Introduction to Stokes structures

I: dimension one

Claude Sabbah

Centre de Mathématiques Laurent Schwartz
École polytechnique, CNRS, Université Paris-Saclay
Palaiseau, France

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Ubiquity of the Stokes phenomenon

Various places where the Stokes phenomenon occurs.

- Asympt. behaviour of sols of the Airy Eqn (Stokes...).
- Global behaviour of vanishing cycles of functions $X \to \mathbb{C}$ in alg. geom. (Pham, Berry...)
- Analogy with the theory of wild ramification in Arithmetic (Deligne...).
- Frobenius manifolds and quantum cohomology (Dubrovin...).
- tt* geometry (Cecotti & Vafa...).
- Geometric Langlands correspondence with wild ramification (Frenkel & B. Gross...).
- Wild character varieties (Boalch...).
- Similarities with the theory of stability conditions on some Abelian categories (Bridgeland, Kontsevich...).

Aim: RH corresp. for merom. ODE's

- Riemann-Hilbert corresp. (categorical) on a punctured Riemann surf. $X^* = X \smallsetminus S$:

$$\left\{\begin{array}{l}\text{Merom. flat bdles}\\ \text{on } (X,S)\\ \text{with } \textbf{\textit{reg. sing.}} \text{ at } S\end{array}\right. \iff \left\{\begin{array}{l}\text{loc. cst.}\\ \text{sheaves of}\\ \text{finite rk on } X^*\end{array}\right. \iff \left\{\begin{array}{c}\text{Lin. repres.}\\ \pi_1(X^*)\\ \downarrow\\ \mathrm{GL}_n(\mathbb{C})\end{array}\right.$$

- Riemann-Hilbert-Birkhoff corresp. (categorical) on a punctured Riemann surf. $X^* = X \smallsetminus S$:

$$\left\{\begin{array}{l}\text{Merom. flat bdles}\\ \text{on } (X,S)\end{array}\right. \iff \left\{\begin{array}{l}\text{Stokes-filt.}\\ \text{loc. syst.}\\ \text{on } \widetilde{X}\end{array}\right. \iff \left\{\begin{array}{l}\text{Generalized}\\ \text{monodromy data}\\ \text{(Stokes data)}\end{array}\right.$$

Other approaches

- Explicit computation of sols (integral formulas)
- realizing Stokes data with effective solutions ($\rightsquigarrow$ theory of multisummation)
- Constructing moduli spaces of diff. eqns and realizing the RHB corresp. by a ***map*** between moduli spaces.
- replacing the group $\mathrm{GL}_n(\mathbb{C})$ with other reductive algebraic groups.
- Extending the categorical approach to the Tannakian aspect ($\rightsquigarrow$ Differential Galois theory).

Stokes phenomenon in dim. one

- $\Delta =$ complex disc, complex coord. z.
- Linear cplx diff. eqn. $\mathrm{d}f/\mathrm{d}z = A(z)\cdot f$,
- $A(z)$ matrix of size $d\times d$, merom. pole at $z=0$.
- Gauge equiv.: $P \in \mathrm{GL}_d(\mathbb{C}(\{z\}))$,

$$\boxed{A \sim B = P[A] := P^{-1}AP + P^{-1}P'}$$

- Norm. form: $\boxed{B = \begin{pmatrix}\varphi_1' & & \\ & \ddots & \\ & & \varphi_d'\end{pmatrix} + \dfrac{C}{z}}$ $\quad \varphi_k \in \frac{1}{z}\mathbb{C}[\frac{1}{z}]$, $C =$ const. non reson.

Theorem (Levelt-Turrittin). Given A, $\exists$ a ***formal*** gauge transf. $\widehat{P} \in \mathrm{GL}_d(\mathbb{C}(\!(z^{1/p})\!))$ s.t. $B = \widehat{P}[A]$ is a normal form.

