Generalized p-angulations in higher dimension

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Aléa – Marseille – 23/03/2017

1 – Motivation and main ideas

2 – Colored triangulations and edge-colored graphs

3 – Generalized p-angulations

4 – Quadrangulations in 4D

5 – Gluings of octahedra

1 – Motivation and main ideas

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Motivation : quantum gravity

Einstein-Hilbert partition function for Euclidean pure gravity in dimension D

$$Z(\lambda, N) = \int_{\mathcal{M}} D[g] e^{-\int d^D x \sqrt{|g|}(2\Lambda - \frac{1}{2\kappa} R)} = \sum_{\substack{T \\ \text{connected} \\ \text{triangulation}}} \lambda^{n_D} \, N^{n_{D-2} - a n_D}$$

λ^{n_D}: # of D simplices

$N^{n_{D-2} - a n_D}$: # of D-2 simplices

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Allow topology fluctuations -> non-classical

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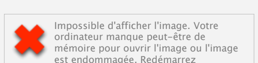
Allow topology fluctuations -> non-classical

- Large N limit (physical limit of small Newton constant) :

  configurations which maximize

- 

  : Continuum limit → quantum space-time

1 – Motivation and main ideas

<u>D=2</u> : continuum limit = Brownian map

Hausdorff dimension 4, homeomorphic to S^2,

Quantum sphere of Liouville quantum gravity (Miller, Sheffield, 2016)

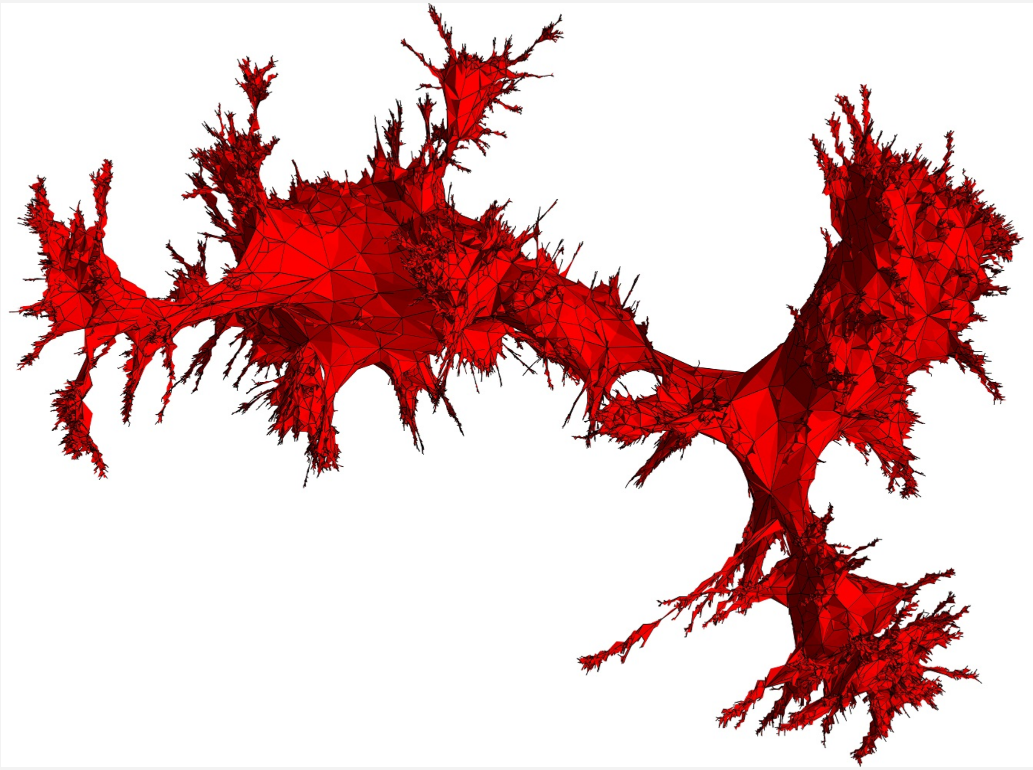


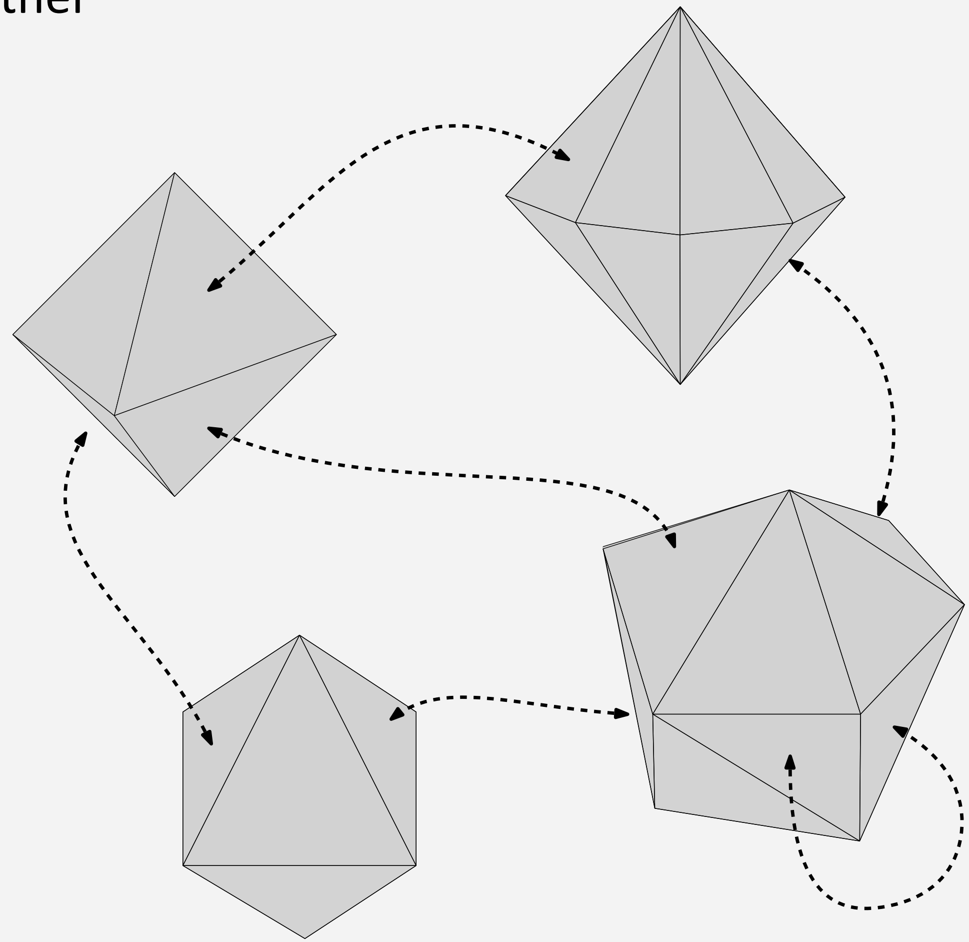
Fig : J. Bettinelli

1 – Motivation and main ideas

D>2 : Basic idea

- Glue building blocks together

"Quanta of space-time"



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with $a \leq \frac{D(D-1)}{4}$ (" < " for interesting cases)

→ *find the coefficient a ? what is their topology ?*

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→ *find the coefficient a ? what is their topology ?*

- Count maximal configurations : generating function has a singularity

→ continuum limit → space-time

→ *critical exponent ? … Hausdorff dimension ? Fractal dimension ? Etc.*

2 – Colored triangulations and edge-colored graphs

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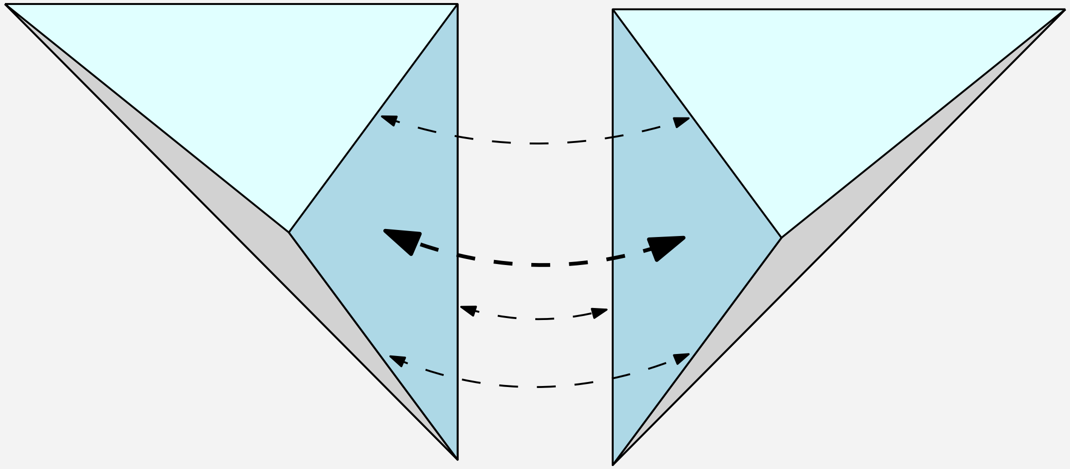
Simplicial pseudo-complexes obtained by gluing D-simplices

Colored faces (D-1 simplices) are glued in a **unique way** :

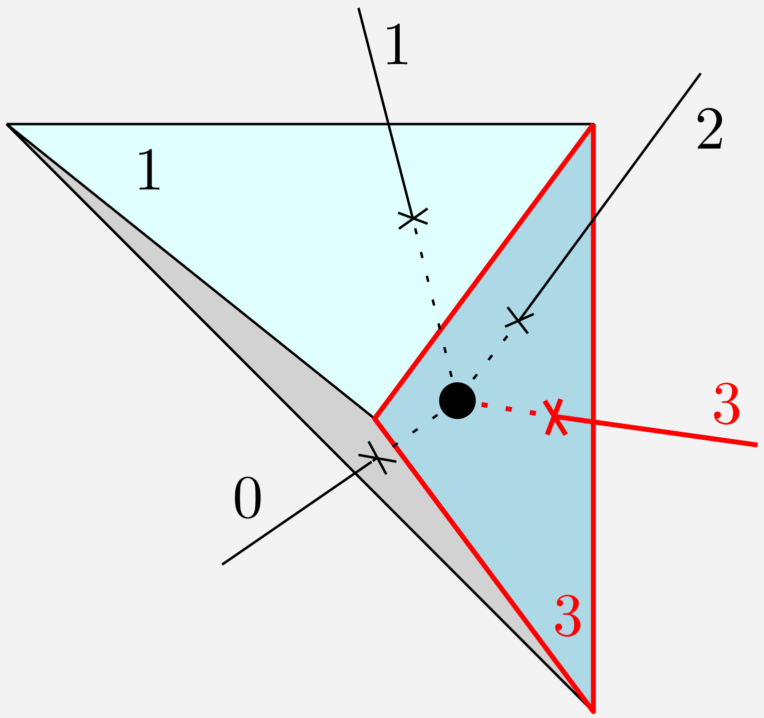
with matching colors on their sub-simplices



2 – Colored triangulations and edge-colored graphs

D-simplices are represented by (D+1)-valent vertices

The colored faces are dual to colored edges



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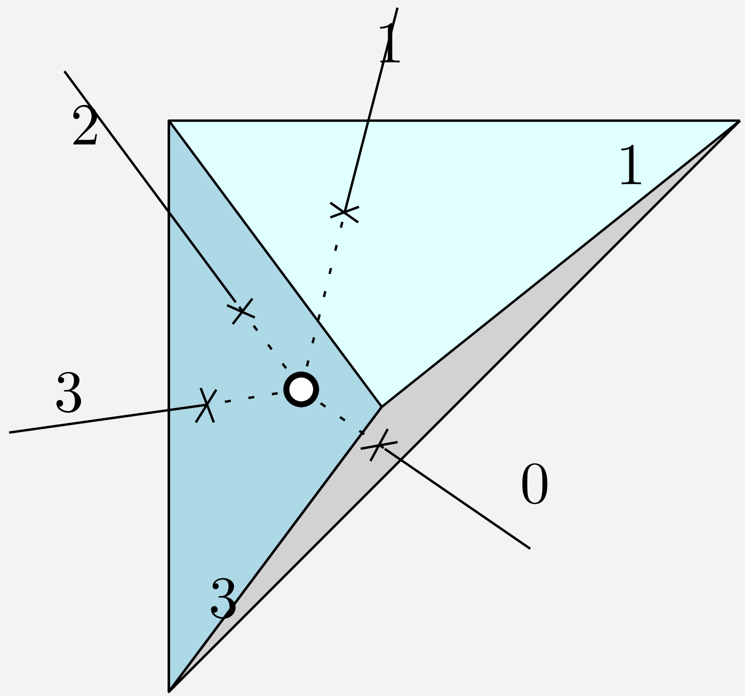
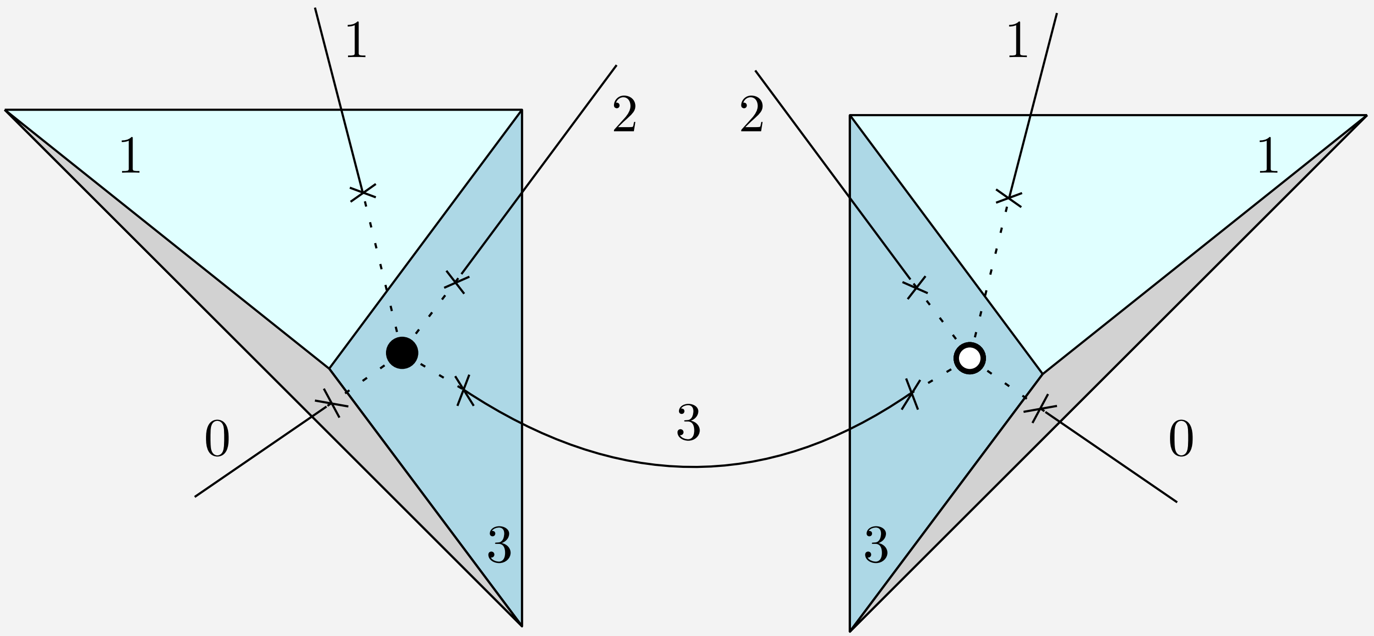
Black vertex / white vertex : opposite ordering of colors around faces



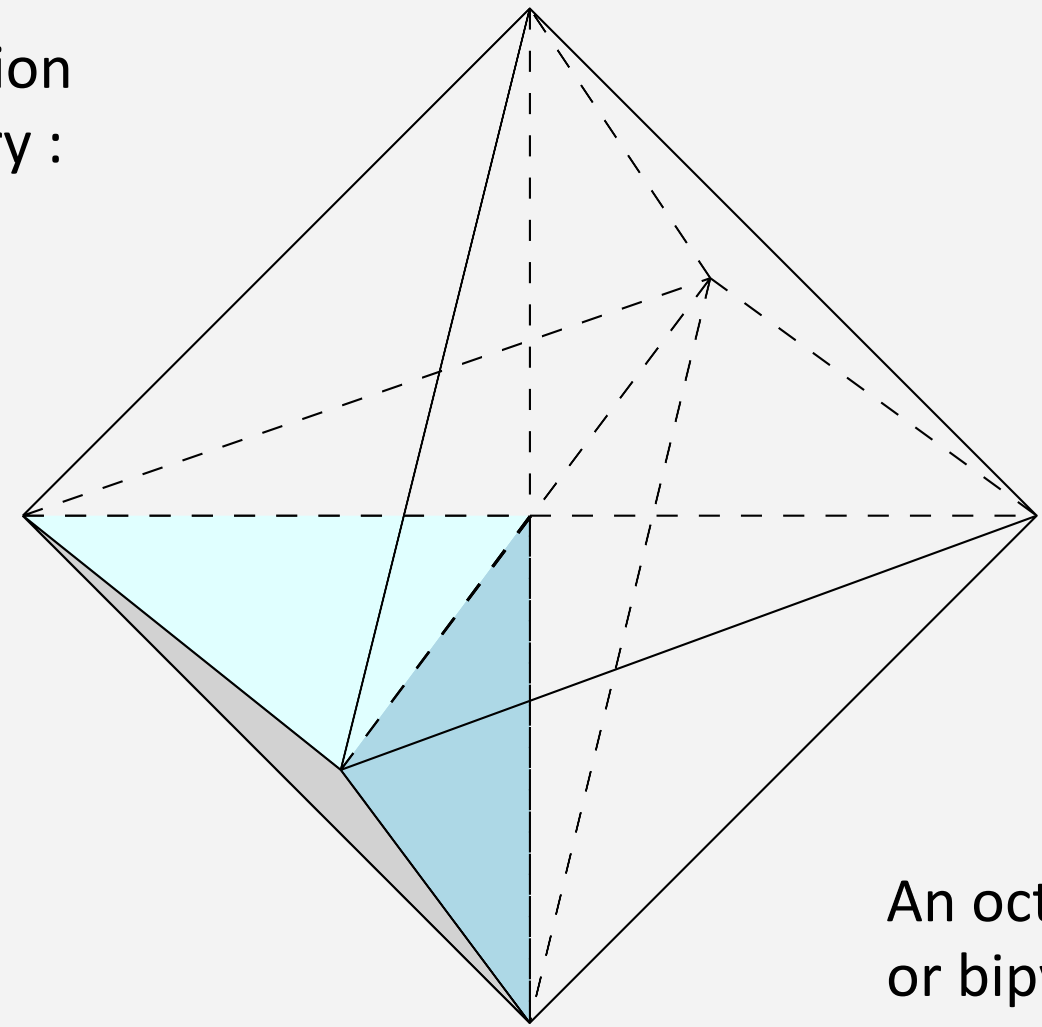
2 – Colored triangulations and edge-colored graphs

