Quantitative Aspects of the Concurrency Theory

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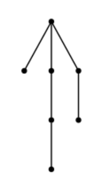
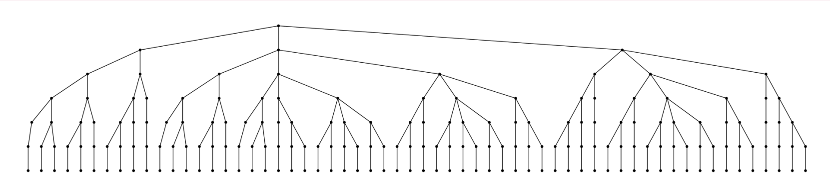
Motivations

When analyzing concurrent processes, the parallel operator is the main source of ***combinatorial explosion****.* [Mi80], [ClGrPe99]





Concurrency theory and combinatorics

In concurrency theory, one manipulates:

- syntactic objects $\Rightarrow$ Processes
- their semantic interpretation $\Rightarrow$ Execution trees

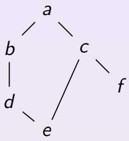
Model checking and testing are classical tools to get information about the processes behaviors.

Our approach

- to consider these objects as combinatorial structures
- to use analytic combinatorics for quantitative studies
- to develop new tools to deal with classical notions of concurrency

Processes and execution trees

A **process** is a specification of events with precedence constraints:



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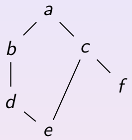
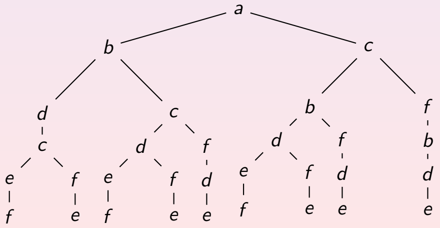
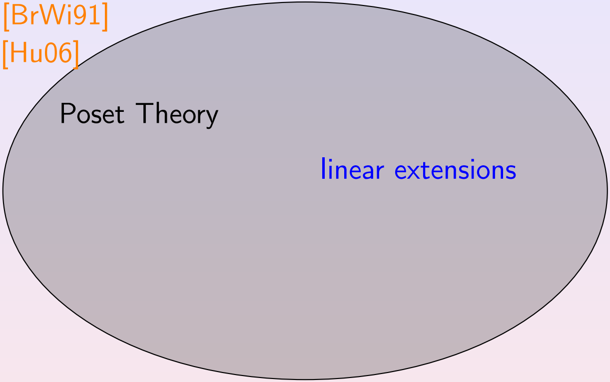


The **execution tree** stores all admissible concurrent runs by sharing prefixes, as in a trie:



Related works

[BrWi91]
[Hu06]



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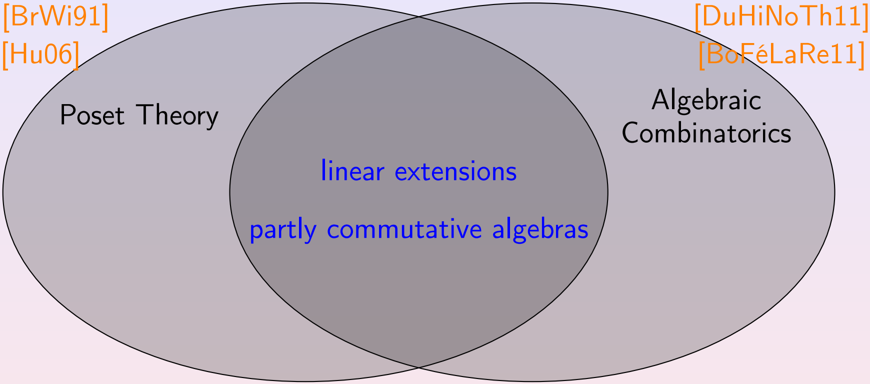

Related works

[BrWi91]
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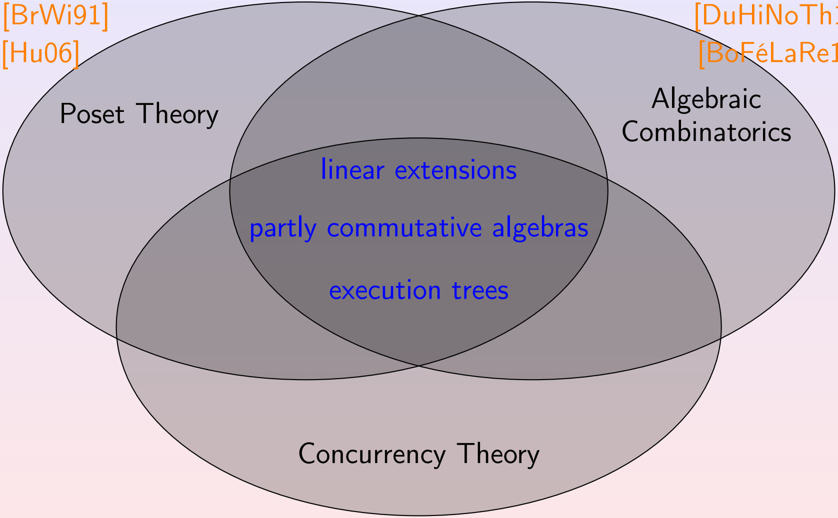
[DuHiNoTh11]
[BoFéLaRe11]



[Mi80] [ClGrPe99] [Fr86]

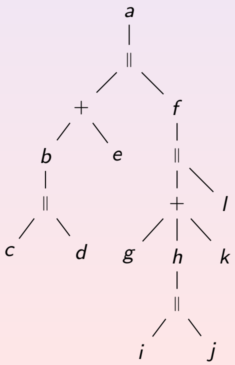
Outline

1. Parallelism and Non-determinism
   - Specification of the processes
   - Counting the average number of executions
2. Parallelism and Synchronization
   - Synchronization
   - Diamond processes and FJ processes
   - Counting the average number of executions
3. Algorithms for Concurrency
   - Counting the exact number of executions
   - Uniform random generation of executions
4. Perspectives

Syntax: process trees

For our process calculus, the grammar of process trees is:

$$\left[\begin{array}{lcll} P & ::= & P_{\|} \mid P_{+} & \text{(process)} \\ P_{\|} & ::= & \alpha \mid \alpha.\,(P \parallel \ldots) & \text{(prefixed parallel)} \\ P_{+} & ::= & P_{\|} + P_{\|} + \ldots & \text{(non-deterministic choice)} \end{array}\right.$$



