THE DISTRIBUTION OF ZETA-ZEROS AND THE REMAINDER TERM OF THE PRIME NUMBER THEOREM

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1. Classical results

Notation: $\psi(x) = \sum'_{n \le x} \Lambda(n)$, $\Lambda(n) = \begin{cases} \log p & \text{if } \ n = p^m, \\ 0 & \text{otherwise.} \end{cases}$

$\varrho = \beta + i\gamma = 1 - \delta + i\gamma$: non-trivial zeros of $\zeta(s)$, $s = \sigma + it$, $\theta = \sup \mathrm{Re}\, \varrho$, $\varepsilon > 0$ arbitrarily small, fixed constant.

Riemann–Von Mangoldt prime number formula (1895):

$$\Delta(x) := \psi(x) - x = -\sum \frac{x^\varrho}{\varrho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2})$$

Oscillation "caused" by a single zero $\varrho = \varrho_0$ is large if

(i) β_0 is large (in particular near to 1)

(ii) $|\gamma_0|$ is small (compared to x)

Phragmén (1891) $\quad \Delta(x) = \Omega(x^{\beta_0 - \varepsilon})$ for any $\varepsilon > 0$

Von Koch (1903) $\quad \Delta(x) = O\big(\sqrt{x}\log^2 x\big)$ on RH

Littlewood (1914) $\quad \Delta(x) = \Omega\big(\sqrt{x}\log\log\log x\big)$

Koch's method yields $\quad \Delta(x) = O(x^\theta \log^2 x)$

2. Semi-classical results and consequences of classical results

A) Lower estimates of

$$(2.1) \qquad S(x) = \max_{u \le x} |\Delta(u)| \quad \text{and} \quad D(x) = \frac{1}{x} \int_0^x |\Delta(u)| du$$

in terms of a single zero ϱ_0 by Turán's power sum method

Turán (1950): If $x > c(\varrho_0)$ then

$$(2.2) \qquad S(x) > x^{\beta_0} \exp\bigl(-\log x \log_3 x / \log_2 x\bigr).$$

Knapowski (1959): If $x > c(\varrho_0)$ then

$$(2.3) \qquad S(x) > x^{\beta_0} \exp\Bigl(-\log x \,/\, \sqrt{\log_2 x}\Bigr).$$

The corresponding consequence for the oscillation of $\Delta(x)$ is

$$\text{(2.4)} \qquad \left|\Delta(x_n)\right| > (C-\varepsilon)\frac{x_n^{\beta_0}}{|\varrho_0|} \quad \text{for a suitable } x_n \to \infty$$

with $C = 1$ (Pintz 1980) and later $C = \pi/2$ (S. G. Révész 1988).

B) The case $\theta < 1$ is simple (e.g. $\theta = 1/2 \iff \mathrm{RH}$)

Namely,

$$\theta' := \inf\left\{\vartheta;\ \Delta(x) = O(x^{\vartheta})\right\} = \theta$$

follows from Phragmén's theorem and $\Delta(x) = O(x^{\theta}\log^2 x)$.

However, for $\theta = 1$ this implies only $\Delta(x) = \Omega(x^{1-\varepsilon})$ and the trivial $\Delta(x) = O(x^{1+\varepsilon})$.

C) Size of $|\Delta(x)|$ in dependence of all zeros ($\theta = 1$)

Theorem (Ingham, 1932)

Suppose $\zeta(s) \neq 0$ *for* $\sigma > 1 - \eta(t)$ *where*

$$(2.5) \qquad \eta(t) \in C^1[1,\infty), \quad \eta'(t) \leq 0, \quad \frac{1}{\eta(t)} = 0(\log t),$$

$$(2.6) \qquad \eta'(t) \to 0 \quad \text{as} \quad t \to \infty,$$

$$(2.7) \qquad \omega_\eta(x) := \inf_{t \geq 1} \bigl(\eta(t) \log x + \log t\bigr).$$

Then

$$(2.8) \qquad \Delta(x) \ll x e^{-(1/2-\varepsilon)\omega_\eta(x)}.$$

Most important special case of Ingham's theorem. If

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{c_1}{\log^{\alpha} t}, \quad t > t_0 \tag{2.9}$$

then

$$\Delta(x) \ll x \exp\left(-c_2(\alpha) \log^{1/(1+\alpha)} x\right). \tag{2.10}$$

Turán (1950): The conversion (2.10) $\Longrightarrow$ (2.9) is true.

Stás (1960/61): Worked out this more explicitly but the zero-free region obtained after the conversion was cca. 1/80 times smaller.

Pintz (1980): for rather general domains one can obtain a conversion up to a factor $(1 + o_x(1))$ in the domain and in the exponent in (2.10), respectively.