A color-*i* edge encodes the gluing of two color-*i* "faces" (D-1 simplices) in the *unique* possible way



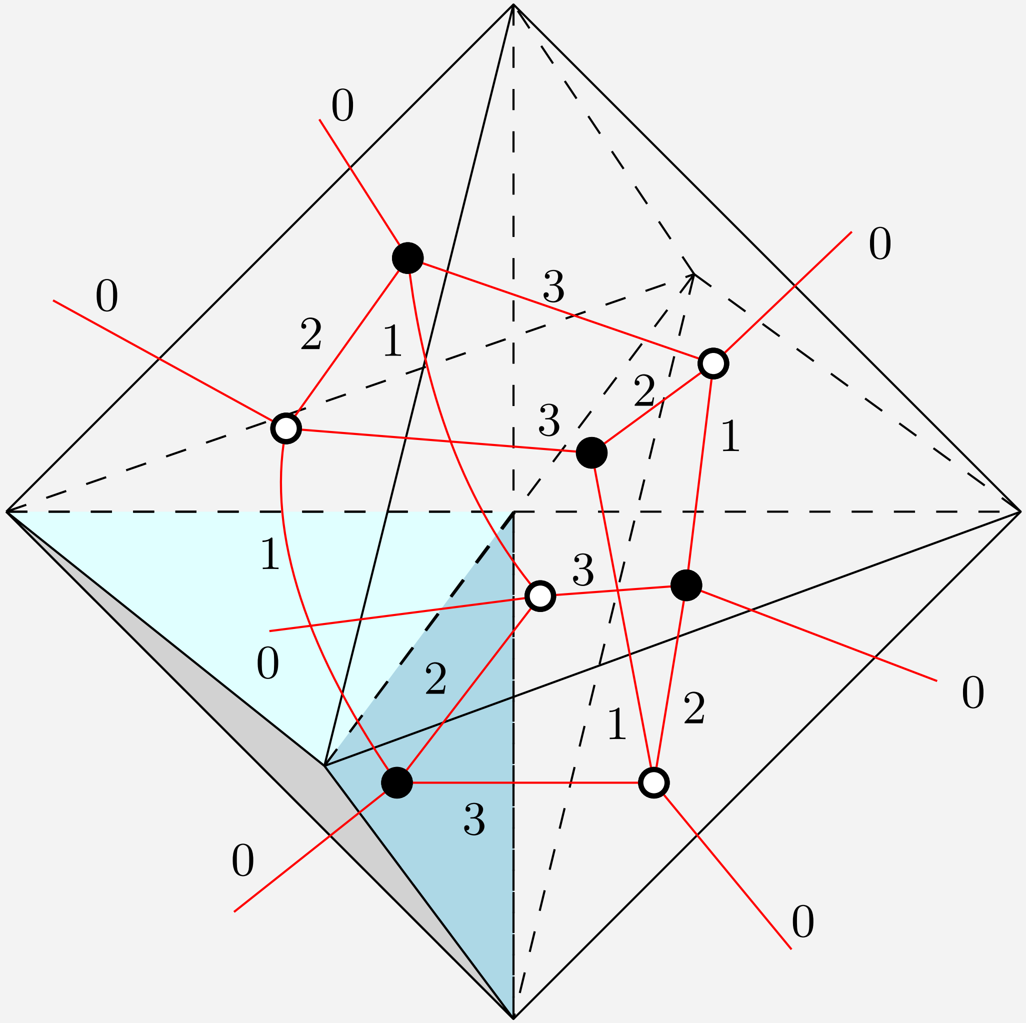
2 – Colored triangulations and edge-colored graphs

3D triangulation
with boundary :



An octahedron,
or bipyramid…

2 – Colored triangulations and edge-colored graphs



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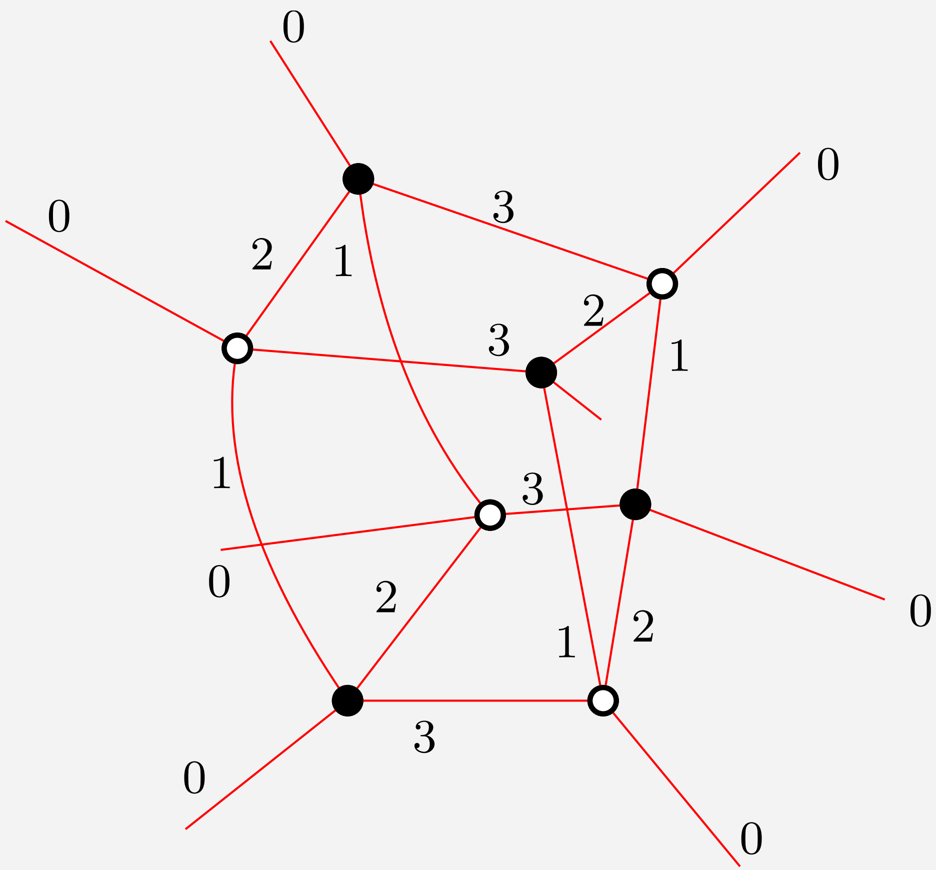

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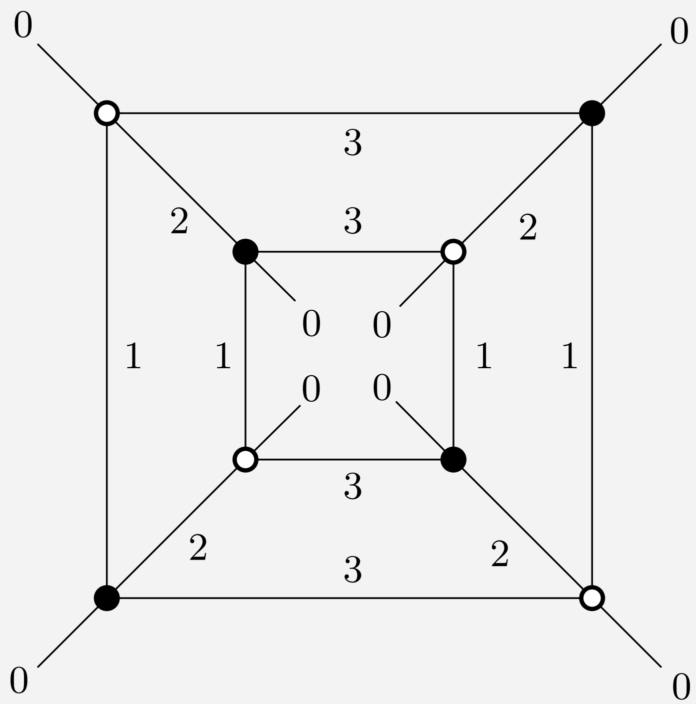

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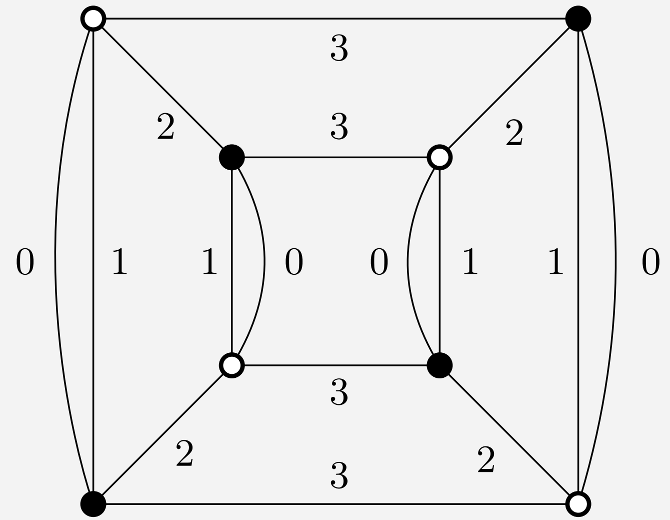


D-dimensional colored triangulation of an orientable pseudo-manifold ⟷ Regular bipartite (D+1)-edge-colored graph

(Pezzana, Ferri, Gagliardi, Casali, Grasselli, Cristofori… '74 until now)

2 – Colored triangulations and edge-colored graphs

Dictionary :

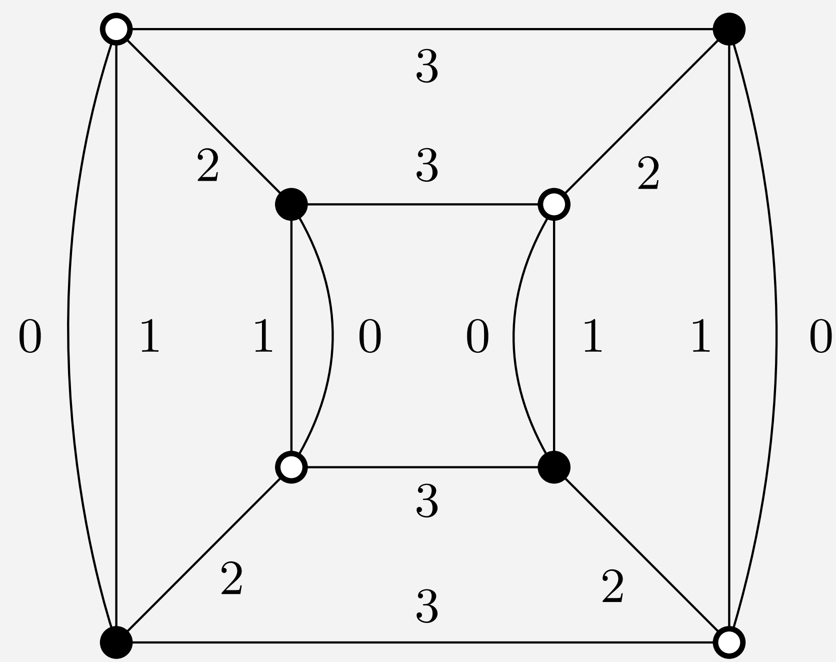
triangulation <-> dual graph

D-simplex <-> vertex

(D-1) simplex <-> edge

(D-2) simplex <-> two-colored cycle

(D-k) simplex <-> sub-graph with k colors only



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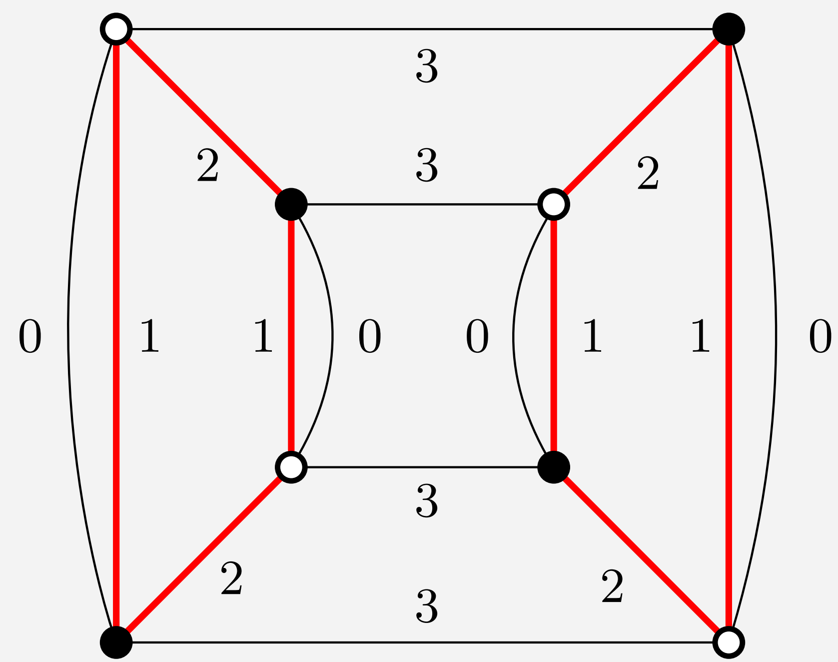

Edges of the triangulation

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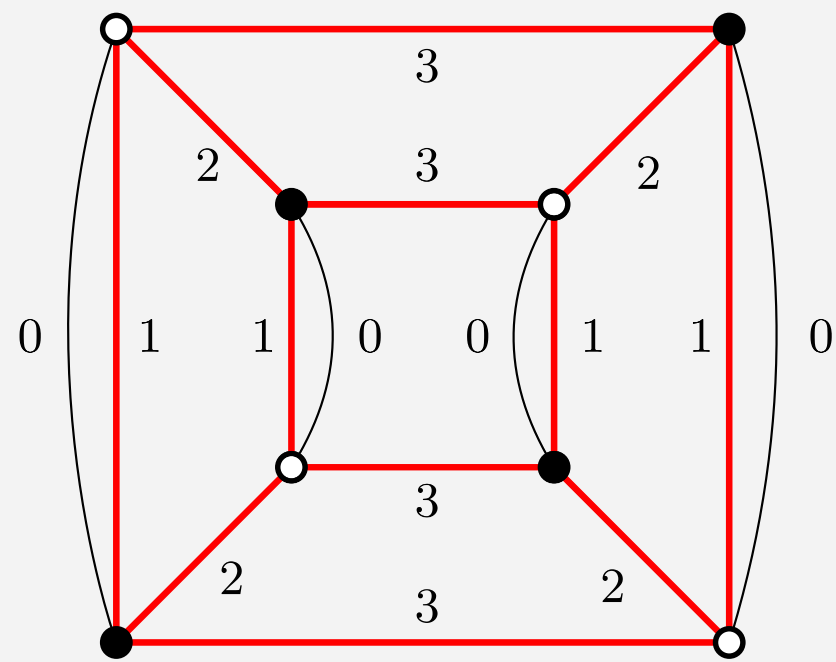

Vertex of the triangulation

2 – Colored triangulations and edge-colored graphs

We are interested in configurations with maximal number of (D-2) simplices at fixed number of D-simplices.

- D=2 : maximal # vertices, fixed # triangles
  → minimize the genus

- D=3 : maximal # edges, fixed # tetrahedra

Dual picture : graphs that maximize the number of two-colored cycles at fixed number of vertices.

