quad \text{vs} \quad \log K / \log \mathrm{N}$$