3. Problems

(i) Are the conditions for $\eta(t)$ necessary?

(ii) Is the estimate (2.8) optimal? (1/2 in the exponent)

(iii) If we have an upper estimate for $|\Delta(x)|$ for all large x can we infer a zero-free region?

(iv) If the answer for (iii) is yes, can we obtain back essentially the same zero-free region?

(v) If we only know an upper estimate for $D(x)$ instead of $S(x)$ (cf. (iii)) do we obtain a zero-free region?

(vi) Is it necessary to use a zero-free region?

Answers

(i) The conditions for $\eta(t)$ can be significantly eased (Pintz 1980).

(ii) $1/2$ can be replaced by 1 in (2.8) (Pintz, 1980).

(iii) YES, one can obtain a zero-free region, which is, however smaller by a factor $1/80$ (Turán, 1950, Stàs, 1961).

(iv) YES (Pintz, 1980).

4. GOALS

I. To obtain a function $\omega(x)$ which depends in a (relatively) simple way on the distribution of ALL zeta-zeros (cf. (vi)) and which describes the behaviour of $S(x)$ and $D(x)$.

II. If we obtain such a function, then the further problem might be answered: are $S(x)$ and $D(x)$ of similar size?

Key point in answers of I and II: Let

$$W(x) := \max_{\varrho;\gamma>0} \frac{x^\beta}{\gamma} : \text{ the contribution of the dominant zero.}$$

5. Results and methods

Notation: $\omega(x) = \log \dfrac{x}{W(x)} = \min\limits_{\varrho;\gamma>0}(\delta \log x + \log \gamma)$

Theorem 1. $\log \dfrac{x}{S(x)} \sim \log \dfrac{x}{D(x)} \sim \omega(x)$ *as* $x \to \infty$.

This includes

Theorem 2. $\Delta(x) \ll xe^{-(1-\varepsilon)\omega(x)}$.

Theorem 3. $S(x) > D(x) \gg xe^{-(1+\varepsilon)\omega(x)}$.

Corollary

$\Delta(x) = \Omega(xe^{-(1+\varepsilon)\omega(x)})$.

Main tools of the proof

(i) A density theorem for the zeta-zeros (first such theorem, due to Carlsson (1920) is sufficient).

(ii) Any zero-free region of type (e.g. Chudakov 1938: $\alpha = 3/4 + \varepsilon$)

$$\sigma > 1 - \frac{c}{\log^{\alpha}(|t| + 2)} \quad \text{with some} \quad \alpha < 1.$$

(The minimum is near to $\sigma > 1 - \dfrac{C \log\log t}{\log t}$, $t > t_0$, due to Littlewood, 1922.)

(iii) Turán's power sum method.

Sketch of proof of Theorem 2

$\Delta(x) \leq xe^{-(1-\varepsilon)\omega(x)}$ for $x > x_0(\varepsilon)$.

We restrict ourselves to the more difficult case $\theta = 1$. Let $\varepsilon' = \varepsilon/6$.

$$\sum\nolimits_1 = \sum_{\substack{\beta \leq 1-\varepsilon' \\ |\gamma| \leq x}} \frac{x^\beta}{|\gamma|} \leq cx^{1-\varepsilon'} \log^2 x \leq \frac{1}{2} xe^{-\omega(x)}.$$

We will use Carlson's density theorem (1920):

$$N(1-\varepsilon', T) := \sum_{\beta \geq 1-\varepsilon', |\gamma| \leq T} 1 \ll_\varepsilon T^{4\varepsilon'}.$$

This implies

$$\sum_{\substack{\beta>1-\varepsilon' \\ e^n<\gamma\le e^{n+1}}} \frac{x^\beta}{\gamma} \ll_\varepsilon \; e^{4n\varepsilon'} \max_{e^n<\gamma\le e^{n+1}} \frac{x}{e^{\delta \log x + \log \gamma}}$$

$$\le e^{-n\varepsilon'} \max_{e^n<\gamma\le e^{n+1}} \frac{x}{e^{(\delta \log x + \log \gamma)(1-5\varepsilon')}}.$$

Summing over all n we obtain Theorem 2.