→ ***« maximal graphs »***

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Colored triangulations verify $$n_{D-2} \leq D + \frac{D(D-1)}{4} n_D$$

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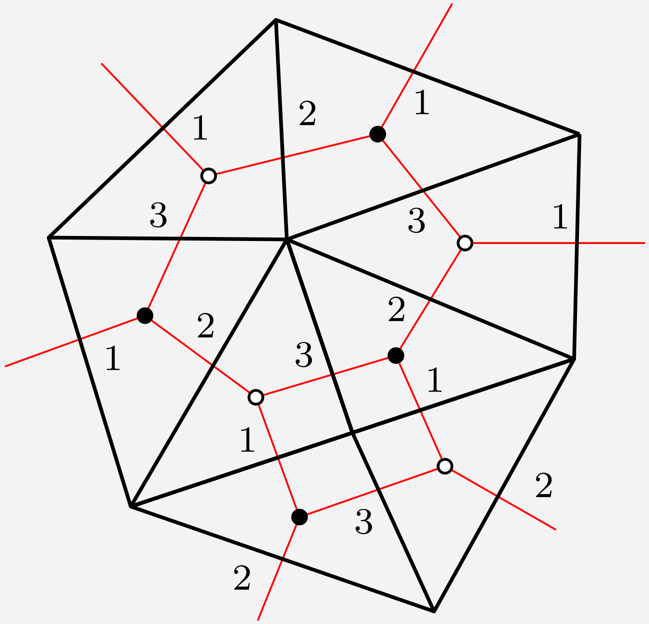
Maximal triangulations : D=2

$$V = 2 + \frac{F}{2} \iff g = 0$$

→ planar triangulations



$$T(n) = \frac{1}{16}\sqrt{\frac{3}{2\pi}} n^{-\frac{5}{2}} \left(\frac{256}{27}\right)^n \propto n^{\gamma-2} \lambda_c^{-n}$$

→ $$\boxed{\gamma = -\frac{1}{2}}$$

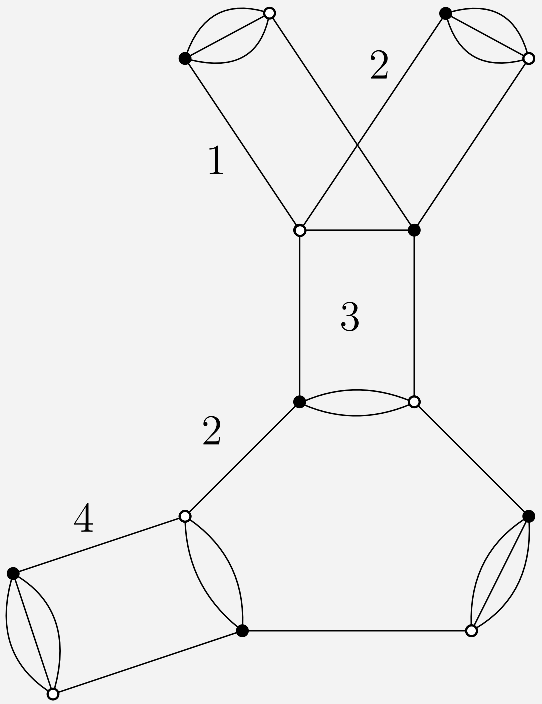
Continuum limit = brownian map

2 – Colored triangulations and edge-colored graphs

Colored triangulations verify $$n_{D-2} \leq D + \frac{D(D-1)}{4} n_D$$

Maximal triangulations : D>2

They are called *melonic graphs*



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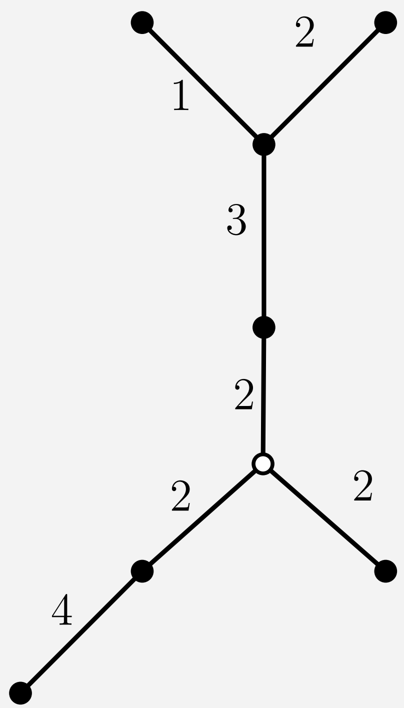
Tree-like structure

→ $$\gamma = \frac{1}{2}$$



Continuum limit = continuous random tree
…not a good space-time candidate…

3 – Generalized p-angulations

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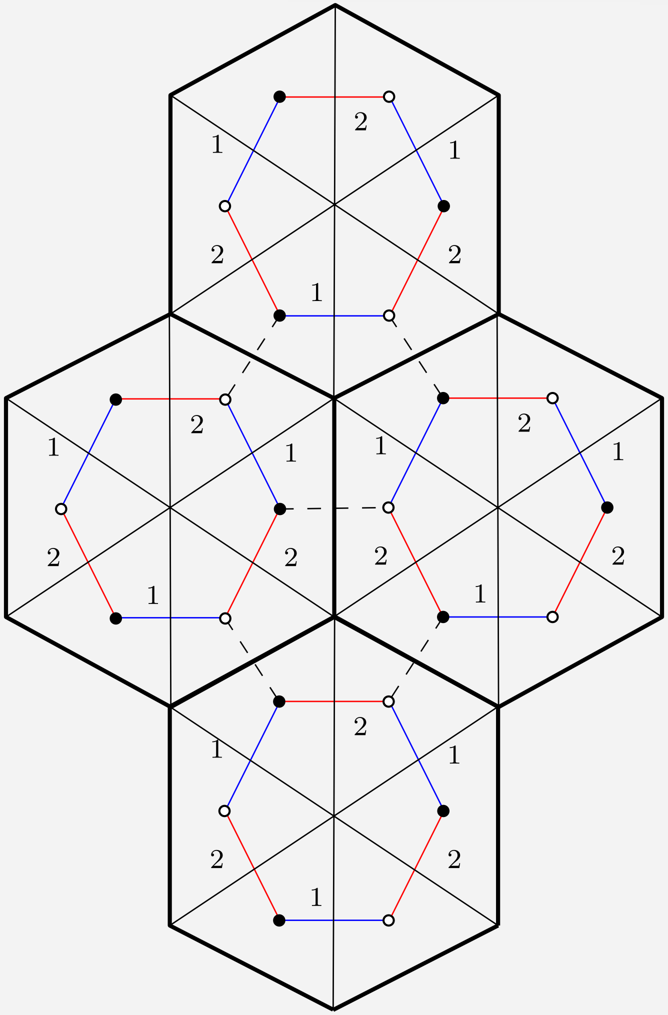
p-angulation in 2D

Maximize the number of vertices at fixed number of p-gons

$$n_{\text{vertices}} \leq 2 + \frac{p-2}{2} n_{p-\text{gons}}$$

→ Selects planar p-angulations, as before for triangulations

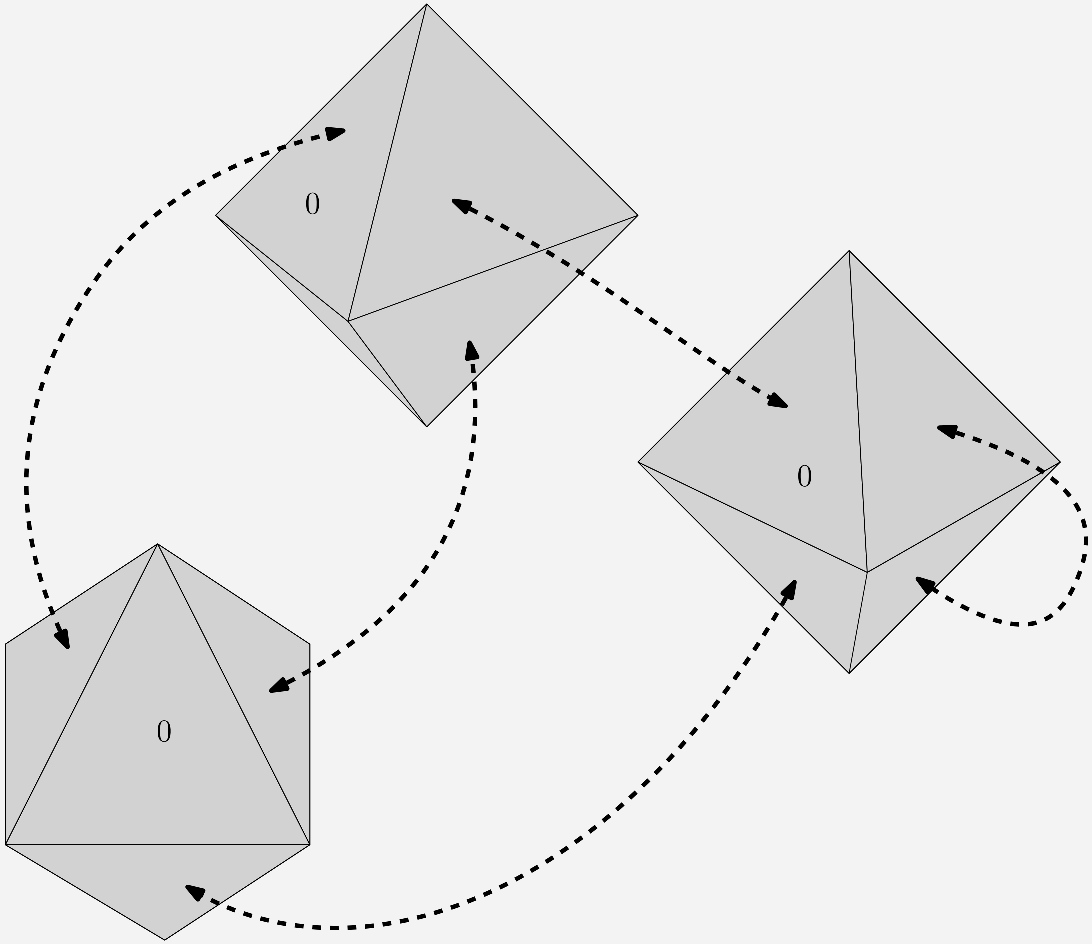
→ Universality (critical exponent, continuum limit…)



hexangulation, locally

3 – Generalized p-angulations

p-angulation in higher dimension

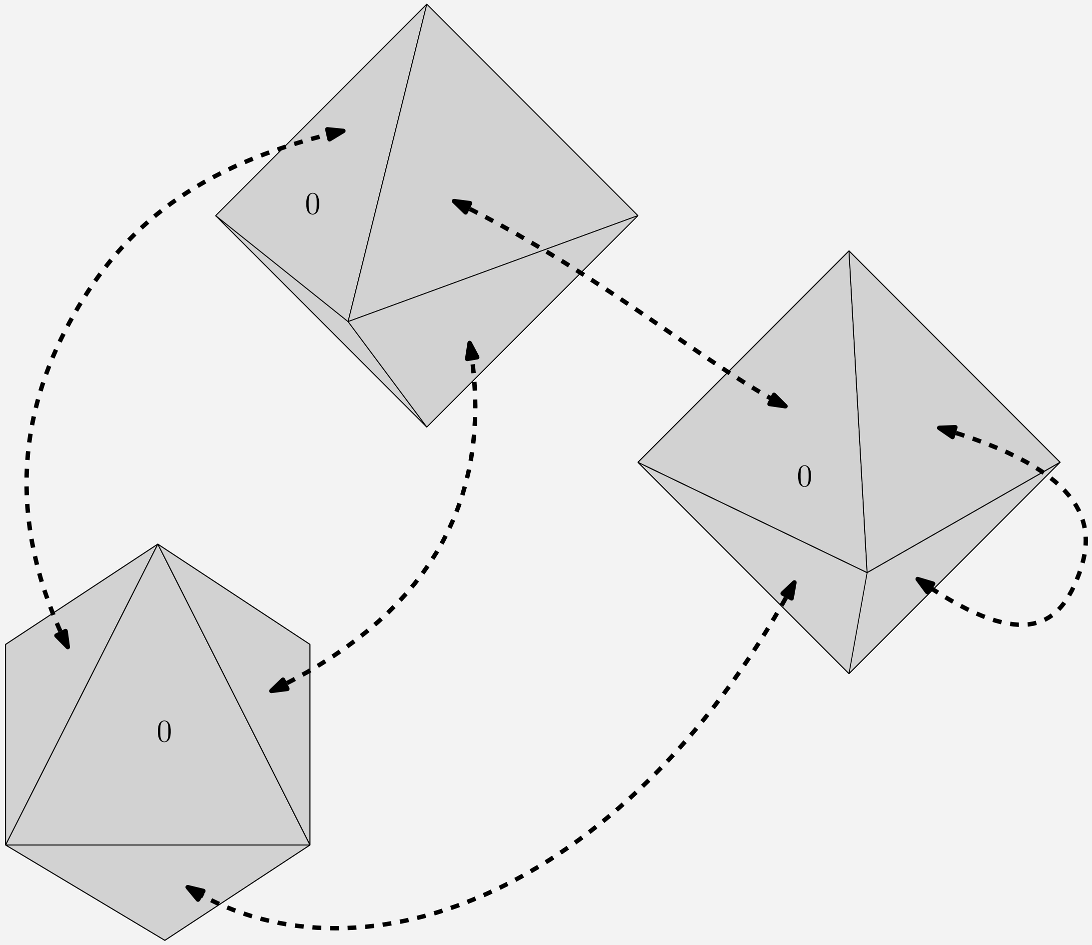


3 – Generalized p-angulations

Gluings of building blocks with *p* external faces of color 0 in dimension D

e.g. :

8-angulation in 3D



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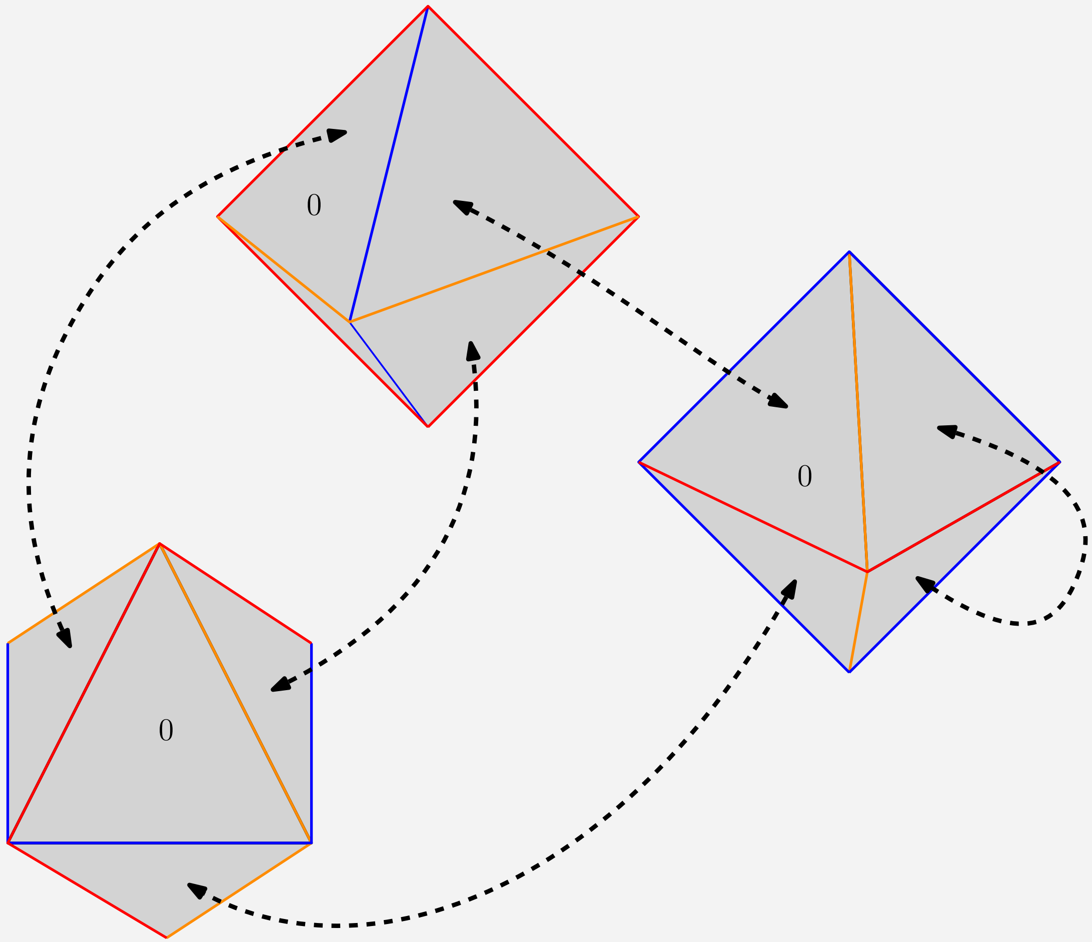
Boundary admits a colored triangulation