Asympt. analysis in dim. one

- Real or. blow-up: $\widetilde{\Delta} = [0,\varepsilon) \times S^1$, coord. $\rho, e^{i\theta}$.

$$\varpi : \left\{\begin{array}{l}\widetilde{\Delta} \to \Delta\\ S^1 \to 0\end{array}\right. \quad (\rho, e^{i\theta}) \longmapsto z = \rho e^{i\theta}$$

- Sheaf $\boxed{\mathscr{A}_{\widetilde{\Delta}} = \ker \overline{z}\partial_{\overline{z}} : \mathscr{C}^\infty_{\widetilde{\Delta}} \to \mathscr{C}^\infty_{\widetilde{\Delta}}}$ $(\mathscr{A}_{\widetilde{\Delta}^*} = \mathscr{O}_{\widetilde{\Delta}^*})$
- Sheaves $\mathscr{A}^{\mathrm{rd}\,0}_{S^1} \subset \mathscr{A}_{S^1} \subset \mathscr{A}^{\mathrm{mod}\,0}_{S^1}$.
- Basic exact sequence:

$$0 \longrightarrow \mathscr{A}^{\mathrm{rd}\,0}_{S^1} \longrightarrow \mathscr{A}_{S^1} \longrightarrow \varpi^{-1}\mathbb{C}[\![z]\!] \longrightarrow 0$$

Asympt. analysis in dim. one

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- Sheaves $\mathscr{A}^{\mathrm{rd}\,0}_{S^1} \subset \mathscr{A}_{S^1} \subset \mathscr{A}^{\mathrm{mod}\,0}_{S^1}$.
- Example: $\varphi = u(z)/z^q$ s.t. $u(z) \in \mathbb{C}[z]$, $q \geqslant 1$, and $u(0) \neq 0$ or $u(z) \equiv 0$. Then $\forall \alpha \in \mathbb{C}$ and $\forall e^{i\theta_o} \in S^1$

$$z^\alpha e^\varphi \in \left\{\begin{array}{l}\mathscr{A}^{\mathrm{rd}\,0}_{\theta_o} \iff \mathrm{Re}(u(0)e^{-ik\theta_o}) < 0,\\ \mathscr{A}^{\mathrm{mod}\,0}_{\theta_o} \iff \text{idem or } u(z) \equiv 0.\end{array}\right.$$

Asympt. analysis in dim. one

Theorem (Hukuhara-Turrittin).
Locally on S^1, $\exists$ a lifting $\widetilde{P} \in \mathrm{GL}_d(\mathscr{A}_{S^1}[1/z])$ of $\widehat{P}$ s.t. $\widetilde{P}[A] = \widehat{P}[A] = B$ normal form.

Corollary. The sheaf on $\widetilde{\Delta}$ of sols of

$$\mathrm{d}f/\mathrm{d}z = A(z)\cdot f$$

having entries in $\mathscr{A}^{\mathrm{rd}\,0}_{\widetilde{\Delta}}$, resp. in $\mathscr{A}^{\mathrm{mod}\,0}_{\widetilde{\Delta}}$, is a real constr. sheaf, constant on any open interval I of S^1 s.t. $\forall k$, $\mathrm{Re}(\varphi_k)$ does not vanish.

Example. $\varphi = z^{-q}u(z)$, $u(0) \neq 0$,

On S^1, $\quad \mathrm{Re}\,\varphi = 0 \iff \theta = \dfrac{1}{q}(\arg u(0) + \pi/2) \mod \mathbb{Z}\cdot\pi/q.$

The Malgrange-Sibuya theorem

Fix a norm. form (***irregular type***), e.g. non-ramified:

$$B = \mathrm{diag}(\varphi_1', \dots, \varphi_d') + \frac{C}{z}.$$

B*-marked connections** (∼: ***holom. gauge equiv.):

$$\mathrm{Iso}(B) = \Big\{(A, \widehat{P}) \mid B = \widehat{P}[A]\Big\}\Big/\sim$$