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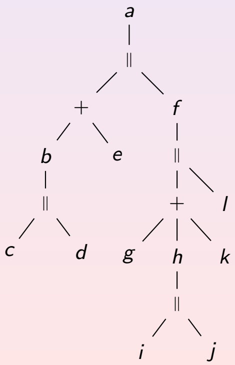

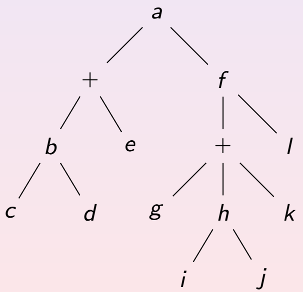

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Process trees and Analytic Combinatorics

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Combinatorial class specification:

$$\left[\begin{array}{lcl} \mathcal{A} & = & \mathcal{A}_{\parallel} \cup \mathcal{A}_{+} \\ \mathcal{A}_{\parallel} & = & \mathcal{Z} \times \mathrm{Seq}\,(\mathcal{A}) \\ \mathcal{A}_{+} & = & \mathcal{A}_{\parallel} \times \mathcal{A}_{\parallel} \times \mathrm{Seq}(\mathcal{A}_{\parallel}) \end{array}\right.$$

$\mathcal{Z}$ marks the actions of the process.

Generating functions system:

$$\left[\begin{array}{lcl} A(z) & = & A_{\parallel}(z) + A_{+}(z) \\ A_{\parallel}(z) & = & z \cdot \frac{1}{1-A(z)} \\ A_{+}(z) & = & A_{\parallel}(z)^2 \cdot \frac{1}{1-A_{\parallel}(z)} \end{array}\right.$$

Explicit generating function:

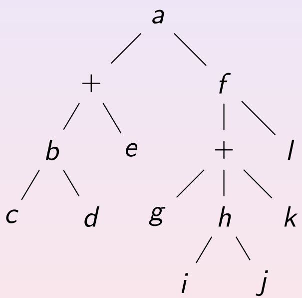
$$A(z) = \frac{1}{2}\left(1 - z - \sqrt{1 - 6z + z^2}\right)$$

By Transfer Theorem [FlOd91]:

$$A_n \sim_{n\to\infty} \sqrt{\frac{3\sqrt{2}-4}{4\pi n^3}}\left(3 - 2\sqrt{2}\right)^{-n}.$$

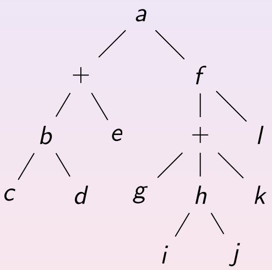
Semantic: execution trees

An **execution** is a complete scheduling of the actions that preserves precedence constraints.

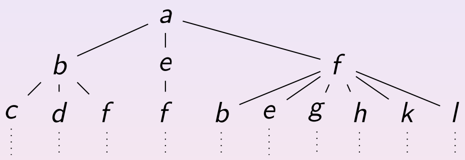


a, e, f, g, l and a, b, f, c, l, d, k are executions,
but a, e, l, f, k and a, e, f, l are not executions.

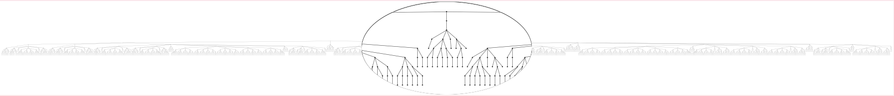
Semantic: execution trees



The executions (or runs) of a process are stored in a prefix tree.

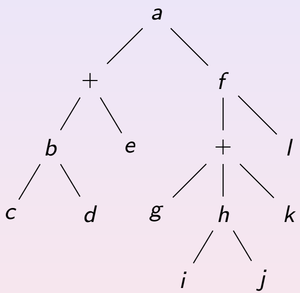


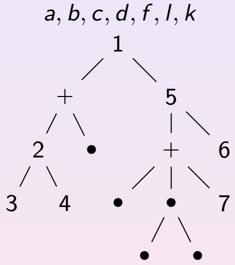
Combinatorial explosion: the complete execution tree contains 1120 leaves.



Counting the executions

- One execution is a partially increasing labeling of the process tree





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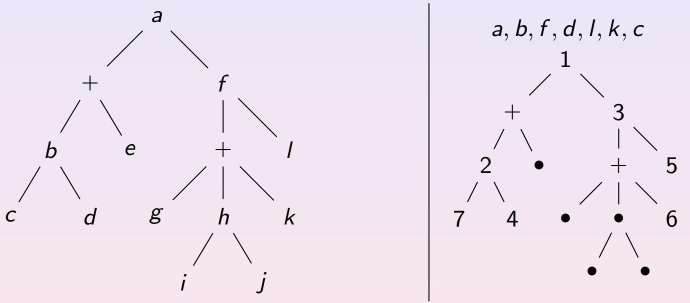

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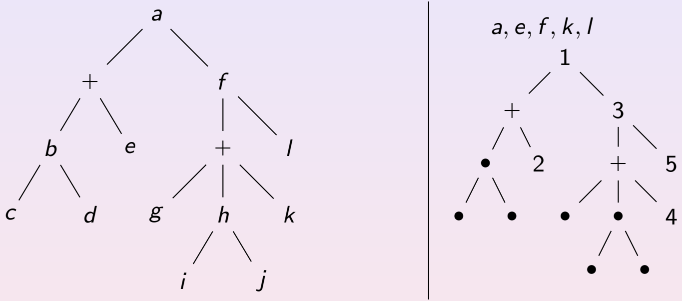

$\star^{\Box}$ is one of the Greene's boxed products.

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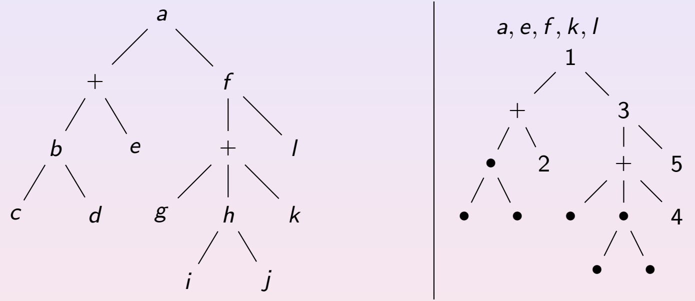

- We can enumerate the cumulative number of partially increasing trees.