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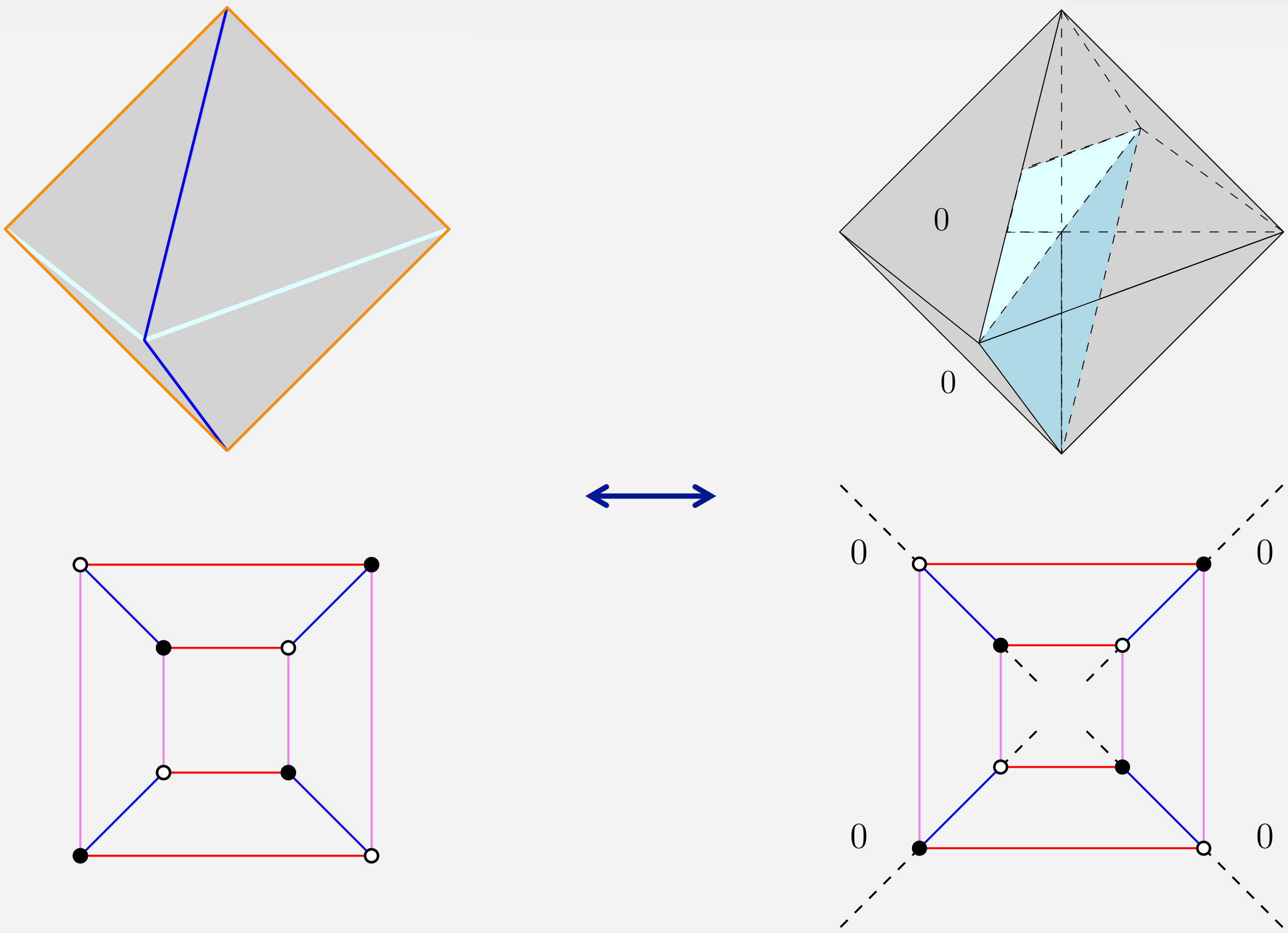
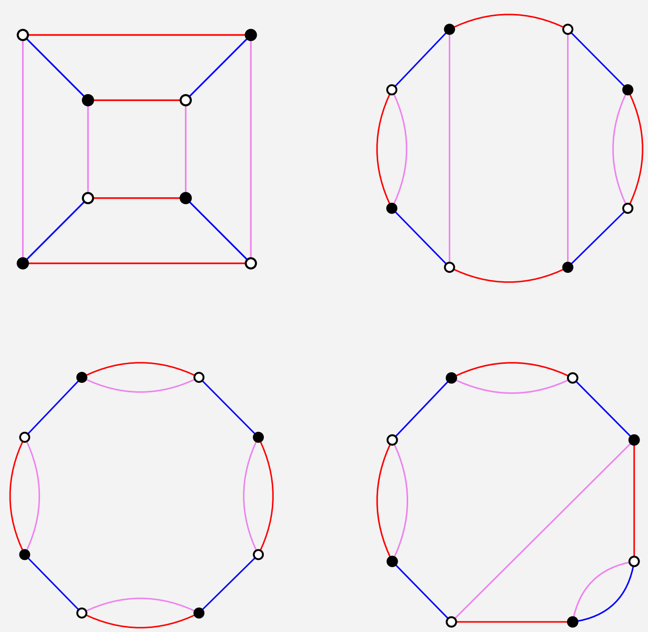


Colored triangulation of the boundary (dim D-1)

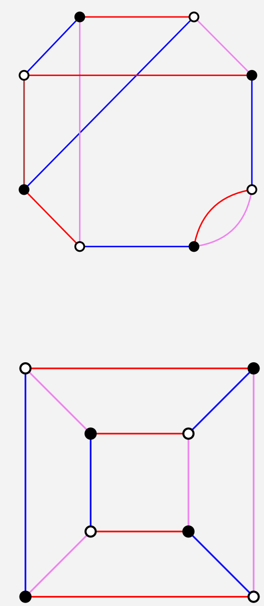
Topological cone with colored facets (dim D)

3 – Generalized p-angulations

Building block (size p, dim D) → Triangulation of its boundary (p vertices, dim D-1)



spherical boundary



toroidal boundary

3 – Generalized p-angulations

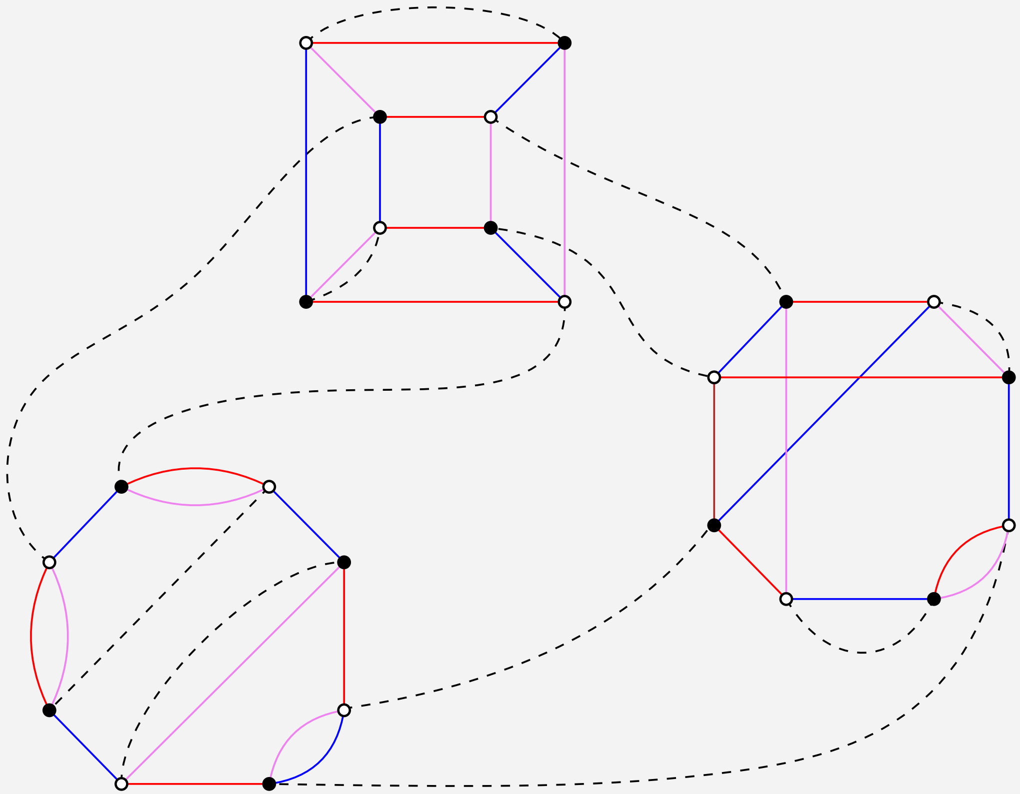
Dual picture

An edge of color 0 (dashed) identifies two faces (D-1 simplices) of color 0

e.g. :

8-angulation in 3D

→ 4-colored graph



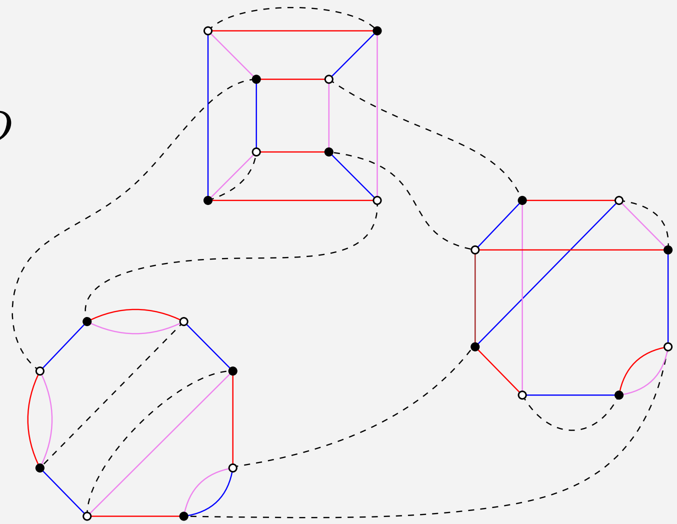
3 – Generalized p-angulations

N.B : building blocks made of D-simplices → $n_{D-2} \leq D + \frac{D(D-1)}{4} n_D$

always true but not saturated!! and finite # gluings per order (Gurau-Schaeffer)

→ Find smallest *a* such that $n_{D-2} \leq D + a n_D$

and = is saturated by infinite # of gluings



4 – Generalized quadrangulations in 4D

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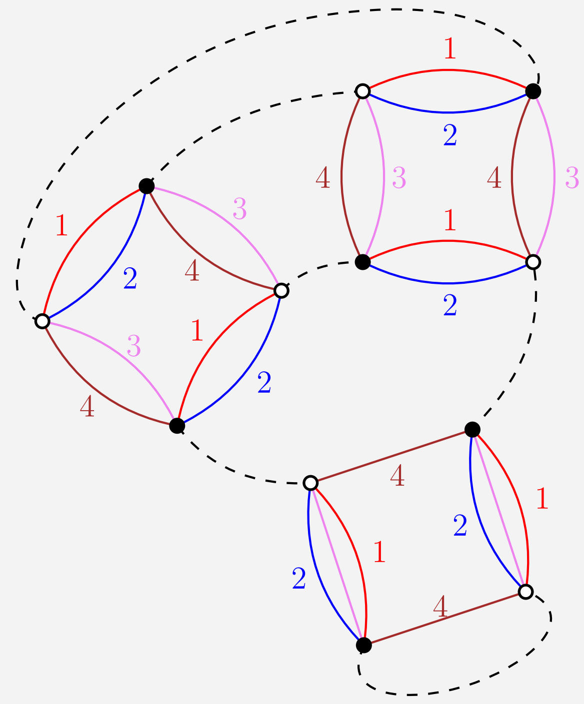
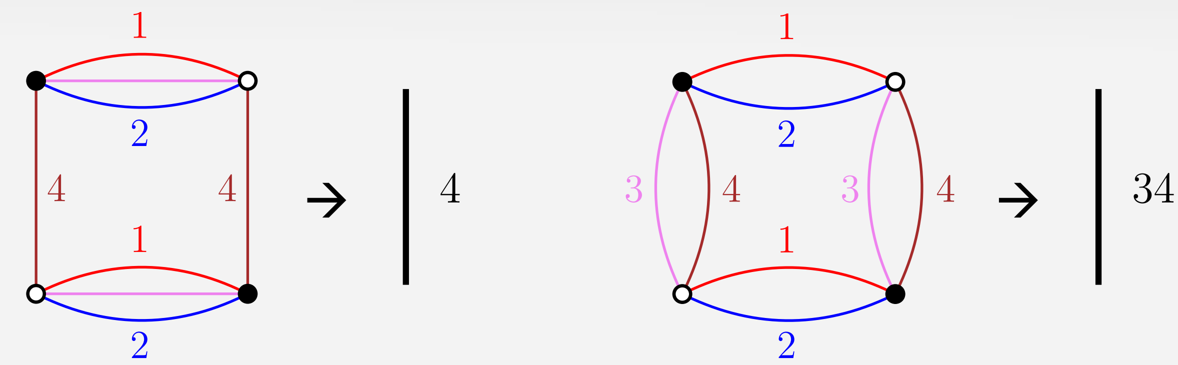
Building blocks

Quadrangulation

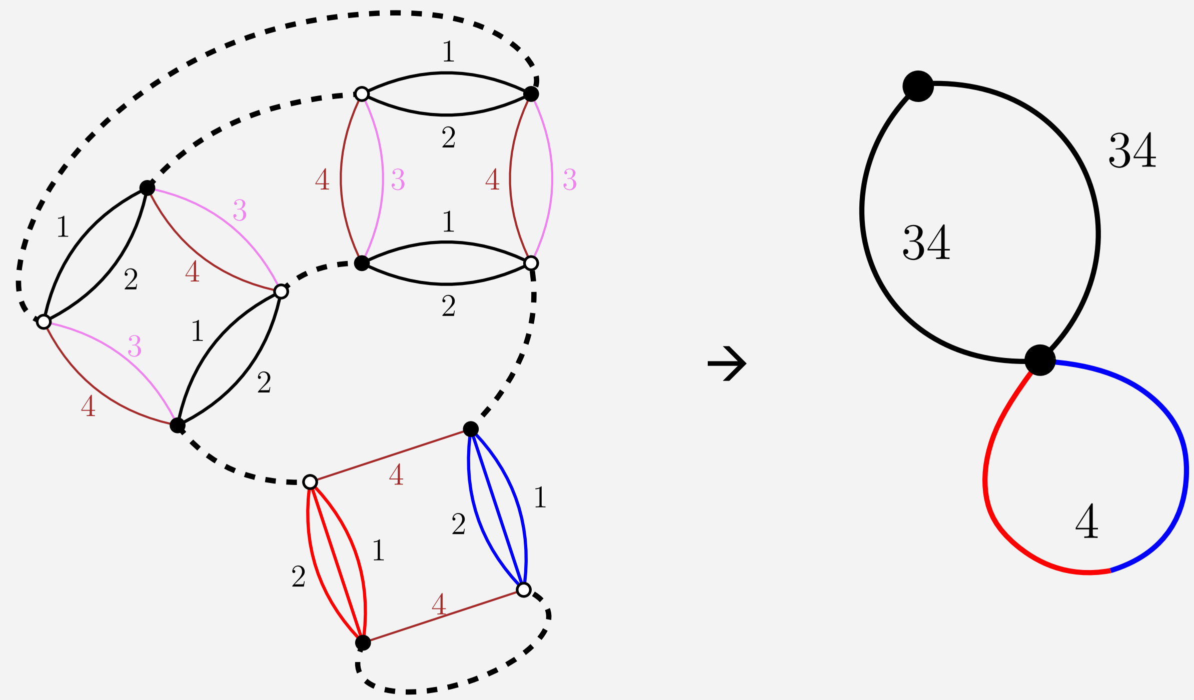


4 – Generalized quadrangulations in 4D

Building blocks



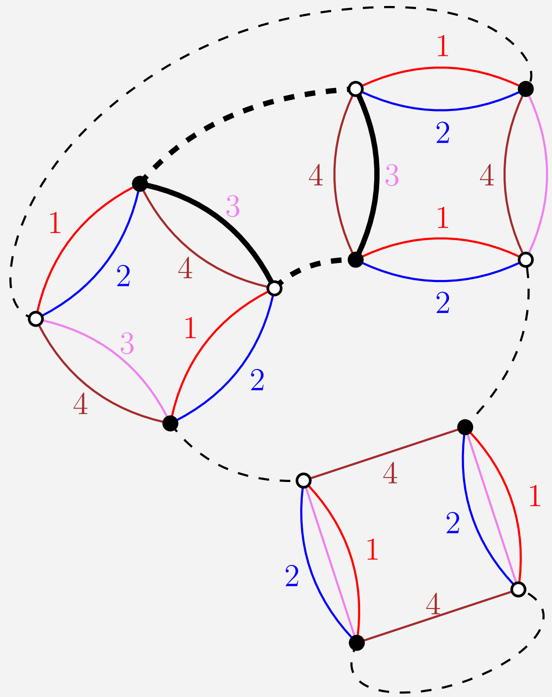
Quadrangulation

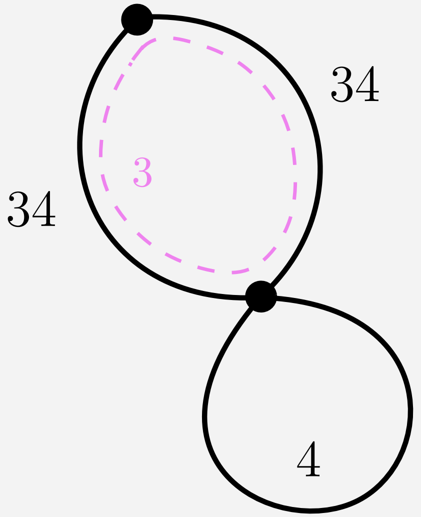


Bijection with combinatorial maps

4 – Generalized quadrangulations in 4D

Bi-colored cycles are faces around one-colored sub-map

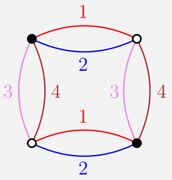


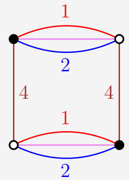


Maximize the sum of faces of one-colored sub-maps

- Trees behave as :

$$n_{D-2}(T) = 4 + \frac{5}{2} n_D^{34} + 3 n_D^4$$





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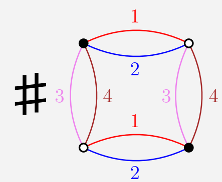
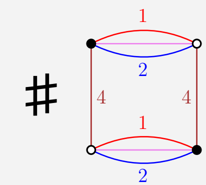
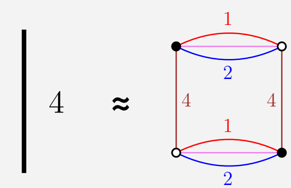




- **Deleting an edge *e*** : *M* maximal $\Longrightarrow$



must be **bridges**

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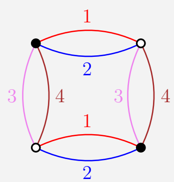
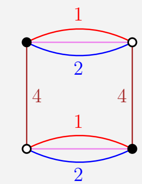
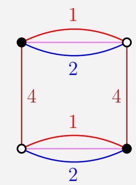
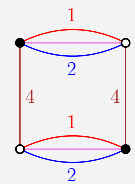




- **Deleting an edge e :** *M* maximal $\Longrightarrow$ | 4 ≈ 

must be **bridges**

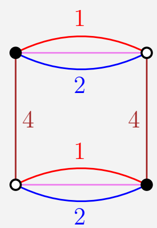
- Once they are : $n_{D-2}(M) = n_{D-2}(T) - 4g(M)$

↳ **Maximal maps** are **planar** and s.t. | 4 ≈ 

are **bridges**

4 – Generalized quadrangulations in 4D

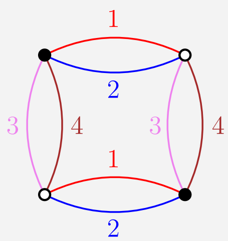
The sharp bounds are

Gluings of 

:

$$n_{D-2} \leq 4 + 3n_D$$

Maximal config. are TREES

Gluings of 

:

$$n_{D-2} \leq 4 + \frac{5}{2} n_D$$

Maximal config. are PLANAR

Maximal gluings have the topology of the 4-sphere

Gluings of **both** :

$$n_{D-2} \leq 4 + \frac{5}{2} n_D^{34} + 3 n_D^4$$

$$\left(3 = \frac{D(D-1)}{4} \right)$$

And maximal Configs : Planar, and $\Big|_4$ are bridges

Generating function : $$F(t,\lambda) = \sum_{M \ \text{max.}} t^{E(M)} \lambda^{E_4(M)}$$

$$\lambda > 3 \quad : \quad F \sim a_1(\lambda) + b_1(\lambda)\sqrt{t_1(\lambda) - t} + \cdots$$

Tree regime $\gamma = \frac{1}{2}$

$$\lambda < 3 \quad : \quad F \sim a_2(\lambda) + b_2(\lambda)(t_2(\lambda) - t) + c_2(\lambda)(t_2(\lambda) - t)^{3/2} + \cdots$$

Planar regime $\gamma = -\frac{1}{2}$

$$\lambda = 3 \quad : \quad F \sim \frac{16}{9} + \frac{128}{3^{5/3}}(\frac{3}{64} - t)^{2/3} + \cdots$$

$\gamma = \frac{1}{3}$ Proliferation of baby universes

4 – Generalized quadrangulations in 4D

→ In D=2, the critical behavior of maximal maps does not depend on the discretization of the boundary *p,* it is *universal*

→ In D=4, the critical behavior of maximal configurations is NOT universal

… it depends on the details of the triangulation of the boundary

This is rather easy (size 4 → combinatorial maps)

Can we do bigger building blocks with any triangulated boundary??