Stokes sheaf $\mathrm{St}(B)$ on S^1:

$$\mathrm{St}(B)_\theta = \Big\{\mathrm{Id} + Q \mid Q \in \mathrm{End}(\mathscr{A}_\theta^{\mathrm{rd}\,0}),\ (\mathrm{Id}+Q)[B] = B\Big\}$$

Theorem (Malgrange-Sibuya).

$$\boxed{\mathrm{Iso}(B) \simeq H^1(S^1, \mathrm{St}(B))}$$

Stokes-filtered loc. syst. (non-ramif. case)

- ***Aim***: To specify the struct. of sol. space of a merom. ODE without
  - making explicit the realization as functions,
  - fixing the normal form.
- The local system $\mathscr{L}$ on S^1: Sols of $\mathrm{d}f/\mathrm{d}z = A(z)f$ on Δ^*, extended to $\widetilde{\Delta} = [0, \varepsilon) \times S^1$ and restricted to $\{0\} \times S^1$. Hence $\mathscr{L} \iff$ monodromy of sols.
- For every $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, a pair of nested subsheaves $\mathscr{L}_{<\varphi} \subset \mathscr{L}_{\leqslant\varphi}$ of $\mathscr{L}$.

$$\boxed{\begin{aligned}\mathscr{L}_{\leqslant\varphi,\theta} &= \{f_\theta \mid e^{-\varphi} f(z) \in \mathscr{A}_\theta^{\mathrm{mod}\,0}\}\\ \mathscr{L}_{<\varphi,\theta} &= \{f_\theta \mid e^{-\varphi} f(z) \in \mathscr{A}_\theta^{\mathrm{rd}\,0}\}\end{aligned}}$$

- Hukuhara-Turrittin $\Rightarrow \mathscr{L}_{<\varphi} = \mathscr{L}_{\leqslant\varphi}$ except if $\varphi = \varphi_k$ for some $k = 1, \dots, d$.

Stokes-filtered loc. syst. (non-ramif. case)

- ***Aim***: To give an intrinsic characterization of the category of Stokes-filtered local systems.

Definition. A (non-ramif.) Stokes-filt. loc. syst. on S^1:

  - A loc. syst. $\mathscr{L}$ on S^1,
  - $\forall \varphi \in z^{-1}\mathbb{C}[z^{-1}]$, an $\mathbb{R}$-const. subsheaf $\mathscr{L}_{\leqslant\varphi} \subset \mathscr{L}$

s.t.

- $\forall \theta \in S^1,\ \mathscr{L}_{\leqslant\psi,\theta} \subset \mathscr{L}_{\leqslant\varphi,\theta} \iff$ $\psi = \varphi$, or $\mathrm{Re}(\psi - \varphi) < 0$ near θ'
- setting $\forall\theta$, $\mathscr{L}_{<\varphi,\theta} = \sum_{\psi <_\theta \varphi} \mathscr{L}_{\leqslant\psi,\theta}$

$\rightsquigarrow$ $\boxed{\mathscr{L}_{<\varphi} \quad \text{and} \quad \mathrm{gr}_\varphi \mathscr{L} := \mathscr{L}_{\leqslant\varphi}/\mathscr{L}_{<\varphi}}$

one asks that $\forall\varphi$,

  - $\mathrm{gr}_\varphi \mathscr{L}$ is a local system on S^1,
  - $\forall\theta$, $\dim \mathscr{L}_{\leqslant\varphi,\theta} = \sum_{\psi \leqslant_\theta \varphi} \mathrm{rk}\,\mathrm{gr}_\psi \mathscr{L}$.

- Remark: can define $(\mathscr{L}, \mathscr{L}_\bullet)$ over $\mathbb{Z}, \mathbb{Q}, \dots$

Stokes-filtered loc. syst. (non-ramif. case)

Let $(\mathscr{L}, \mathscr{L}_\bullet)$ be a non-ramif. Stokes-filt. loc. syst.

