Large Character Sums

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Introduction

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Theorem (Pólya, Vinogradov 1918)

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Theorem (Paley, 1932)

There exist **infinitely many** $q \ge 1$ such that

$$M(\chi_q) \gg \sqrt{q} \log\log q,$$

where χ_q is **the quadratic character** $\left(\frac{\cdot}{q}\right)$.

Theorem (Granville and Soundararajan, 2007)

Assume GRH. Let χ be a primitive character modulo q. Then

$$M(\chi) \leq \begin{cases} \left(\frac{2e^{\gamma}}{\pi} + o(1)\right)\sqrt{q}\log\log q & \text{if } \chi \textbf{ is odd } (\chi(-1) = -1), \\ & \end{cases}$$

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- Studied the distribution of $M(\chi)$ as χ varies among non-principal characters modulo a large prime q.
- Their results give **strong evidence** for the Granville-Soundararajan conjecture.

Application to the least quadratic non-residue

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- GRH $\Longrightarrow$ $n_p \begin{cases} \ll \log^2 p & \text{(Ankeny, 1950)}, \\ \leq \log^2 p & \text{(L., Li and Soundararajan, 2015)}. \end{cases}$
- Montgomery (1971): Assume GRH. There exist infinitely many p for which $n_p \gg \log p \log\log p$. He also **conjectured** that this is **best possible**.

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- Granville and Soundararajan's Conjecture for $M(\chi) \Longrightarrow$

$$n_p \ll (\log p)^{1.37+\varepsilon}.$$

Even order character sums

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Assume GRH. Let $k \geq 1$. There exist **infinitely many** positive integers q and primitive characters $\chi \pmod{q}$ of **order** $2k$ such that

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Moreover, there exist an **even** character χ of order $2k$, and conductor $q \leq Q$, such that

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Theorem 2 (L. and Mangerel, 2017+)

There are arbitrarily large q and primitive characters $\chi \pmod q$ of order g such that

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$$\sum_{n \le \alpha q} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{1 \le |n| \le q} \frac{\overline{\chi}(n)}{n} \left(1 - e\left(-\alpha n\right)\right) + O\left(\log q\right).$$

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Improving the Pólya-Vinogradov inequality $\Longleftrightarrow$ Obtaining non-trivial information on the exponential sum $\sum_{1 \le |n| \le q} \frac{\chi(n) e(n\alpha)}{n}$, for $\alpha \in [0, 1]$.

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Assume **GRH**. For all $\alpha \in [0, 1]$ we have

$$\sum_{n \leq q} \frac{\chi(n)}{n} e(n\alpha) = \sum_{\substack{n \leq q \\ P(n) \leq (\log q)^{20}}} \frac{\chi(n)}{n} e(n\alpha) + o(1).$$

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Conditionally on GRH, we deduce

$$M(\chi) \ll \sqrt{q} \sum_{\substack{n \leq q \\ P(n) \leq (\log q)^{20}}} \frac{1}{n} \ll \sqrt{q} \exp\left(\sum_{p \leq (\log q)^{20}} \frac{1}{p} \right) \ll \sqrt{q} \log \log q.$$

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If $\alpha \approx \frac{b}{m}$ lies in a major arc, then there is $N = N(\alpha, q, b, m)$ such that

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The case $\alpha = b/m \in \mathbb{Q}$

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$$M(\chi) \ll_{g,\varepsilon} \begin{cases} \sqrt{q}(\log q)^{1-\delta_g/2+\varepsilon} & \text{unconditionally,} \\ \sqrt{q}(\log\log q)^{1-\delta_g/2+\varepsilon} & \text{under GRH.} \end{cases}$$

Goldmakher's thesis

$$\mathcal{M}(f;y,T) := \min_{|t|\le T} \mathbb{D}(f,n^{it};y)^2 = \min_{|t|\le T} \operatorname{Re} \sum_{p\le y} \frac{1-f(p)p^{-it}}{p}.$$

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For all $f \in \mathcal{F}$ and $T \ge 1$, we have

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- We obtain **precise estimates** for $\mathcal{M}(\chi\overline{\psi}; y, T)$, where $\chi \pmod q$ has order g, and $\psi \pmod m$ is an odd primitive character with $m \leq (\log y)^A$.

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- This secondary term is of size $(\log\log y)/k^2$ where k is the order of ψ. It is responsible for the extra saving of $(\log\log q)^{-1/4}$ ($(\log_3 q)^{-1/4}$ on GRH) in Pólya-Vinogradov.

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Let $\alpha \in (0,1)$. Let $\chi \pmod{q}$ be of order g. Let $\psi \pmod{m}$ be odd, with $m \leq (\log y)^{4\alpha/7}$. Then

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Proposition (L. and Mangerel, 2017+)

Assume **GRH**. Let N be large, and $y \le (\log N)/10$. Let $\psi \pmod m$ be odd with $(\log_2 y)^{\varepsilon} \le m \le \log y$. There exists a $\chi \pmod q$ of order g with $q \le N$, such that

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For all $f \in \mathcal{F}$ and $T \geq 1$, we have

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- The logarithmic version of Halász's Theorem is **not optimal**.

Theorem 3 (L. and Mangerel, 2017+)

Let $f \in \mathcal{F}$ and $x \geq 2$. Then, for any real number $0 < T \leq 1$ we have

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Corollary (L. and Mangerel, 2017+)

Let $\alpha \in (0, 1]$. If $f \in \mathcal{F}$ satisfies (1), then f close to n^{it} for some $|t| \ll (\log x)^{-\alpha}$.

Omega results for $M(\chi)$

L. and Mangerel (2017+): There are arbitrarily large q and $\chi \pmod q$ of order g such that $M(\chi) \gg \sqrt{q}\,(\log_2 q)^{1-\delta_g}\,(\log_3 q)^{-\frac{1}{4}}\,(\log_4 q)^{O(1)}$.

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- This method result in a **loss** of a factor of $\log_3 q$.

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- Maximize the short sum by controlling $\chi(p)$ for the **small** primes p.
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- Maximize this truncated Euler product for **many** χ.

Thank you for your attention !