5 – Gluings of octahedra

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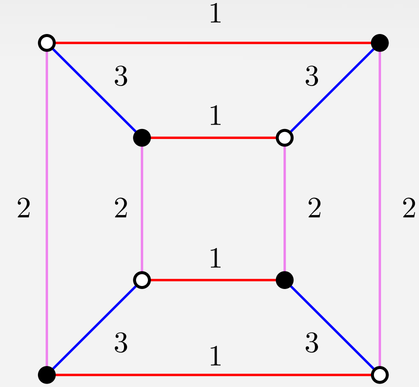
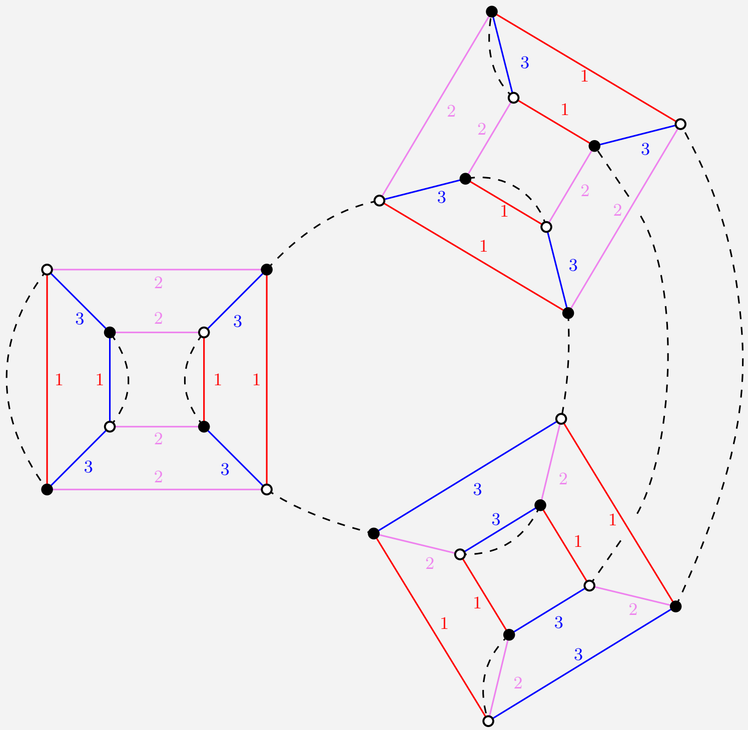
Building blocks



Gluings of octahedra

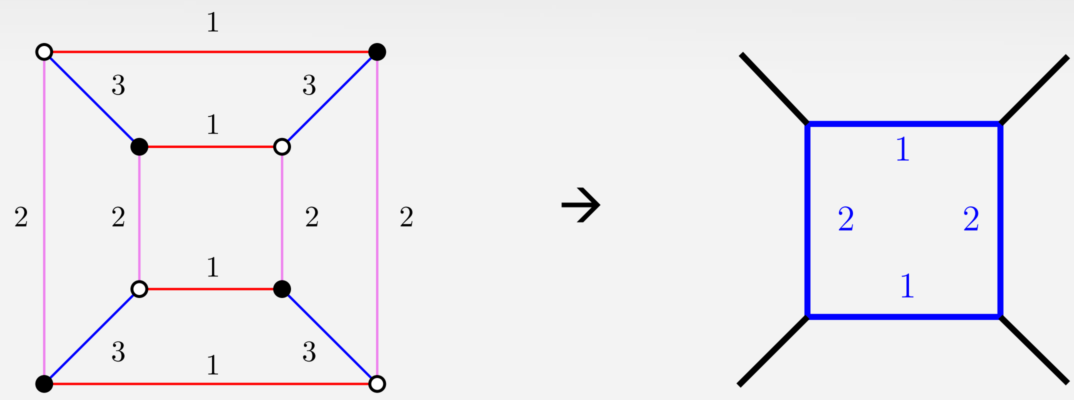


5 – Gluings of octahedra

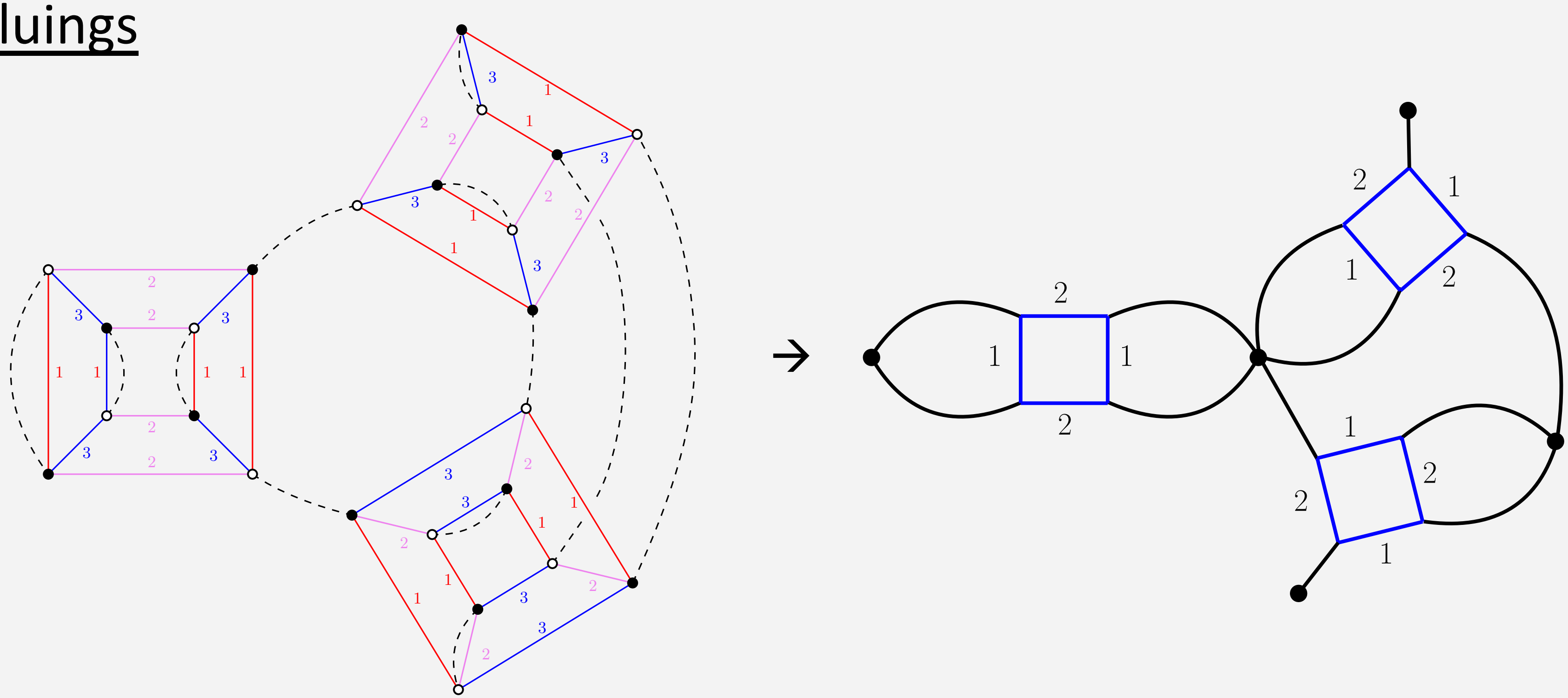
Proofs also rely on a bijection (with "stuffed" hyper-maps)

5 – Gluings of octahedra

Building blocks



Gluings



5 – Gluings of octahedra

And 3D gluings of octahedra verify (w.r.t. their constituting tetrahedra)

$$n_{\text{edges}} \leq 2 + \frac{11}{8} n_{\text{tetrahedra}}$$

Compare with 3D gluings of melonic 8-gons $n_{\text{edges}} \leq 3 + \frac{3}{2} n_{\text{tetrahedra}}$

5 – Gluings of octahedra

Maximal triangulations are in bijection with a family of trees.

The generating function of maximal maps with one marked corner is s.t.

$$G(z) = 1 + 3zG(z)^4 \rightarrow G(z) = \frac{4}{3} - \sqrt{\frac{2048}{243}\left(\frac{9}{256} - z\right)} + \cdots$$

Maximal triangulations are shown to have the topology of the 3-sphere.

Conclusions

Conclusions

- Colored triangulations provide a good framework for combinatorics

- Bijection which generalizes Tutte's bijection for any D-dimensional *p*-angulation (Bonzom, LL, Rivasseau 2015)

  It precisely represents topologies by superposed hyper-maps

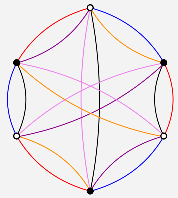
- Maximal configurations exhibit different critical behaviors (≠ 2D)

- A lot to be explored!

What next?

1 - Are there building blocks s.t. n_{D-2} is a non linear function of n_D for maximal gluings?

(Possible candidate in D=6)

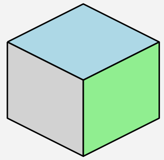


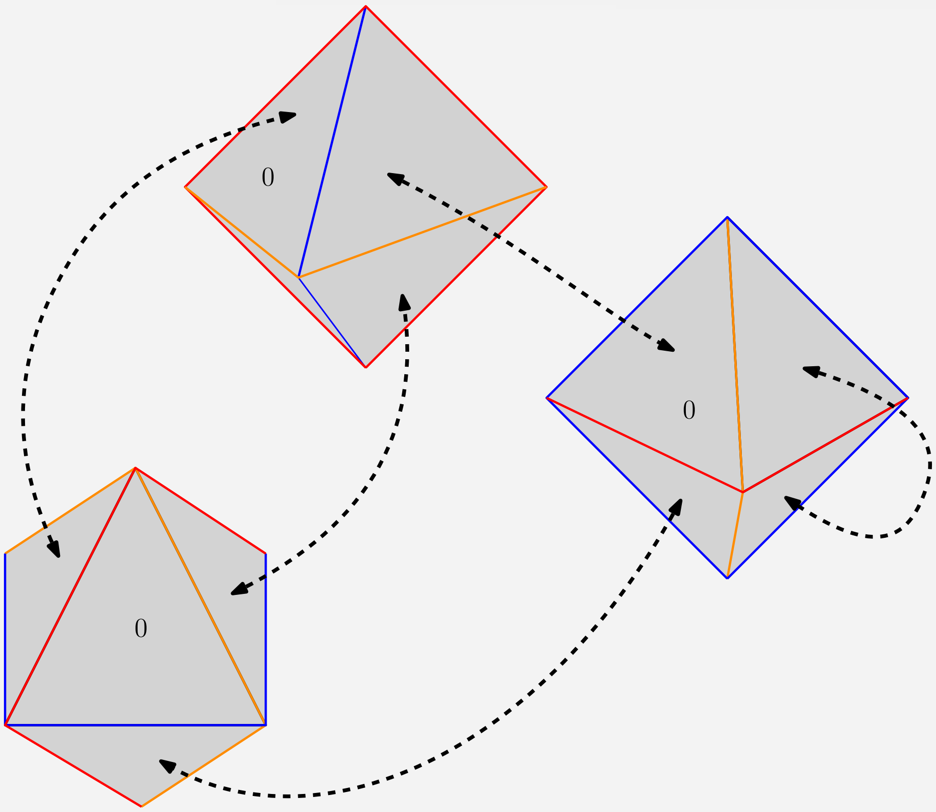
2 - Can we exhibit building blocks with more interesting maximal maps?

3 - Exact counting of gluings of a single building block (→ Unicellular maps)

(Harer-Zagier formula ? Chapuy's identity ?)

4 - Gluings of building blocks with colored faces and no internal structure

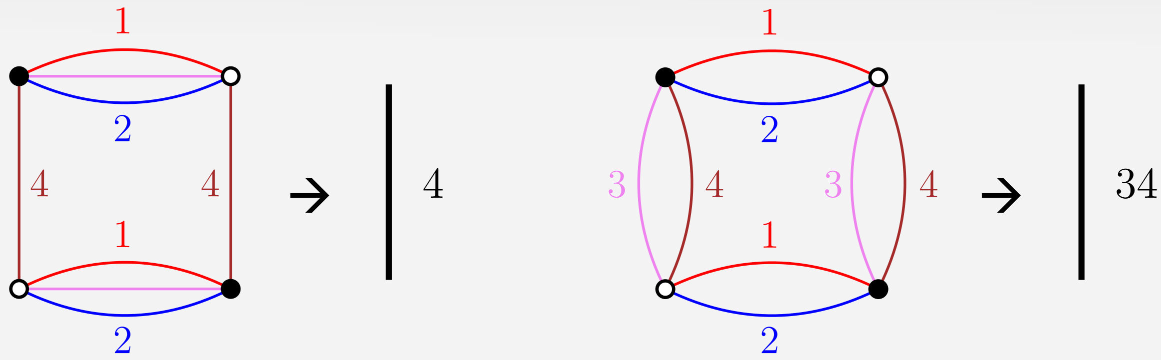




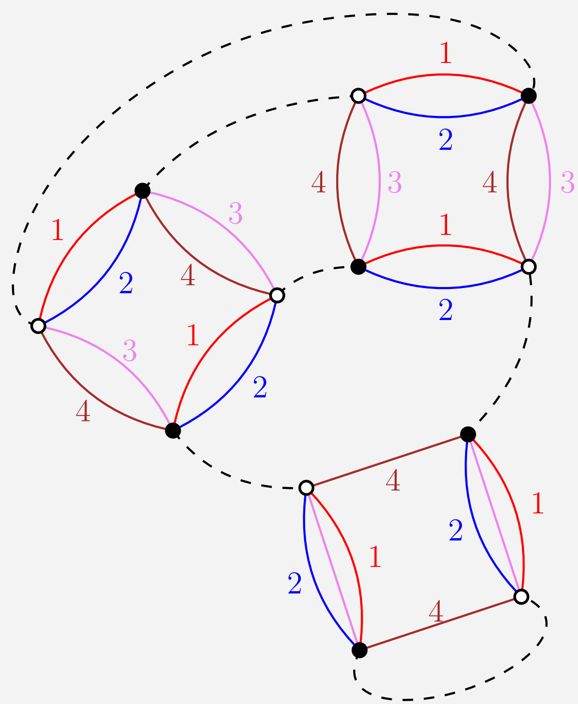
THANK YOU FOR YOUR ATTENTION!!