$$\left[\begin{array}{lcl} \mathcal{B} & = & \mathcal{B}_{\|} \cup \mathcal{B}_{+} \\ \mathcal{B}_{\|} & = & \mathcal{Z} \times \mathcal{W}^{\square} \star \mathrm{Seq}(\mathcal{B}) \\ \mathcal{B}_{+} & = & (\mathcal{B}_{\|} \times \mathcal{A}_{\|} \times \mathrm{Seq}(\mathcal{A}_{\|})) \cup (\mathcal{A}_{\|} \times \mathrm{Seq}(\mathcal{A}_{\|}) \times \mathcal{B}_{\|} \times \mathrm{Seq}(\mathcal{A}_{\|})) \end{array}\right.$$

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Average quantitative results

Theorem: Average number of executions

Asymptotically, the average number of executions induced by all the process trees of a given size n is: $\Theta(\alpha^n \cdot n!)$, with $\alpha \approx 0.34315\ldots$

more precisely:

$$\frac{e}{\sqrt{3\sqrt{2}-4}} \left(6-4\sqrt{2}\right)^n n! \left(1+\frac{12\sqrt{2}-7}{48(3\sqrt{2}-4)n}+O(n^{-2})\right).$$

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The worst case is: $(n-1)!$

In the pure $\|$ model, it is $\Theta(0.5^n \cdot n!)$.
There is some cutting-effect induced by the choice operator.

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Theorem: Expressivity of the $+$ operator

Asymptotically, the average number of choices induced by all the process trees of a given size n is: $\alpha \cdot \beta^n$, with $\alpha \approx 1.4408\dots$ and $\beta \approx 1.11062\dots$

Key-ideas of the proof

- One computation is a partially increasing labeling of the process tree.
- In order to compute the average number of computations paths, we enumerate the cumulative number of partially increasing trees.

$$\begin{cases} \mathcal{B} & = \mathcal{B}_{\|} \cup \mathcal{B}_{+} \\ \mathcal{B}_{\|} & = \mathcal{Z} \times \mathcal{W}^{\square} \star \mathrm{Seq}(\mathcal{B}) \\ \mathcal{B}_{+} & = (\mathcal{B}_{\|} \times \mathcal{A}_{\|} \times \mathrm{Seq}(\mathcal{A}_{\|})) \cup (\mathcal{A}_{\|} \times \mathrm{Seq}(\mathcal{A}_{\|}) \times \mathcal{B}_{\|} \times \mathrm{Seq}(\mathcal{A}_{\|})) \end{cases}$$

- We get a functional system satisfied by the generating functions.

$$\begin{cases} B(z,w) & = B_{\|}(z,w) + B_{+}(z,w) \\ B_{\|}(z,w) & = z \cdot \int_0^w \frac{1}{1-B_{\|}(z,t)-B_{+}(z,t)}\, dt \\ B_{+}(z,w) & = \frac{B_{\|}(z,w)\cdot A_{\|}(z)}{1-A_{\|}(z)} \cdot \left(1 + \frac{1}{1-A_{\|}(z)}\right). \end{cases}$$

Key-ideas of the proof: assisted proof using Maple

- One computation is a partially increasing labeling of the process tree.
- In order to compute the average number of computations paths, we can enumerate the cumulative number of partially increasing trees.
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- The bivariate generating function $B(z, w)$ is holonomic (i.e. D-finite).

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$$
\begin{aligned}
&(12719889n + 22375188n^2 + 2931297n^3 - 8201443n^4 - 4090482n^5 - 613040n^6 + 4704n^7 + 6528n^8)\mathbf{b(n)}\\
&+ (-117440010 - 58477377n + 426337222n^2 + 536891121n^3 + 159366704n^4\\
&\ - 45499352n^5 - 32369040n^6 - 5041408n^7 + 37632n^8 + 52224n^9)\mathbf{b(n+1)}\\
&+ (-1840352130 - 5243895651n - 2584708853n^2 + 4298075862n^3 + 5852298713n^4\\
&\ + 2893886286n^5 + 671061648n^6 + 58982112n^7 - 2742912n^8 - 626688n^9)\mathbf{b(n+2)}\\
&+ (11807380260 - 6756975753n - 65434988818n^2 - 85941513865n^3 - 52354768885n^4\\
&\ - 17114870064n^5 - 2908711504n^6 - 174499904n^7 + 14642688n^8 + 1984512n^9)\mathbf{b(n+3)}\\
&+ (-9389674170 + 12063284850n + 57817477934n^2 + 62197895398n^3 + 31811971363n^4\\
&\ + 8697304778n^5 + 1194267120n^6 + 42809376n^7 - 7325568n^8 - 626688n^9)\mathbf{b(n+4)}\\
&+ (2198255760 - 3582298935n - 13638796887n^2 - 12881202246n^3 - 5723588930n^4\\
&\ - 1322436008n^5 - 141002224n^6 - 731008n^7 + 1029888n^8 + 52224n^9)\mathbf{b(n+5)}\\
&+ (-158436540 + 294800982n + 958932073n^2 + 795995131n^3 + 298779511n^4\\
&\ + 53795162n^5 + 3103408n^6 - 291168n^7 - 32640n^8)\mathbf{b(n+6)}\\
&+ (2402190 - 4905717n - 13874707n^2 - 9958586n^3 - 2996489n^4 - 356944n^5 + 7248n^6 + 3264n^7)\mathbf{b(n+7)}\\
&= 0
\end{aligned}
$$

$$b(0), \ldots, b(6) = 0, 1, 3, 12, 63, 418, 3460.$$

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- We convert automatically the P-recurrence into a linear homogeneous differential equation for $B(z)$.

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- Using the functional equation satisfied by the generating function, we prove that the guess is correct.

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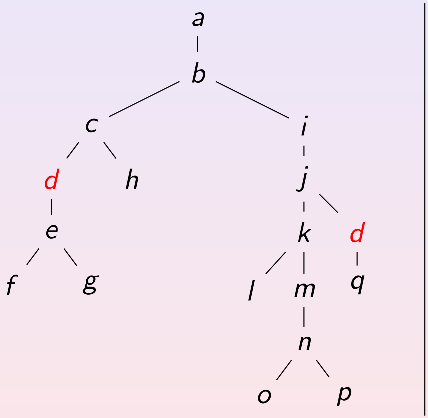
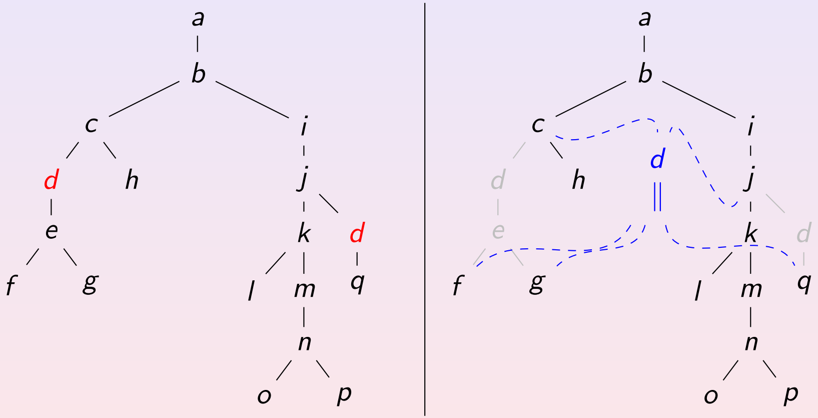
- Using the functional equation satisfied by the generating function, we prove that the guess is correct.
- We use the saddle point method to conclude.