Sketch of proof of Theorem 3

$$D(x) \geq xe^{-(1+\varepsilon)\omega(x)} \quad \text{for} \quad x > x_0(\varepsilon).$$

This will follow in the crucial case $\theta = 1$ from the following

Theorem 4. *Let* $0 < \varepsilon < a$, $\zeta(\varrho_0) = \zeta(1 - \delta_0 + i\gamma_0) = 0$ $(\gamma_0 > 0)$, $\delta_0 < \varepsilon^{10}$. *For* $x > \gamma_0^{1/\varepsilon^{10}}$ *we have*

$$D(x) \geq \frac{1}{x} \int\limits_{x^{1-\varepsilon\delta_0}\gamma_0^{-\varepsilon}}^{x} |\Delta(u)| du \geq \frac{1}{(x^{\delta_0}\gamma_0)^{\varepsilon}} \cdot \frac{x^{\beta_0}}{\gamma_0}.$$

Notation: $L = \log x$, $\omega = \delta_0 L + \log \gamma_0$, $\alpha = \frac{\omega}{L} < 2\varepsilon^{10}$. μ will be a real number, to be chosen later with

$$\mu \in [L - 6\varepsilon_1\omega, L - 5\varepsilon_1\omega] = \big[L(1-6\varepsilon_1\alpha), L(1-5\varepsilon_1\alpha)\big],$$

$$M = 5\varepsilon_1\alpha\mu \in [4\varepsilon_1\omega, 5\varepsilon_1\omega], \quad k = 5\varepsilon_1^2\alpha\mu = \varepsilon_1 M.$$

This implies $\omega \geq c\sqrt{L} \geq c\varepsilon_1^{-5}$, $k \geq c\varepsilon_1^{-3}$.

We will use two identities:

$$\int_1^\infty \Delta(u)\frac{d}{du}(u^{-s}du) = \frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} =: H(s) \quad (\sigma > 1),$$

$$\frac{1}{2\pi i}\int_{(2)} e^{As^2+Bs}ds = \frac{1}{2(\pi A)^{1/2}}\exp\left(-\frac{B^2}{4A}\right).$$

These imply our basic identity

$$U = U(\mu) := \frac{1}{2\pi i} \int\limits_{(2)} H(s+\varrho_0) e^{ks^2+\mu s} ds$$

$$= \frac{1}{2\sqrt{\pi k}} \int\limits_{1}^{\infty} \frac{\Delta(u)}{u^{1+\varrho_0}} \exp\left(-\frac{(\mu - \log u)^2}{4k}\right) \left(-\varrho_0 + \frac{\mu - \log u}{2k}\right) du$$

$$U := U_1 + U_2 + U_3 := \int\limits_{1}^{e^{\mu-M}} + \int\limits_{e^{\mu-M}}^{e^{\mu+M}} + \int\limits_{e^{\mu+M}}^{\infty} .$$

It is easy to show that

(i) $U_1, U_3 \le e^{-\omega/5}$ (negligible) because $\dfrac{M^2}{4k} = \dfrac{M}{4\varepsilon_1} > \omega$;

(ii) $|U_2| \le \dfrac{\gamma_0}{\varepsilon_1}\left(\dfrac{e^{11\varepsilon_1\omega}}{x}\right)^{1+\beta_0} \displaystyle\int\limits_{xe^{-11\varepsilon_1\omega}}^{x} |\Delta(u)|du$, because

$\mu - M \ge 11\varepsilon_1\omega$,
$\mu + M \le L = \log x$;

(iii) $U_2 = \displaystyle\sum_{\varrho} e^{k(\varrho-\varrho_0)^2+\mu(\varrho-\varrho_0)} + O(e^{-\omega})$;

(iv) $U_2 = E_2(\mu) + O(e^{-\omega}) :=$
$$\sum_{\substack{|\gamma-\gamma_0|\le\varepsilon_1^{-1}\\ \beta\ge 1-2\alpha}} \exp\left\{5\varepsilon_1^2\alpha\mu(\varrho-\varrho_0)^2 + (\varrho-\varrho_0)\mu\right\} + O(e^{-\omega}).$$

Crucial result to use:

Turán's second main theorem (continuous form):

If $a, d > 0$, $\alpha_1, \dots, \alpha_n \in C$, Re $\alpha_1 = 0$. Then

$$\max_{a \le t \le a+d} \left| \sum_{i=1}^{n} e^{\alpha_i t} \right| \ge \left(\frac{1}{8e\left(\frac{a+d}{a}\right)} \right)^n .$$

The lower estimate depends strongly on n. Using Vinogradov's method to estimate $\zeta(s)$ near the line $\sigma = 1$ we obtain by Jensen's inequality for the number of terms

$$n \le c\varepsilon_1^{-1} \log \frac{1}{\alpha} \alpha^{1/3} \omega.$$

This yields to

$$E_2(\mu) \geq e^{-\varepsilon_1 \omega}, \quad |U(\mu)| \geq e^{-\varepsilon_1 \omega}/2$$

and finally to

$$D(x) \geq \frac{1}{x} \int\limits_{xe^{-\varepsilon\omega}}^{x} |\Delta(u)| du \geq e^{-\varepsilon\omega} \frac{x^{\beta_0}}{\gamma_0}.$$