4 – Generalized quadrangulations in 4D

Building blocks

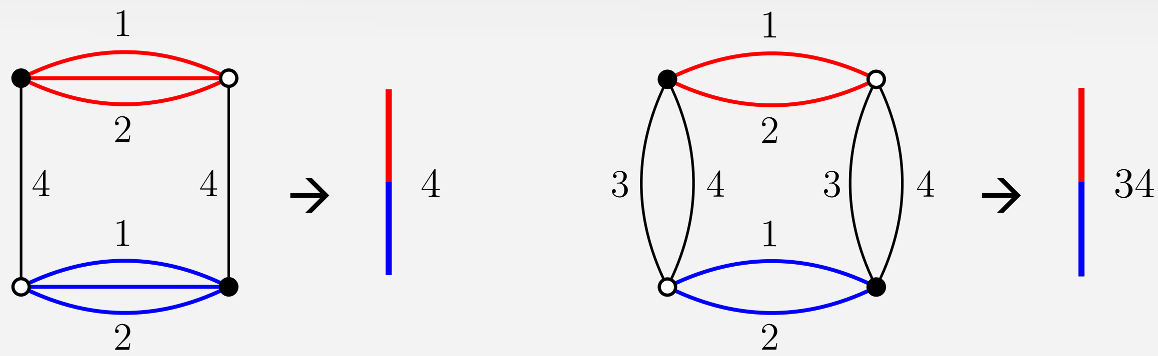


Quadrangulation

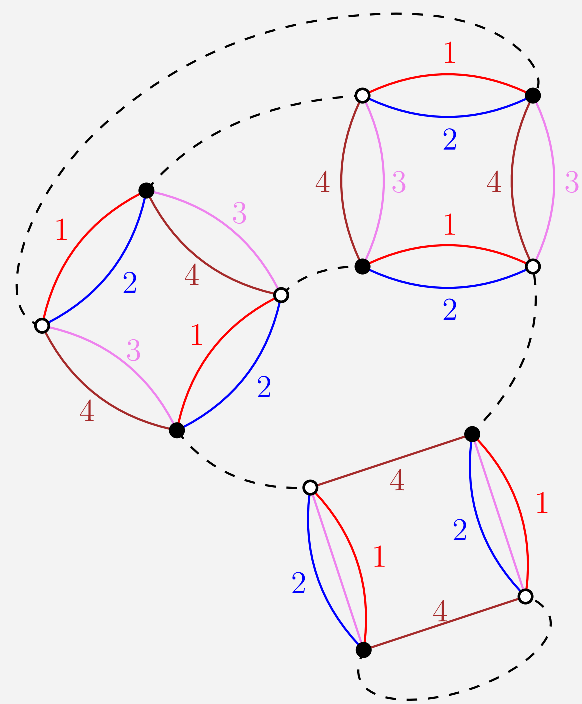


4 – Generalized quadrangulations in 4D

Building blocks

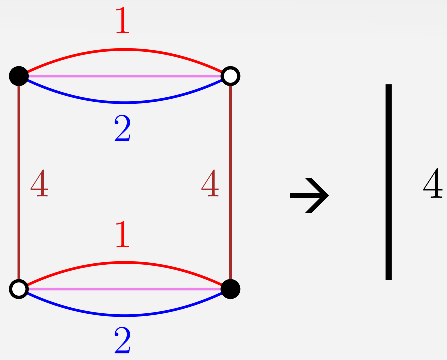


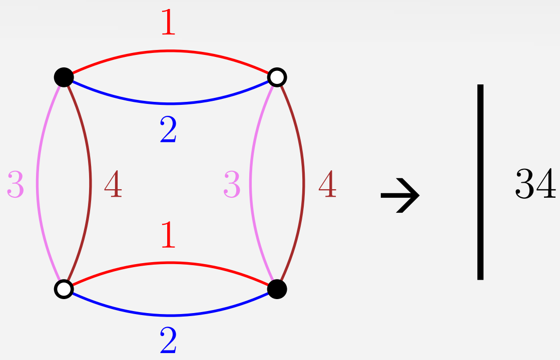
Quadrangulation



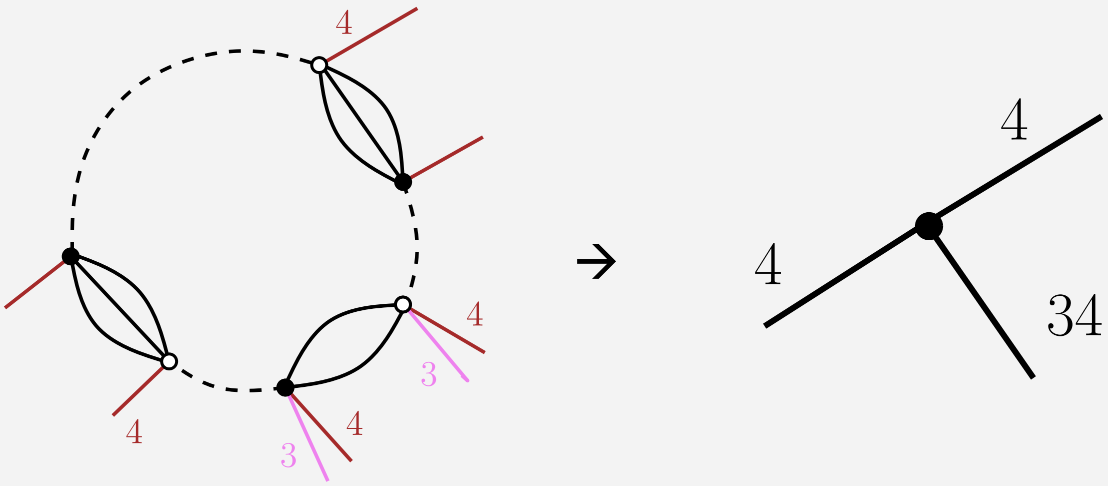
4 – Generalized quadrangulations in 4D

Building blocks

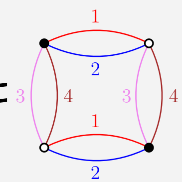


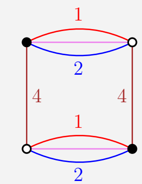


Cycles 0-Pairs

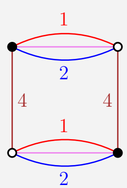


- Trees behave as : $n_{D-2}(T) = 4 + \frac{5}{2} n_D^{34} + 3 n_D^4$

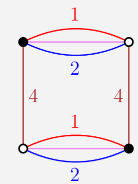
\# 

\# 

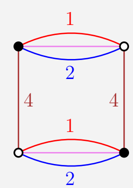
- Deleting an edge *e* : $n'_{D-2} = n_{D-2}(M) + 4 - 2 I_2(e)$

M maximal $\Rightarrow$ 

must be bridges

≤ 1 for $\Big|_4 \approx$ 

- Once they are : $n_{D-2}(M) = n_{D-2}(T) - 4 g(M)$

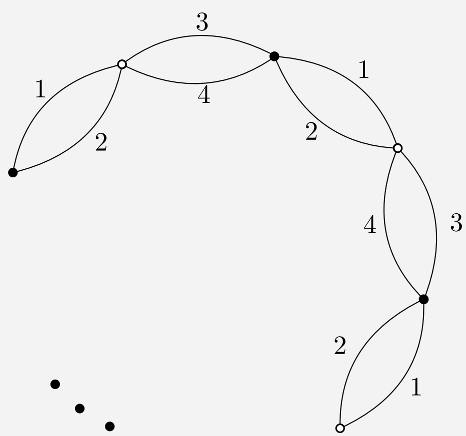
→ Maximal maps are planar and s.t. $\Big|_4 \approx$ 

are bridges

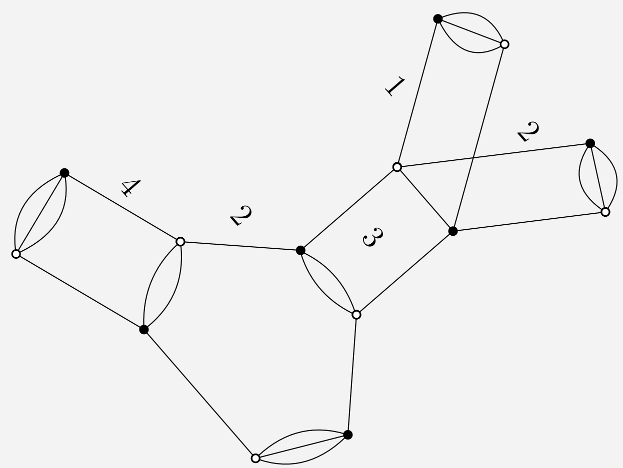
4 – Generalized quadrangulations in 4D

These results can be extended to blocks of any size, in any even dimension :

"Necklaces" + "Melonic" graphs



+



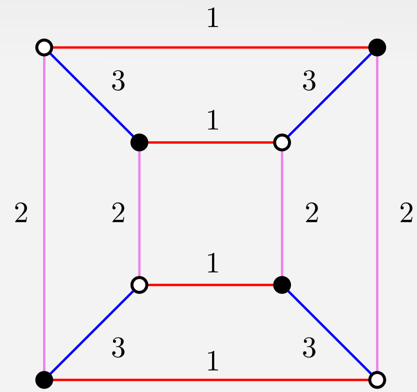
(and their connected sums)

$$n_{D-2} \leq 4 + 2(1 + \frac{1}{p})n_D$$

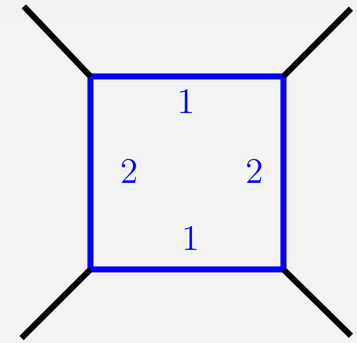
$$n_{D-2} \leq 4 + 3n_D$$

4 – Gluings of octahedra

Building blocks

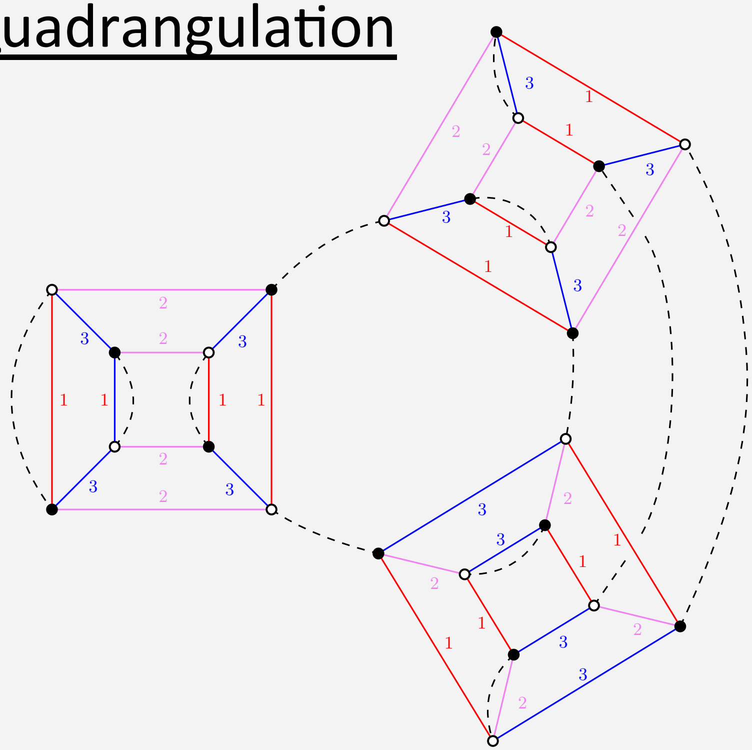


→



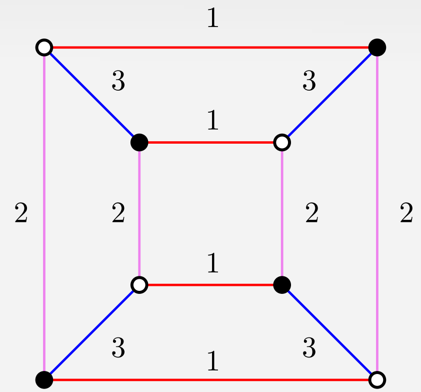
(hyper-edge with internal structure)

Quadrangulation

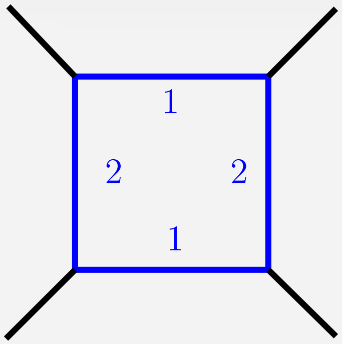


5 – Gluings of octahedra

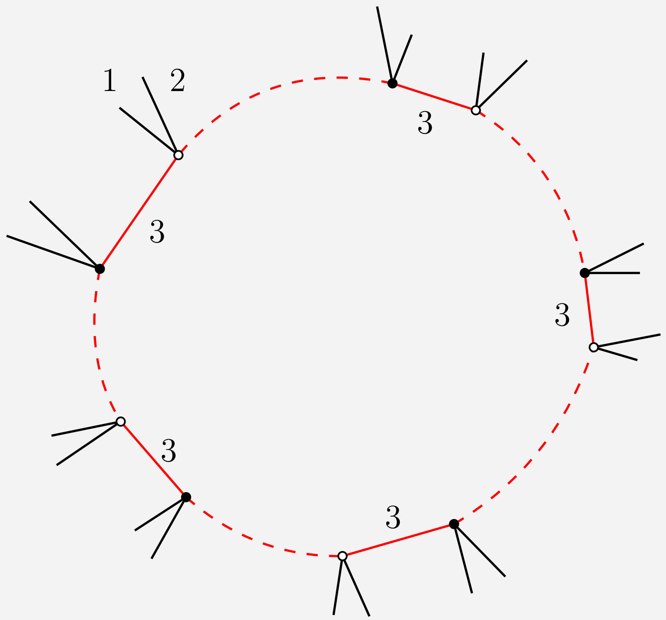
Building blocks



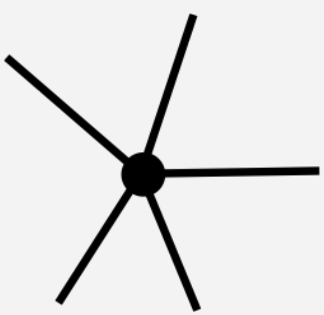
→



Bicolored cycles 03

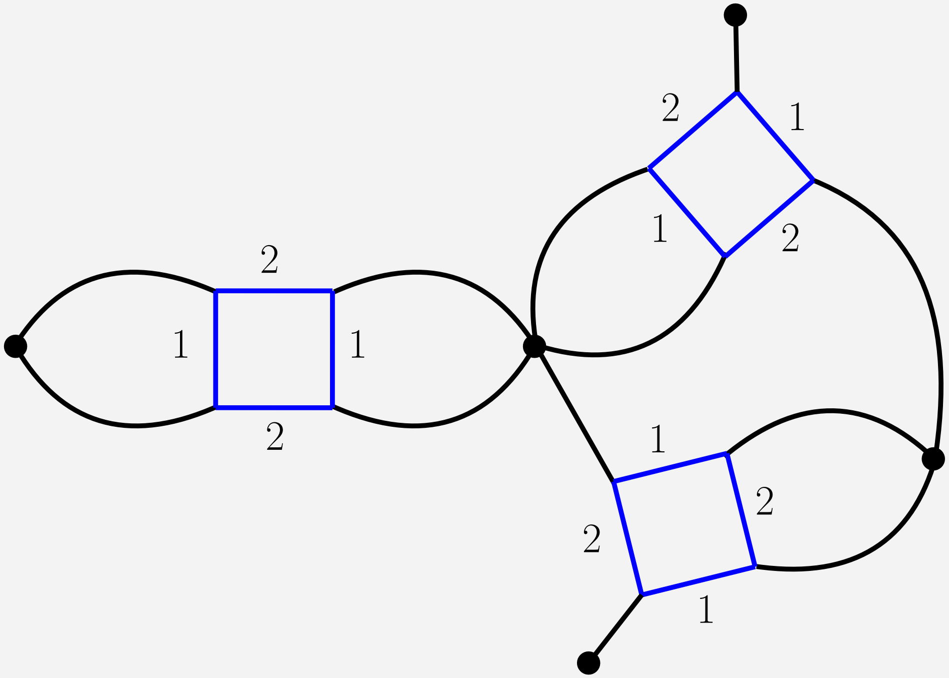


→



2 – Bijection with hypermaps

Edge in triangulation
<-> Two-colored cycle in graph
<-> Face around combinatorial map of single color



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Edge in triangulation
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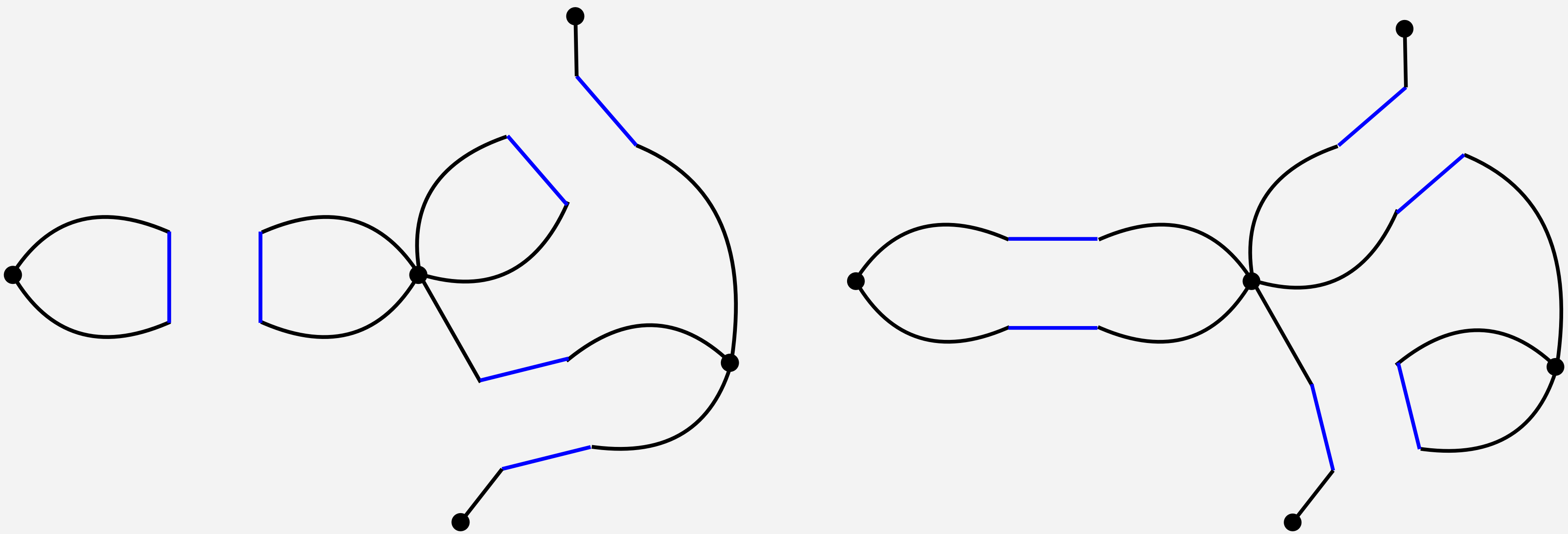