- $\Phi := \{\varphi \mid \mathrm{rk}\,\mathrm{gr}_\varphi \mathscr{L} \neq 0\}$ is ***finite*** and $\sum_{\varphi\in\Phi} \mathrm{rk}\,\mathrm{gr}_\varphi \mathscr{L} = \mathrm{rk}\,\mathscr{L}$.
- $\forall \varphi \in \Phi$, $\forall\theta$, $\boxed{\mathscr{L}_{\leqslant\varphi,\theta} \overset{(*)}{\simeq} \bigoplus_{\psi \leqslant_\theta \varphi} \mathrm{gr}_\psi \mathscr{L}_\theta.}$
- ***Level structure***

Levels of B (hence A) : $\{q_1 < \cdots < q_r\}$

$q_i :=$ pole ord. of some $\psi - \varphi$, $\quad \varphi \neq \psi \in \Phi$.

  - $\#\mathrm{Levels}(A) = 1$. $\rightsquigarrow$ theory of summability. $2q$ Stokes directions for ***each*** (φ, ψ).
  - $\#\mathrm{Levels}(A) > 1$. $\rightsquigarrow$ theory of multisummability. Principal and Secondary Stokes directions.

Stokes-filtered loc. syst. (non-ramif. case)

Theorem

- $\forall$ open $I \subset S^1$ which $\ni$ at most one Stokes dir. $\forall$ pair in Φ, then $(*)$ holds on I (e.g. $|I| \leqslant \pi/q_r + \varepsilon$).
- Any morphism $\lambda : (\mathscr{L}, \mathscr{L}_\bullet) \to (\mathscr{L}', \mathscr{L}'_\bullet)$ ***graded*** on I w.r.t. some iso $(*)$ and $(*)'$, hence is ***strict***, *i.e.*, $\forall\varphi,\quad \lambda(\mathscr{L}_{\leqslant\varphi}) = \mathscr{L}'_{\leqslant\varphi} \cap \lambda(\mathscr{L})$.
- Uniqueness of the splitting if $\#\mathrm{Level}(A) = 1$ and moreover $|I| = \pi/q + \varepsilon$.

Duality.

- The exact sequences

$$0 \longrightarrow \mathscr{L}_{\leqslant\varphi} \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}^{>\varphi} \longrightarrow 0$$
$$0 \longrightarrow \mathscr{L}_{<-\varphi} \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}^{\geqslant-\varphi} \longrightarrow 0$$

are switched by duality $\mathscr{H}\!om_{\mathbb{C}}(\bullet, \mathbb{C})$.

$\Rightarrow \mathrm{gr}_\varphi(\mathscr{L}^\vee) \simeq (\mathrm{gr}_{-\varphi} \mathscr{L})^\vee$. $\quad \mathscr{E}xt^k(\bullet, \mathbb{C}) = 0$ if $k \geqslant 1$.

Deligne's RH correspondence

Theorem (Deligne's RH corresp.).

Merom. flat bdles on $(\Delta, 0)$	norm. form $\longmapsto$	Merom. flat bdles of norm. form on $(\Delta, 0)$
$\downarrow \wr$		$\downarrow \wr$
Stokes-filt. loc. syst. on S^1	gr $\longmapsto$	graded Stokes-filt. loc. syst. on S^1

Stokes data (non-ramif. case, pure level)

Case $\#\mathrm{Level}(A) = 1$ (level $= q$)

- ***Stokes data***
  - $(L_\ell)_{\ell\in\mathbb{Z}/2q\mathbb{Z}}$: $\mathbb{C}$-vect. spaces,
  - Isoms $S_\ell^{\ell+1} : L_\ell \xrightarrow{\sim} L_{\ell+1}$
  - Exhaustive filtrations $\begin{cases} F_\bullet L_{2\mu} & \nearrow \\ F^\bullet L_{2\mu+1} & \searrow \end{cases}$
- ***Opposedness property***:

$$L_{2\mu} = \bigoplus_k F_k L_{2\mu} \cap S_{2\mu-1}^{2\mu}(F^k L_{2\mu-1})$$

$$L_{2\mu+1} = \bigoplus_k F^k L_{2\mu+1} \cap S_{2\mu}^{2\mu+1}(F_k L_{2\mu})$$

Stokes data (non-ramif. case, pure level)