Outline

Synchronization

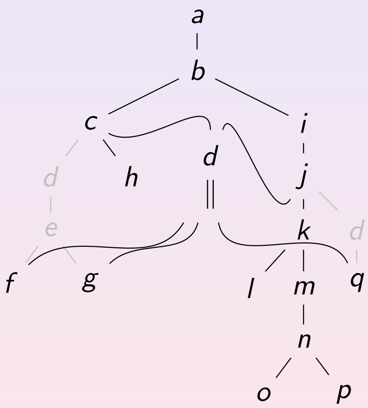


Synchronization

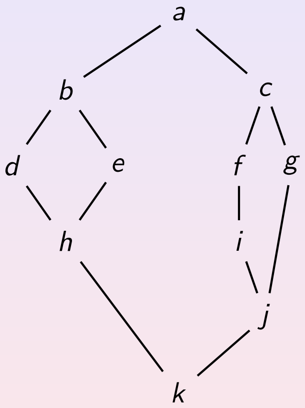


Synchronization

Partially increasing trees substituted by partially increasing DAGs



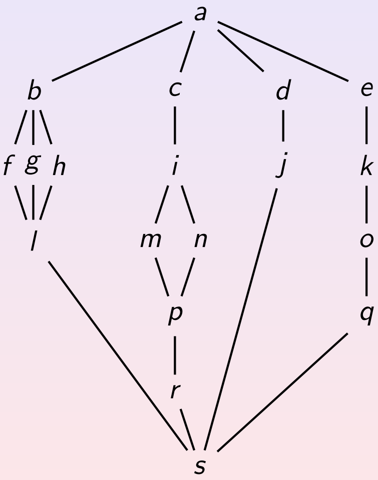
Diamond processes [BoDiFoGeHw16]



Unlabeled binary diamonds

$$\mathcal{S} = \mathcal{Z} + \left(\mathcal{Z} \times (\mathcal{E} + \mathcal{S}^2) \times \mathcal{Z}\right)$$

Diamond processes [BoDiFoGeHw16]



Unlabeled plane diamonds

$$\mathcal{S} = \mathcal{Z} + (\mathcal{Z} \times \mathrm{Seq}(\mathcal{S}) \times \mathcal{Z})$$

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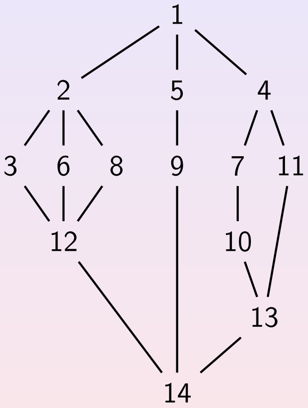


Unlabeled diamonds

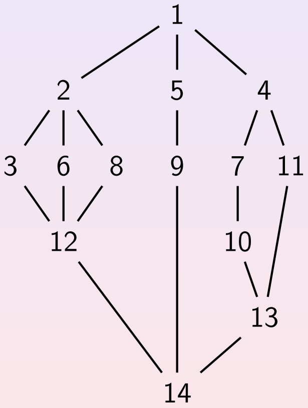
$$\mathcal{S} = \mathcal{Z} + (\mathcal{Z} \times G(\mathcal{S}) \times \mathcal{Z})$$

Increasing labelings

$$\mathcal{I} = \mathcal{Z}^{\square} + (\mathcal{Z}^{\square} \star (G(\mathcal{I}) \star \mathcal{Z}^{\blacksquare}))$$

$\star^{\square}; \star^{\blacksquare}$ are Greene's boxed products.

Generating functions for diamonds



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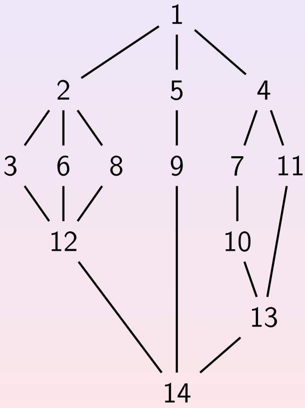
Functional equation
(via Symbolic method)

$$I(z) = z + \int_0^z \int_0^t G(I(u))\ du\ dt$$

Generating functions for diamonds



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Differential equation (non-linear)

$$I'' = G(I)$$
$$I(0) = 0$$
$$I'(0) = 1$$

Related Works

Varieties of increasing Trees [BeFlSa92]

- Increasing trees: generating functions satisfying a first order differential equation $f' = \phi(f)$
- Study of some parameters: root degree, path length, profile, ...

[KuPa12+]

- *Bilabelled increasing trees and hook-length formulas* '12
- *Combinatorial analysis of growth models for series-parallel networks* '16

- A model introduced by H. Mahmoud: the bucket trees
- The same scheme of generating functions: $f'' = \phi(f)$

Exactly solvable cases

$$\mathcal{F} = \mathcal{Z} + \mathcal{Z}^{\square} \star G(\mathcal{F}) \star \mathcal{Z}^{\blacksquare}$$

$G(X)$	$F(z)$	$f_n = n![z^n]F(z)$
$1 + X^2$	$K\wp(\rho - z; \omega_1, \omega_2)$ Weierstrass $\wp$ function	$6\frac{(n+1)!}{\rho^{n+2}} \sum_{(k,l)\in\mathbb{Z}^2} \frac{1}{\left(1 + \frac{k\omega_1}{\rho} + \frac{l\omega_2}{\rho}\right)^{n+2}}$
$1 + X^3$	$\frac{P(\mathrm{sn}(Mz;k))}{Q(\mathrm{sn}(Mz;k))}$ Jacobi sinus function	$\frac{\sqrt{2}\ n!}{\rho^{n+1}} \sum_{(k,l)\in\mathbb{Z}^2} \frac{1}{(1+C_{k,l})^{n+1}} - \frac{1}{(2+C_{k,l})^{n+1}}$
e^X	$\int \tan(z) + \frac{1}{\cos(z)}$	$\frac{2^{n+1}\ (n-1)!}{\pi^n} \sum_{j=-\infty}^{+\infty} \frac{1}{(1+4j)^n}$
$\frac{1}{(1-X)^3}$	$1 - \sqrt{1-2z}$	$\prod_{k=1}^{n}(2k-1) = (2n-1)!!$