Color 1 : 5 faces

Color 2 : 3 faces

Color 3 : 5 faces

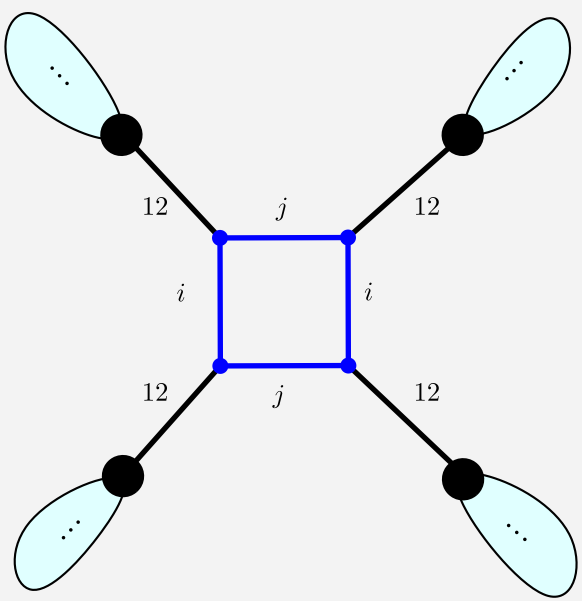
3 – Maximal gluings of octahedra

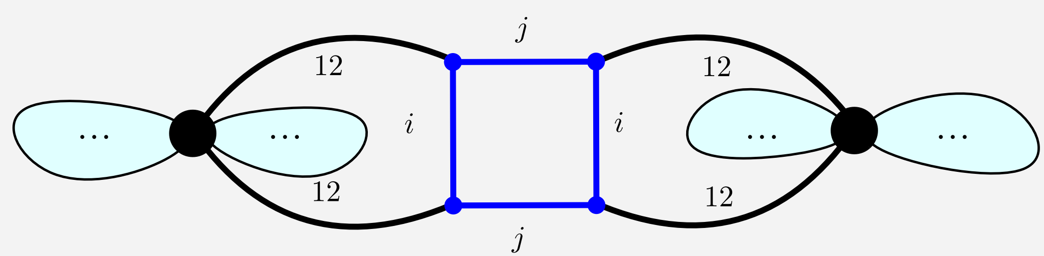
Maximizing maps :

(more complicated than for quadrangulations… see V. Bonzom & L.L 2016)

→ Planar

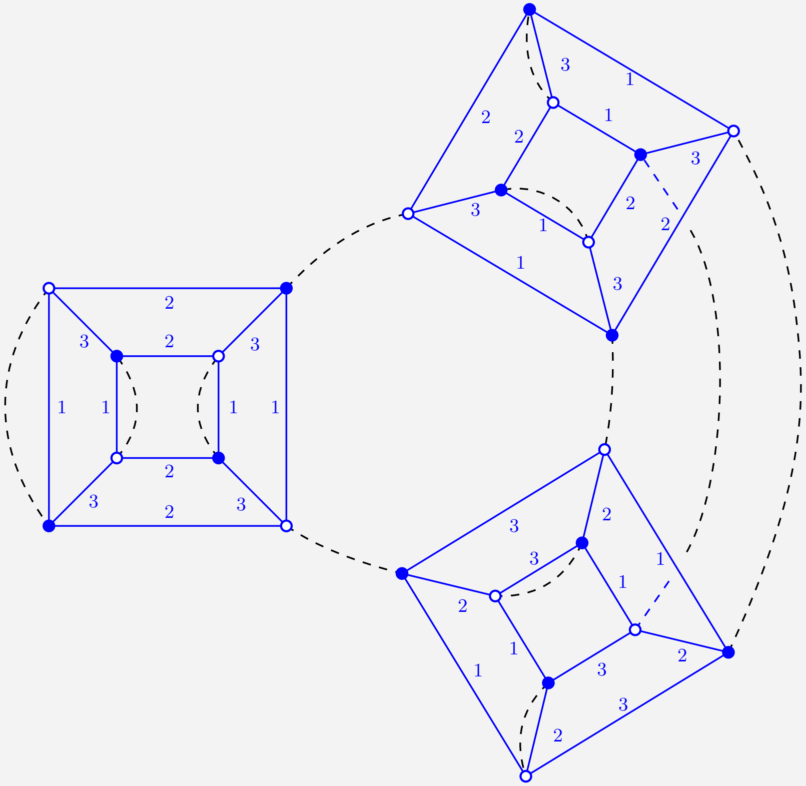
→ Each blue sector locally s.t.

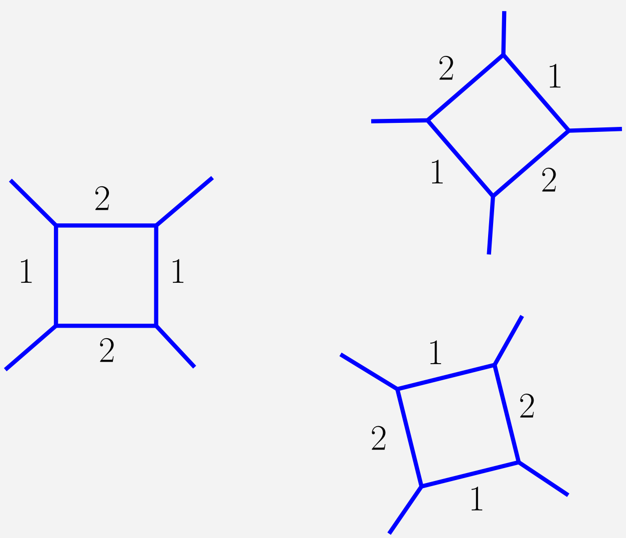




2 – Bijection with stuffed Walsh maps

Building block :

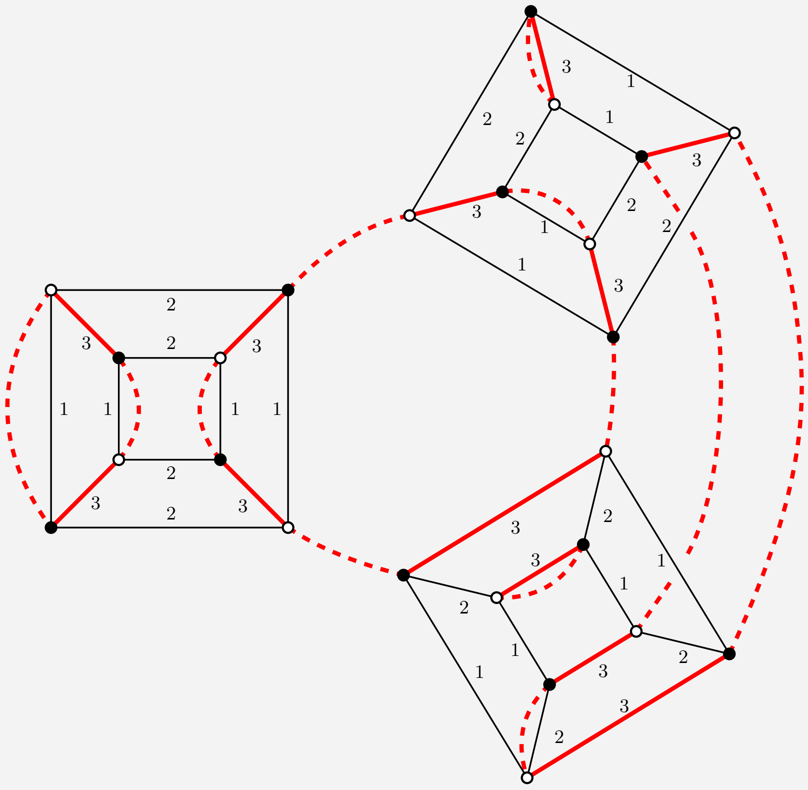




2 – Bijection with stuffed Walsh maps

Glue building blocks together?

⇔ Cycles that alternate edges of color 0 and 3



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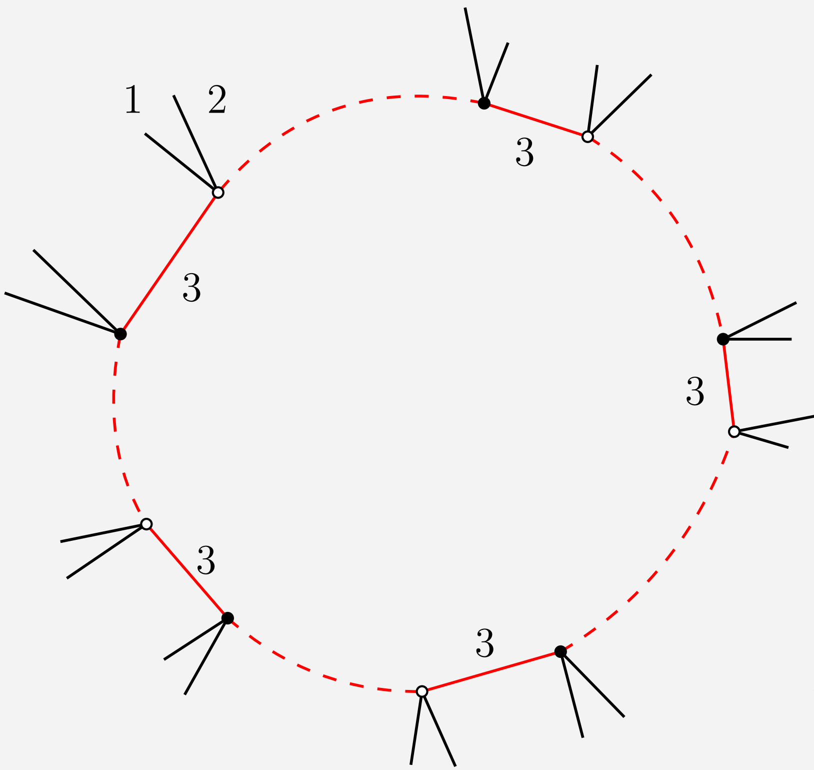
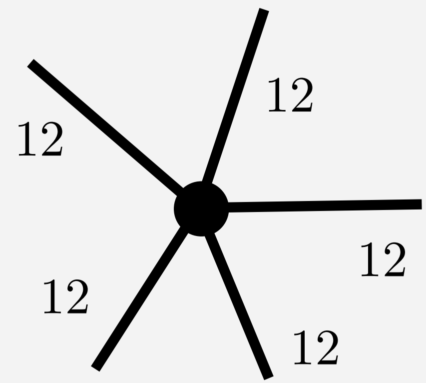


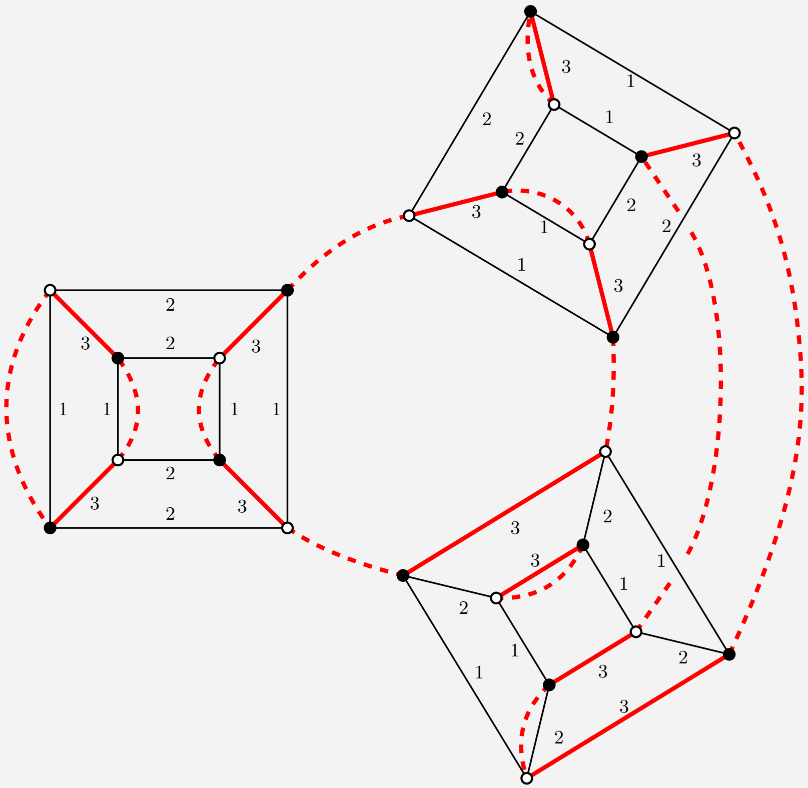


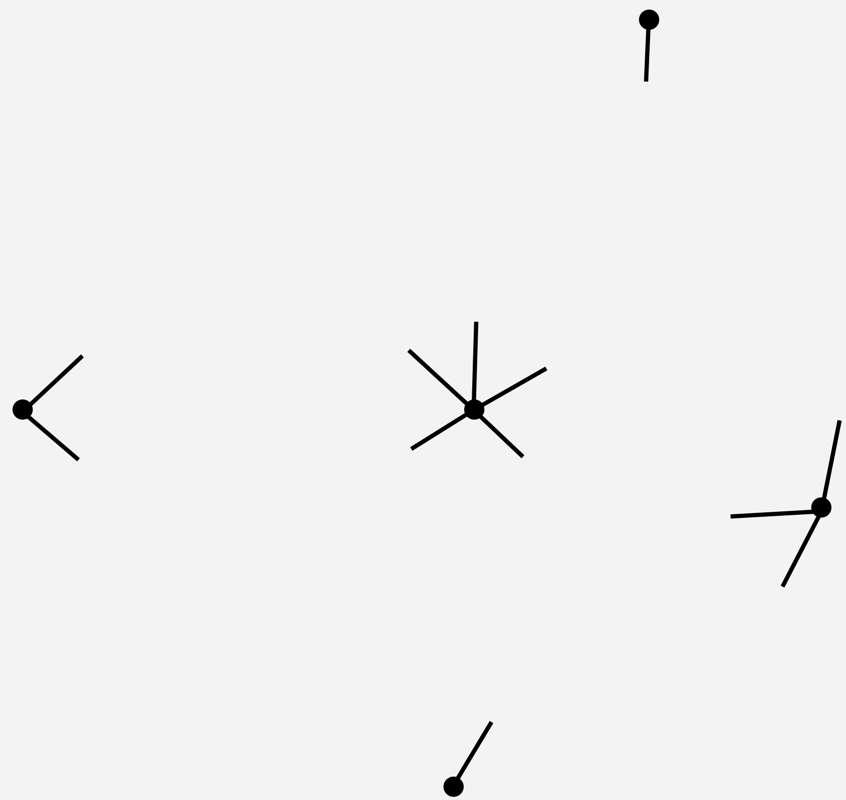
Color 3 edge = half an edge around a black vertex

2 – Bijection with stuffed Walsh maps

Glue building blocks together?