Case $\#\mathrm{Level}(A) = 1$ (level $= q$)

- Opposed filtrations $\Rightarrow$ ***unique*** splittings

$$\tau_{2\mu} : L_{2\mu} \xrightarrow{\sim} \mathrm{gr}^F L_{2\mu} = \bigoplus_k \mathrm{gr}_k^F L_{2\mu}$$

$$\tau_{2\mu+1} : L_{2\mu+1} \xrightarrow{\sim} \mathrm{gr}_F L_{2\mu+1} = \bigoplus_k \mathrm{gr}_F^k L_{2\mu+1}$$

- $\rightsquigarrow$ ***Stokes multipliers***

$$\Sigma_\ell^{\ell+1} := \tau_{\ell+1} \circ S_\ell^{\ell+1} \circ \tau_\ell^{-1} : \mathrm{gr}_F L_\ell \longrightarrow \mathrm{gr}_F L_{\ell+1}$$

  - $\Sigma_\ell^{\ell+1}$ block lower/upper triangular,
  - diag. blocks $(\Sigma_\ell^{\ell+1})_{jj}$ are isos.

Stokes data (non-ramif. case, pure level)

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$(\mathscr{L}, \mathscr{L}_\bullet)$ Stokes-filt. loc. syst. pure level q
$\overset{\theta_0}{\Longleftrightarrow}$ Stokes data of pure level q.

- Fix $\theta_0 \in S^1$ not a Stokes dir. $\Rightarrow$ numbering of Φ s.t.

$$\varphi_1 <_{\theta_0} \cdots <_{\theta_0} \varphi_r$$

- $2q$ Stokes dirs $(\theta_\ell := \theta_0 + \ell\pi/q)_{\ell\in\mathbb{Z}/2q\mathbb{Z}}$ on S^1.

$$\Longrightarrow \begin{cases} \varphi_1 <_{\theta_{2\mu}} \cdots <_{\theta_{2\mu}} \varphi_r \\ \varphi_r <_{\theta_{2\mu+1}} \cdots <_{\theta_{2\mu+1}} \varphi_1 \end{cases}$$

- $\Longrightarrow (\mathscr{L}_{\leqslant\varphi_j,\theta_\ell})_j$: $\begin{cases} \text{filt. } \nearrow \text{ if } \ell = 2\mu \\ \text{filt. } \searrow \text{ if } \ell = 2\mu+1 \end{cases}$

Stokes data (non-ramif. case, pure level)

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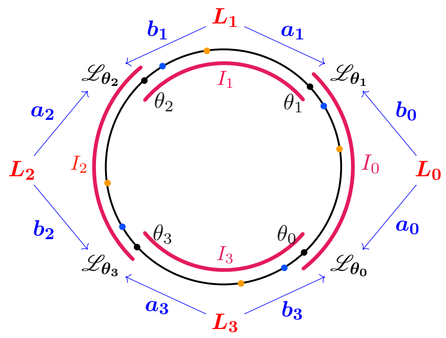
$(\mathscr{L}, \mathscr{L}_\bullet)$ Stokes-filt. loc. syst. pure level q (e.g. $q = 2$)
$\overset{\theta_0}{\Longleftrightarrow}$ Stokes data of pure level q. $L_i = \Gamma(I_i, \mathscr{L})$



Stokes data (non-ramif. case, pure level)