Bijection: Many thanks to Alea Young Researchers

General settings

An important lemma (adapted from *Varieties of increasing Trees*)

Let f be the solution of

$$f''(z) = G(f(z)),$$

then, the real dominant singularity ρ of f is given by

$$\rho = \int_0^{\rho_{\mathcal{G}}} \frac{\mathrm{d}t}{\sqrt{f'(0)^2 + 2\int_0^t G(v)\mathrm{d}v}}.$$

General plane diamonds

Theorem

Let $\mathcal{D}$ be the class of general plane inc. diamonds specified by

$$\mathcal{D} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathrm{Seq}(\mathcal{D}) \star \mathcal{Z}^{\blacksquare}.$$

Then, the number d_n of size n diamonds is

$$d_n = \frac{\rho^{1-n}\ n!}{n^2\sqrt{2\log n}} \left(\sum_{0 \leqslant k < K} \frac{P_k(\log\log n)}{(\log n)^k} + \mathcal{O}\left(\frac{(\log\log n)^K}{(\log n)^K} \right) \right),$$

when n tends to ∞.

The sequence d_n is also known as A032035 in OEIS which also enumerates increasing rooted (2,3)-cactus graphs with $n - 1$ nodes.

Many thanks to Alea Young Researchers

General plane diamonds: key ideas

- Define $g = D' - 1$, and note that $g' = e^{g^2/2+g}$
- $z = \int_0^{g(z)} e^{-\frac{t^2}{2}-t}\mathrm{d}t \;=\; \sqrt{2e}\int_{\frac{1}{\sqrt{2}}}^{\frac{g(z)+1}{\sqrt{2}}} e^{-u^2}\mathrm{d}u.$
- $\rho - z = \sqrt{2e}\int_{\frac{g(z)+1}{\sqrt{2}}}^{\infty} e^{-t^2}\mathrm{d}t = \sqrt{\frac{e\pi}{2}}\left(1 - \mathrm{erf}\left(\frac{g(z)+1}{\sqrt{2}}\right)\right).$
- $\rho - z \underset{z\to\rho}{\sim} \sqrt{e}\frac{\exp(-\frac{f'(z)^2}{2})}{f'(z)}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{f'(z)^{2n}}.$
- $\log(1-f(z)) \underset{z\to\rho}{\sim} \log(\rho - z) - \frac{1}{2}\log\log(\frac{1}{1-f(z)}) + \frac{1}{2}\log 2$
-
$$\begin{aligned} f(z) =& 1 - \sqrt{2}(\rho - z)\sqrt{\log\frac{1}{\rho - z}}\left(1 - \frac{\log\log\frac{1}{\rho-z} + \log 2}{4\log\frac{1}{\rho-z}}\right) \\ &+ O\left((\rho - z)\left(\log\frac{1}{\rho - z}\right)^{-\frac{3}{2}}\left(\log\log\frac{1}{\rho - z}\right)^2\right)\end{aligned}$$

Improving the expressivity for synchronization (1)

Definition

$$\mathcal{D} = \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{D}\backslash\mathcal{Z}) + \mathcal{Z}^{\square} \star (\mathcal{E} + \mathcal{D}^2) \star \mathcal{Z}^{\blacksquare}$$

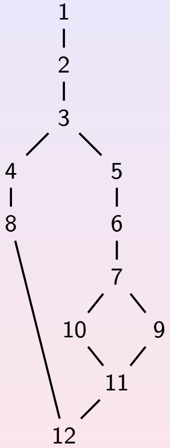
$$D''(z) = D'(z) + D^2(z)$$
with $D(0) = 0$ and $D'(0) = 1$

Psi-series expansion

When z tends to ρ, we prove

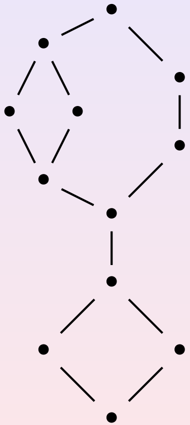
$$D(z) = \sum_{k \geqslant 0} Z^{k-2} \sum_{0 \leqslant \ell \leqslant \lfloor \frac{k}{6} \rfloor} d_{k,\ell} \log^{\ell}(Z),$$

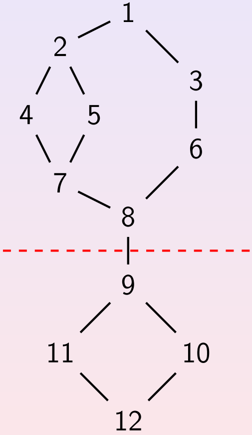
with $Z = 1 - \frac{z}{\rho}$.



Psi-series method for equality of random trees and quadratic convolution recurrences. [ChFeHwMa14]

Improving the expressivity for synchronization (2)



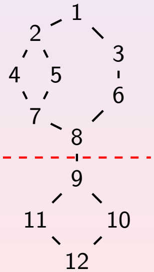


Ordered product

Definition

Let $\mathcal{A}$ and B be labeled combinatorial classes. The ordered product $\mathcal{A} \boxed{\star} \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is the subclass of $\mathcal{A} \star \mathcal{B}$ such that the labels of the $\mathcal{A}$-structures are **smaller** than the labels of the $\mathcal{B}$-structures.

$$\mathcal{A} \boxed{\star} \mathcal{B} = \{(\alpha, f_{|\alpha|}(\beta)) \mid \alpha \in \mathcal{A},\ \beta \in \mathcal{B},\ f_{|\alpha|}(\cdot) \text{ shifts the labels from } \beta \text{ by } |\alpha|\}$$



Generating series

Let C be the generating function of $\mathcal{A} \boxed{\star} \mathcal{B}$, then

$$C(z) = \sum_{n \geqslant 0} \sum_{k=0}^{n} a_k b_{n-k} \frac{z^n}{n!}$$
$$= \int_0^z A(t)B'(z-t)\mathrm{d}t + B(0)A(z) = A(z) * B(z)$$

Similar to the ordinal product on species: [BeLaLe98], [DaPaRoSo12].