⇔ Cycles that alternate edges of color 0 and 3

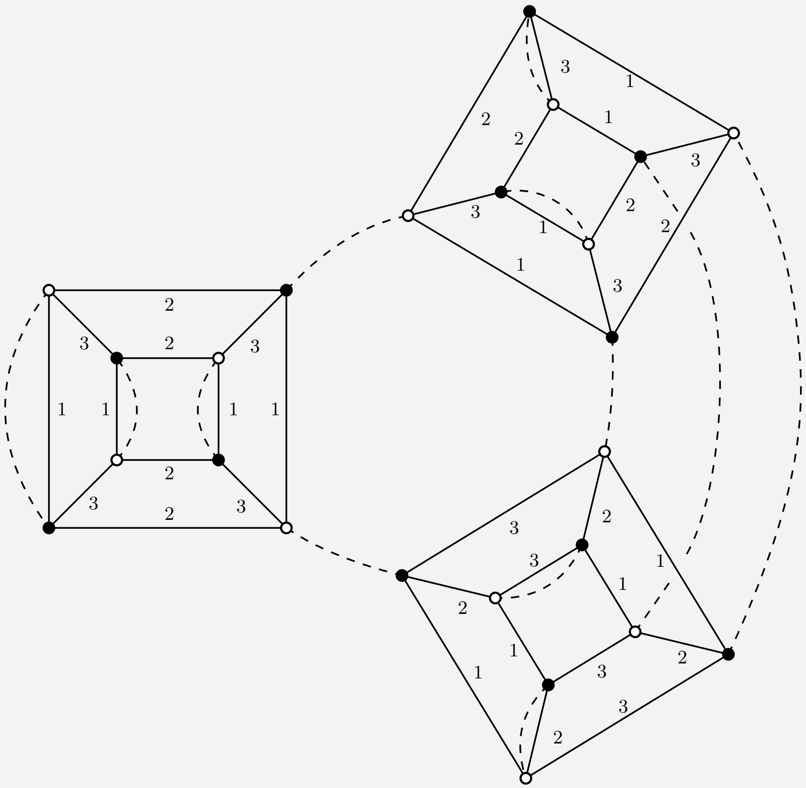


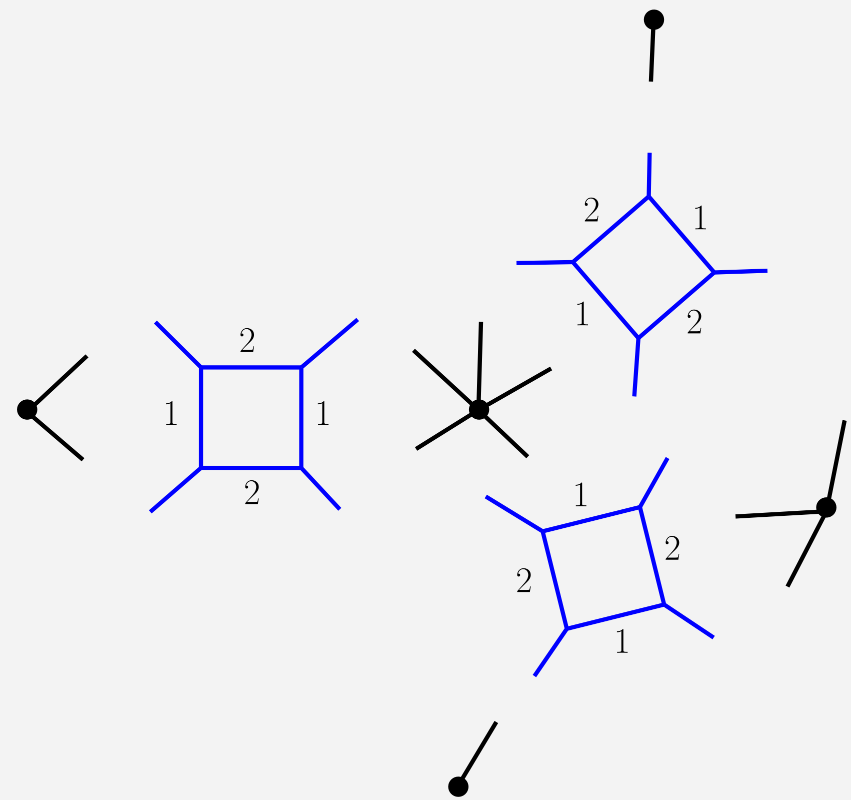


2 – Bijection with stuffed Walsh maps

Color 3 edge = two half edges : *one* around a blue sector, *one* around a black vertex.

→ contract them to form an edge!





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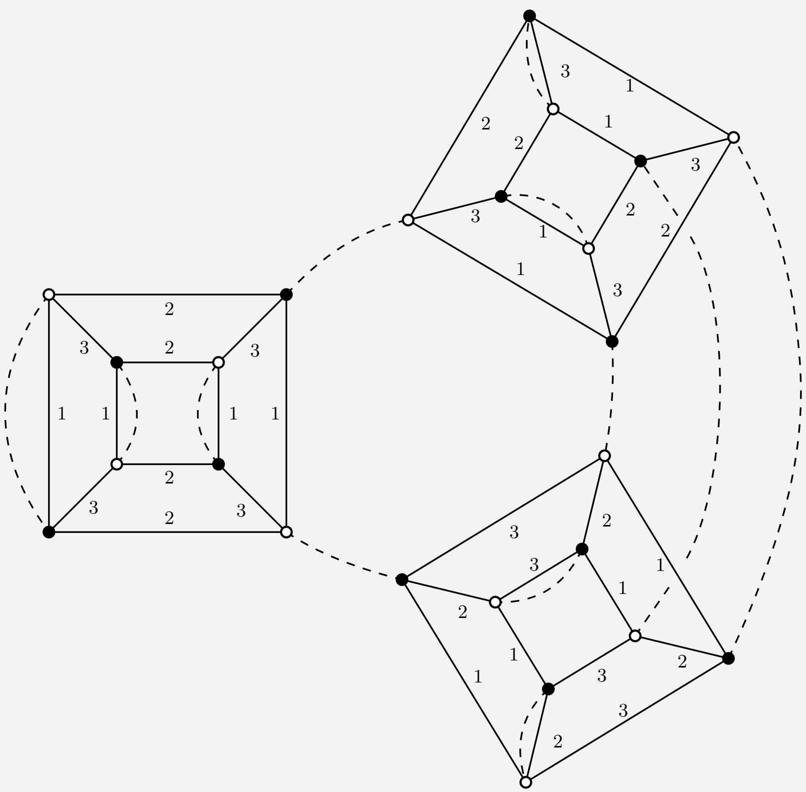
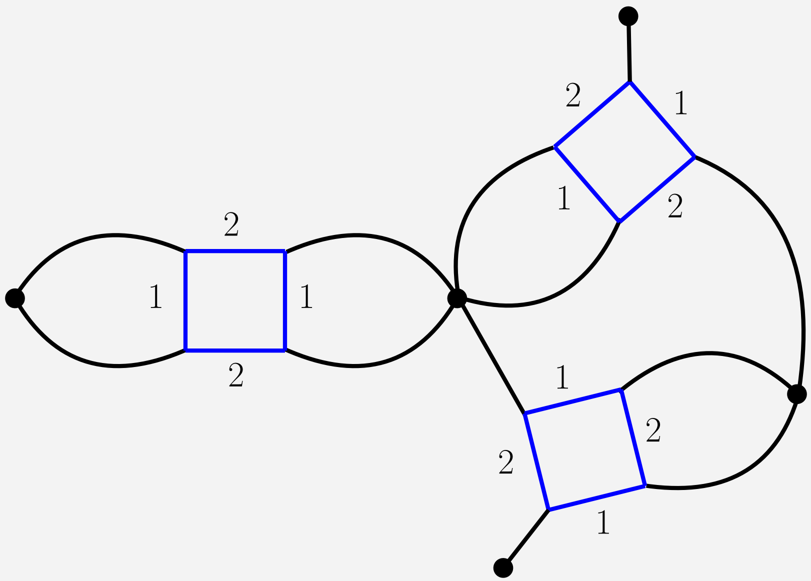
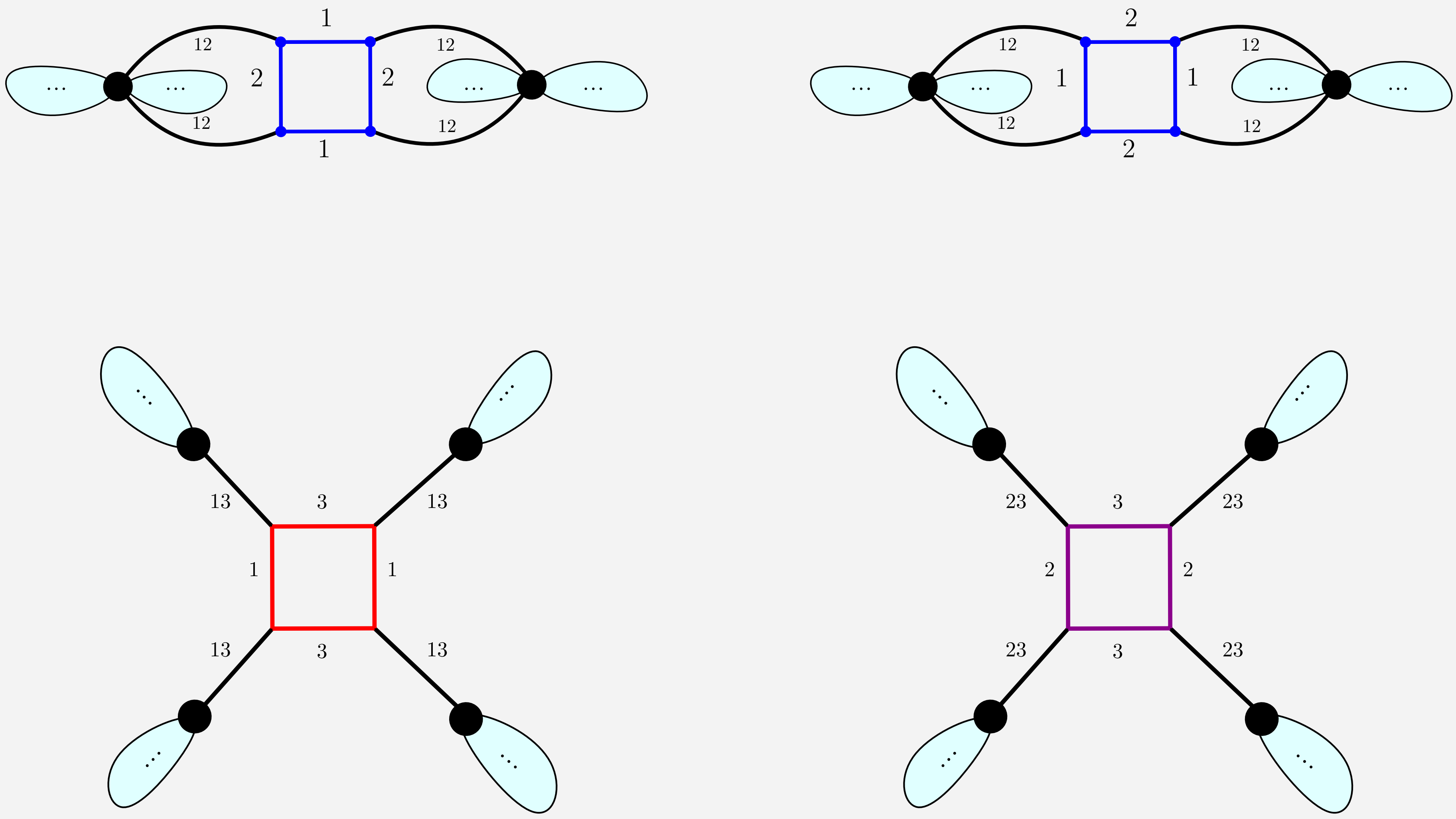
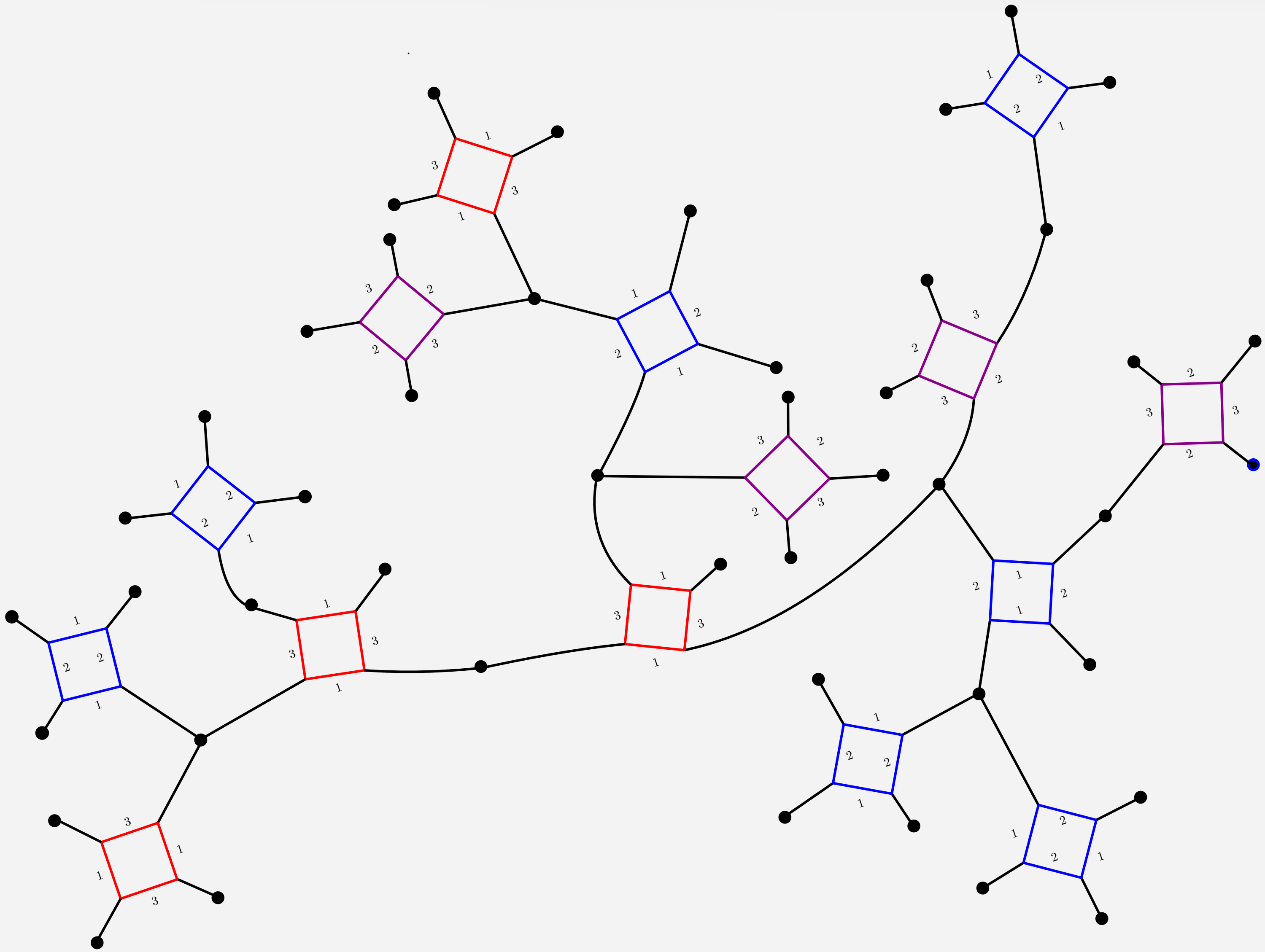




3 – Maximal gluings of octahedra

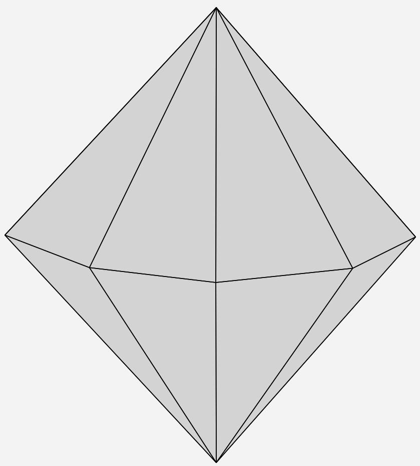


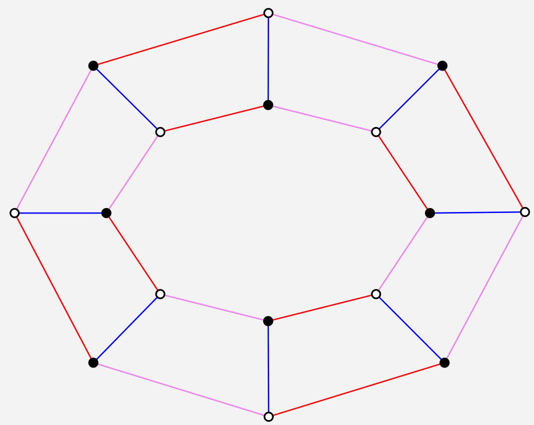
5 – Gluings of octahedra

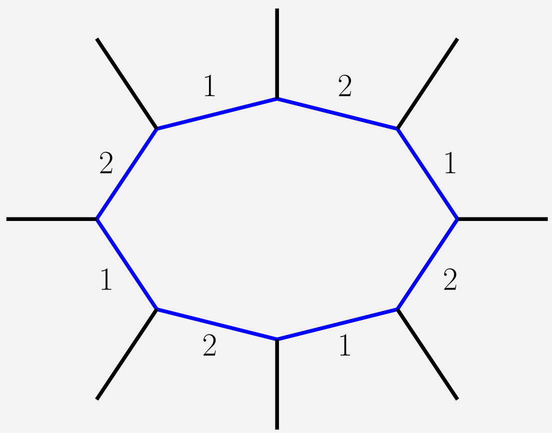


5 – Gluings of octahedra

These results generalize to the infinite family of bi-pyramids (and connected sums)







$$n_{D-2} \leq 3 + (\frac{3}{2} - \frac{1}{2p})n_D$$

Compare with 3D gluings of melonic p-gons $n_{\text{edges}} \leq 3 + \frac{3}{2} n_{\text{tetrahedra}}$

	type of p-gon	D	size	sharp bound	critical exponent
A.	2D p-gon (∞)	2	p	$n_{\text{vertices}} \leq 2 + \frac{p-2}{2} n_{p-\text{gons}}$	-1/2
B.	“melonic” (∞)	$\forall$	even	$n_{D-2} \leq D + \frac{D(D-1)}{4} n_D$ (Gurau)	1/2
		3	even	$n_{\text{edges}} \leq 3 + \frac{3}{2} n_{\text{tetrahedra}}$	
		4	even	$n_{D-2} \leq 4 + 3 n_D$	
C.	“necklaces” (∞)	even	even	$n_{D-2} \leq 4 + 2(1 + \frac{1}{p}) n_D$ (Bonzom, Delepouve, Rivasseau, 2015)	-1/2
D.	4-gons	even	4	$n_{D-2} \leq D + (\frac{D(D-1)}{4} - \frac{\alpha(D-1-\alpha)}{4}) n_D$	1/2 , -1/2, 1/3
E.	6-gons	3	6	B. or K_{33} : $n_{\text{edges}} \leq 3 + n_{\text{tetra}}$ (Bonzom & L.L, 2015)	1/2
		4	6	Various (L.L & J. Thürigen, IP)	1/2 , -1/2, 1/3
F.	Bi-pyramids (∞)	3	8	$n_{\text{edges}} \leq 3 + (\frac{3}{2} - \frac{1}{2p}) n_{\text{tetrahedra}}$ (Bonzom & L.L, 2016)	1/2

1/3