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$(\mathscr{L}, \mathscr{L}_\bullet)$ Stokes-filt. loc. syst. pure level q (e.g. $q = 2$)
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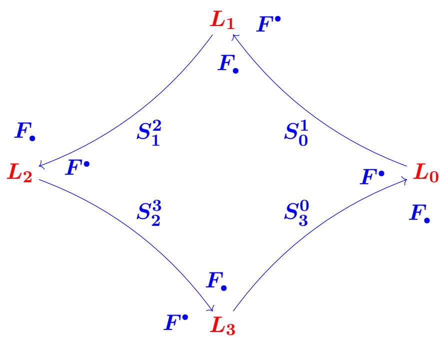

Stokes data (non-ramif. case)

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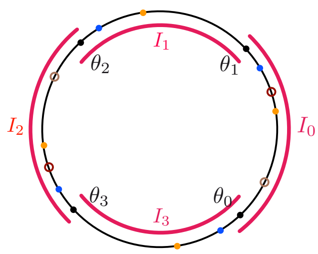
$(\mathscr{L}, \mathscr{L}_\bullet)$ Stokes-filt. loc. syst. max level q_r
(e.g. $q_1 = 1$, $q_2 = 2$)



$\longrightarrow$ more difficult to describe the char. properties of Stokes data $(L_\ell, S_\ell^{\ell+1})$

Examples

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- ***How*** to compute Stokes data?
- Use of the Fourier transf. (Marco's talk):
  more complicated $\Longleftarrow$ simpler.
- Explicit procedures (..., Mochizuki), but difficult to obtain closed formulas for Stokes data.
  - e.g. Airy diff. eq. (ramified)

$$(\partial_y^2 - y)u = 0 \quad \overset{z=1/y}{\rightsquigarrow} \quad A(z) = -\frac{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{z^4} + \frac{\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}}{z}$$

  - Unramified case, $q = 1$: Fourier transform of a reg. sing. diff. eq. on $\mathbb{P}^1$.
  - Gaussian type: irreg. sing. at ∞, unramif., $q = 2$.
  - FT of $\mathscr{E}^\varphi$: irreg. sing. at ∞, possibly ramif.

Dubrovin's conjecture

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- X: smooth proj. Fano var. which admits a full exceptional collection:
  - $(E_1, \ldots, E_m) \in \mathsf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ generate as a triang. category,
  - for $i \neq j$, $\mathrm{Ext}^k(E_i, E_j) = 0$ except $i < j$ and $k = k(i, j)$,
  - $\mathrm{Ext}^k(E_i, E_i) = 0$ except $k = \begin{cases} 0, & \mathrm{Hom} = \mathbb{C}, \\ \dim X \end{cases}$
- $\rightsquigarrow S_X = (S_{ij}) \in \mathsf{M}_m(\mathbb{Z})$,

$$S_{ij} = \chi(E_i, E_j) := \sum_k (-1)^k \dim \mathrm{Ext}^k(E_i, E_j).$$

Dubrovin's conjecture

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- $f : U \to \mathbb{C}$ a tame reg. fnct. on a smooth affine var.
- $\rightsquigarrow$ finite $\#$ of crit. pts.
- GM syst.:

$$\mathrm{GM} := \left(\frac{\Omega^{\max}(U)[z, z^{-1}]}{(z\mathrm{d} + \mathrm{d}f)\Omega^{\max -1}(U)[z, z^{-1}]}, \quad z^2\partial_z - f\right)$$

- GM at $z = 0$: one level, $q = 1$, Stokes matr. $S_f^\pm$, can be chosen with entries in $\mathbb{Z}$.
  Diag. blocks $\leftrightarrow$ monodr. of vanishing cycles of f.

Dubrovin's conjecture

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Dubrovin's conjecture.
If f is the Landau-Ginzburg potential mirror to X good Fano, then $\exists$ choices a bases of vanishing cycles of f s.t.

$$\boxed{S_f^+ = S_X}$$

- $X = \mathbb{P}^n$ (Dubrovin, Guzzetti)
- $\ldots$