Increasing Sets of Increasing Diamonds

Specification

Unlabeled Binary Diamonds: $\mathcal{T} = \mathcal{Z} + \mathcal{Z} \times (\mathcal{E} + \mathcal{T} + \mathcal{T}^2) \times \mathcal{Z}$

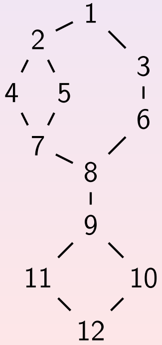
Increasing Binary Diamonds: $\mathcal{D} = \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{E} + \mathcal{D} + \mathcal{D}^2) \times \mathcal{Z}^{\blacksquare}$

Increasing Sets of Increasing Binary Diamonds: $\mathcal{S} = \mathrm{Set}^{\boxed{\star}}(\mathcal{D})$

Number of increasing labelings of binary diamonds:

$$n![z^n]D(z) \underset{n\to\infty}{\sim} 6\frac{(n+1)!}{\rho^{n+2}}$$

with $\rho = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{\frac{2}{3}t^3+t^2+2t}} \approx 3.1721709321\ldots$



Increasing Sets of Increasing Diamonds

Specification

Unlabeled Binary Diamonds: $\mathcal{T} = \mathcal{Z} + \mathcal{Z} \times (\mathcal{E} + \mathcal{T} + \mathcal{T}^2) \times \mathcal{Z}$

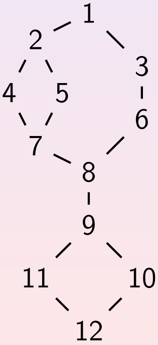
Increasing Binary Diamonds: $\mathcal{D} = \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{E} + \mathcal{D} + \mathcal{D}^2) \times \mathcal{Z}^{\blacksquare}$

Increasing Sets of Increasing Binary Diamonds: $\mathcal{S} = \text{Set}^{\boxed{\star}}(\mathcal{D})$

Number of increasing labelings of sets of binary diamonds:

$$n![z^n]S \underset{n\to\infty}{\sim} n![z^n]D \underset{n\to\infty}{\sim} 6\frac{(n+1)!}{\rho^{n+2}}$$

using a transfer theorem
based on the theorem for supercritical sequences [FlSo93].



Increasing Sets of Increasing Diamonds

Specification

Unlabeled Binary Diamonds: $\mathcal{T} = \mathcal{Z} + \mathcal{Z} \times (\mathcal{E} + \mathcal{T} + \mathcal{T}^2) \times \mathcal{Z}$

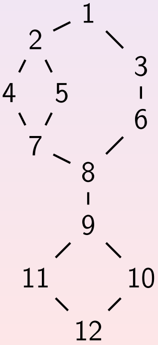
Increasing Binary Diamonds: $\mathcal{D} = \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{E} + \mathcal{D} + \mathcal{D}^2) \times \mathcal{Z}^{\blacksquare}$

Increasing Sets of Increasing Binary Diamonds: $\mathcal{S} = \mathrm{Set}^{\boxed{\star}}(\mathcal{D})$

Average nb. of executions of sets of binary diamonds:

$$\frac{[z^n]\ S(z)}{[z^n]\ (1 - T(z))^{-1}} \underset{n\to\infty}{\sim} 6\ T'(\sigma)\ \frac{\sigma^{n+1} \cdot (n+1)!}{\rho^{n+2}}$$

with $\sigma = \frac{1}{6}(\sqrt{13} - 1)$ is a solution of $T(z) = 1$, and thus $T'(\sigma) = \sqrt{13}/\sigma$ $\qquad \sigma/\rho \approx 0.13689\ldots$.



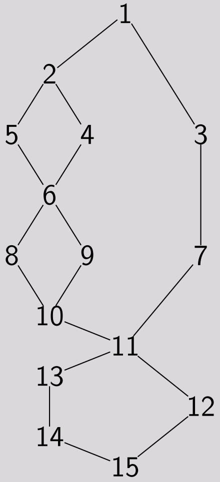
Improving the expressivity for synchronization (3)

Fork-Join processes:

$$\mathcal{F} = \mathcal{Z} + \mathcal{Z} \times \mathcal{F} + \mathcal{Z} \times \mathcal{F}^2 \times \mathcal{F}$$

Increasing labelings of FJ processes:

$$\mathcal{P} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{P} + \mathcal{Z}^{\square} \star ((\mathcal{P} \star \mathcal{P}) \boxed{\star} \mathcal{P})$$



FJ processes are almost equivalent to Series-Parallel Posets.

Fork-Join graphs

Functional equation for the generating function

$$P(z) = z + \int_0^z P(t)\ \mathrm{d}t + \int_0^z \int_0^t P^2(u) P'(t-u)\ \mathrm{d}u\ \mathrm{d}t.$$

Theorem

The average number of executions in size n Fork-Join processes is

$$\Theta\left(n^{5/2}\ \alpha^n\ n!\right),$$

with $\alpha \approx 0.14967\ldots$, when n tends to ∞.

Proof method: working on the **recurrence equation** and use some **bootstrap**.

Combinatorial processes

$$
\begin{tikzcd}
\text{Diamonds} \arrow[d] & & \\
\text{Fork-Join processes} \arrow[dr] & & \text{Trees} \arrow[dl] \\
& \text{Series Parallel processes} \arrow[d] & \\
& \text{Cycle-free processes} \arrow[d] & \\
& \text{All processes} &
\end{tikzcd}
$$

Outline

1. Parallelism and Non-determinism
   - Specification of the processes
   - Counting the average number of executions
2. Parallelism and Synchronization
   - Synchronization
   - Diamond processes and FJ processes
   - Counting the average number of executions
3. Algorithms for Concurrency
   - Counting the exact number of executions
   - Uniform random generation of executions
4. Perspectives

Parallelism and Synchronization

Uniform random generation of executions | Counting

Diamonds [BoDiFoGeHw16]
$O(n \log n)$ | $O(n)$

↓

Fork-Join processes [BoDiGePe17]
$O(n\sqrt{n})$ | $O(n)$

$(\|)$-Trees [BoGePe16]
$O(n \log n)$ | $O(n)$

↘ ↙

Series Parallel processes
$O(n\sqrt{n})$ [BoDiGePe17] | $O(n)$ [Mö87]

↓

Cycle-free processes
$O(n^3 \log n)$ [Hu06] | $O(?)$: Conjecture by [Ha89]

↓

All processes
$O(n^3 \log n)$ [Hu06] | ♯P-complet [BrWi91]

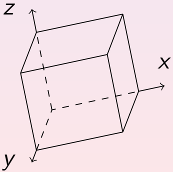
n = process size

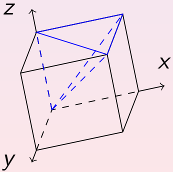
Geometric interpretation of Posets / Processes

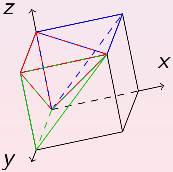
Theorem [St86]

The number of linear extensions of a given poset $\mathcal{P}$ of size n is equal to $n!\mathrm{Vol}(P)$ with P is the polytope associated to $\mathcal{P}$.

partial order	$x \quad y \quad z$	$y \downarrow x \downarrow z$	$x \downarrow z \quad y$
volume	$3! \int_0^1 \int_0^1 \int_0^1 \mathrm{d}z\ \mathrm{d}x\ \mathrm{d}y = 6$	$3! \int_0^1 \int_0^z \int_0^x \mathrm{d}y\ \mathrm{d}x\ \mathrm{d}z = 1$	$3! \int_0^1 \int_0^1 \int_0^z \mathrm{d}x\ \mathrm{d}z\ \mathrm{d}y = 3$

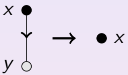




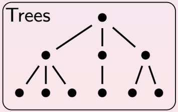


BITC process decomposition

B(ottom)

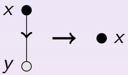


$$\Psi' = \int_x^1 \Psi . dy$$



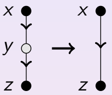
BITC process decomposition

B(ottom)

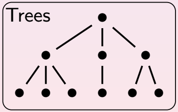


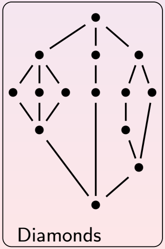
$$\Psi' = \int_x^1 \Psi.dy$$

I(ntermediate)



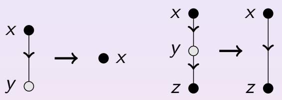
$$\Psi' = \int_x^z \Psi.dy$$



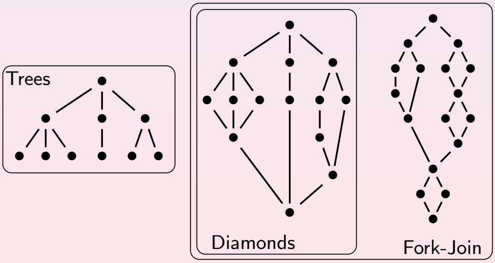


BITC process decomposition

B(ottom) I(ntermediate)

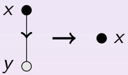


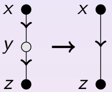
$$\Psi' = \int_x^1 \Psi.dy \qquad \Psi' = \int_x^z \Psi.dy$$



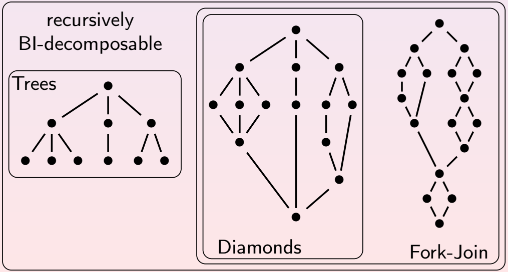
BITC process decomposition

B(ottom) I(ntermediate)



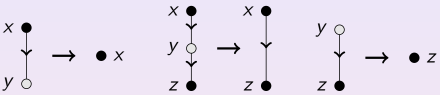


$$\Psi' = \int_x^1 \Psi.dy \qquad \Psi' = \int_x^z \Psi.dy$$

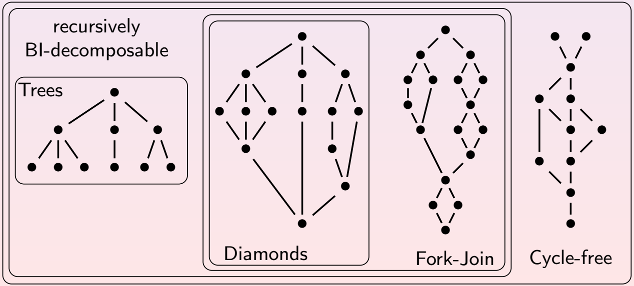


BITC process decomposition

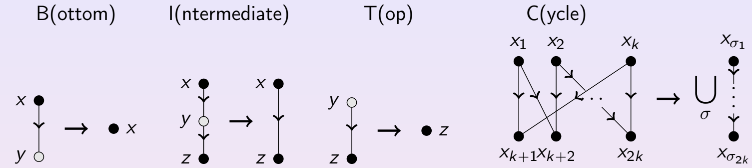
B(ottom) I(ntermediate) T(op)



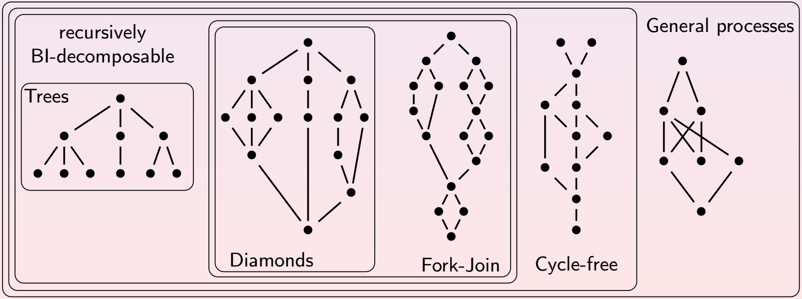
$$\Psi' = \int_x^1 \Psi.dy \qquad \Psi' = \int_x^z \Psi.dy \qquad \Psi' = \int_0^z \Psi.dy$$



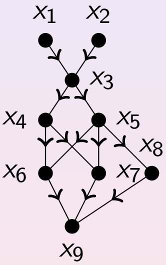
BITC process decomposition



$$\Psi' = \int_x^1 \Psi.dy \qquad \Psi' = \int_x^z \Psi.dy \qquad \Psi' = \int_0^z \Psi.dy \qquad \Psi' = \sum_{\substack{\sigma \text{ a total} \\ \text{order}}} \Psi_\sigma$$

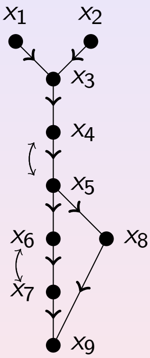


Example (1)



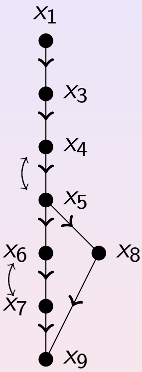
$$\Psi = 1$$

$$\xrightarrow{C_{(4,5,6,7)}}$$



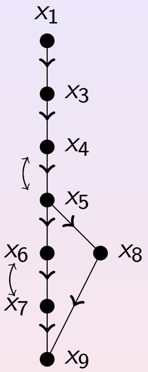
$$\Psi = \begin{array}{l} \Psi_{(4,5,6,7)} \\ +\Psi_{(5,4,6,7)} \\ +\Psi_{(4,5,7,6)} \\ +\Psi_{(5,4,7,6)} \end{array}$$

$$\xrightarrow{T_2}$$



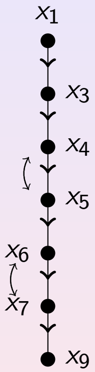
$$\Psi = \int_0^{x_3} \left(\begin{array}{l} \Psi_{(4,5,6,7)} \\ +\Psi_{(5,4,6,7)} \\ +\Psi_{(4,5,7,6)} \\ +\Psi_{(5,4,7,6)} \end{array} \right) \mathrm{d}x_2$$

Example (2)



$$\Psi = \int_0^{x_3} \left(\begin{array}{l} \Psi_{(4,5,6,7)} \\ +\Psi_{(5,4,6,7)} \\ +\Psi_{(4,5,7,6)} \\ +\Psi_{(5,4,7,6)} \end{array} \right) \mathrm{d}x_2$$

$$\xrightarrow{I_8}$$



$$\Psi = \int_{x_5}^{x_9} \int_0^{x_3} \left(\begin{array}{l} \Psi_{(4,5,6,7)} \\ +\Psi_{(5,4,6,7)} \\ +\Psi_{(4,5,7,6)} \\ +\Psi_{(5,4,7,6)} \end{array} \right) \mathrm{d}x_2 \mathrm{d}x_8$$

$$\xrightarrow{B} \quad \ldots$$

Example (3)

$$
\begin{aligned}
Vol(P) &= Vol(P_{(4,5,6,7)}) + Vol(P_{(5,4,6,7)}) + Vol(P_{(4,5,7,6)}) + Vol(P_{(5,4,7,6)}) \\
&= \int_0^{x_3} \left(Vol(P'_{(4,5,6,7)}) + Vol(P'_{(5,4,6,7)}) + Vol(P'_{(4,5,7,6)}) + Vol(P'_{(5,4,7,6)}) \right) \mathrm{d}x_2 \\
&= \int_{x_7}^{1} \int_{x_5}^{x_9} \int_0^{x_3} \left(Vol(P'''_{(4,5,6,7)}) + Vol(P'''_{(5,4,6,7)}) + Vol(P'''_{(4,5,7,6)}) + Vol(P'''_{(5,4,7,6)}) \right) \mathrm{d}x_2 \; \mathrm{d}x_8 \; \mathrm{d}x_9 \\
&= \cdots \\
&= \int_0^1 \int_{x_1}^1 \int_{x_3}^1 \int_{x_4}^1 \int_{x_5}^1 \int_{x_6}^1 \int_{x_7}^1 \int_{x_5}^{x_9} \int_0^{x_3} 1 \; \mathrm{d}x_2 \; \mathrm{d}x_8 \; \mathrm{d}x_9 \; \mathrm{d}x_7 \; \mathrm{d}x_6 \; \mathrm{d}x_5 \; \mathrm{d}x_4 \; \mathrm{d}x_3 \; \mathrm{d}x_1 + \cdots + \cdots + \cdots \\
&= \frac{6+6+8+8}{9!} = \frac{28}{9!}.
\end{aligned}
$$

Hook length formulas (1)

Hook length in ($\|$)-trees [Kn73]

Let T be a process tree (parallelism). The number of executions of T equals:

$$\Psi_T = \frac{|T|!}{\prod_{R \text{ subtree of } T} |R|}.$$

Hook length in ($\|, +$)trees

Let T be a size n process tree (parallelism, non-determinism). The number of executions Ψ_T of T is computed in $O(n^2)$ time complexity.

Do not expand the polynomials in the integrations.

Hook length formulas (2)

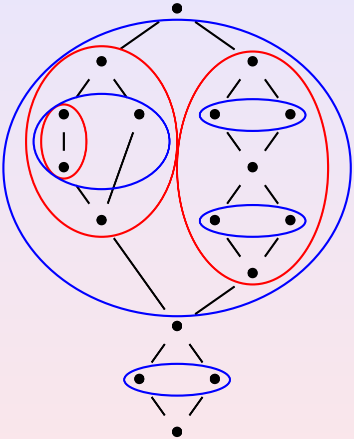
Hook length in Series-Parallel Processes [Mö87]

Let P and Q two Series Parallel posets, then

- $\Psi_{P.Q} = \Psi_P \cdot \Psi_Q$
- $\Psi_{P\|Q} = \binom{|P|+|Q|}{|P|} \cdot \Psi_P \cdot \Psi_Q$

Hook length in Series-Parallel Processes

$$\Psi_P = \frac{\prod_{Y \in \mathcal{P}a} |Y|!}{\prod_{X \in \mathcal{S}e} |X|!}$$



$$\Psi_P = \frac{3!\ 2!\ 2!\ 12!\ 2!}{2!\ 5!\ 7!} = 19008.$$

Perspectives (1)

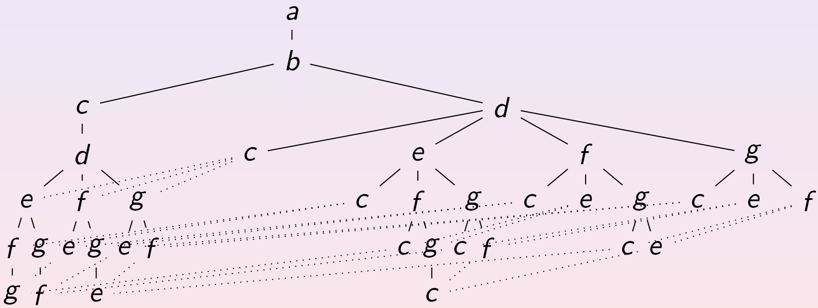
- Parameters in increasing diamonds

Diamonds	Root arity	Path length
$D'' = \exp(D)$	$\Theta(\log n)$	$\Theta(n \log n)$
$D'' = (1-D)^{-3}$	$\Theta(\sqrt{n})$	$\Theta(n \log n)$
$D'' = (1-D)^{-1}$	$\Theta(n/\sqrt{\log n})$	under analysis

- Is the execution counting for cycle-free processes polynomial?

Perspectives (2)

- Compactification of execution trees :
  average size : $\Theta\left(n\, \beta^n\right)$, with $\beta \approx 1.1634\ldots$.



- Uniform random sampling of executions.

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