# Introduction to Stokes Structures

## IV: explicit computations of one dim'l cases via Fourier-Laplace

## CIRM, April 14, 2017

Marco Hien
Universität Augsburg

Universität Augsburg University

Aim:

- illustrate how the two approaches to the irregular Riemann-Hilbert problem

| Stokes-filtered local systems, Deligne-Malgrange-Sabbah | Enhanced ind-sheaves, d'Agnolo-Kashiwara |
|---|---|
| cp. Claude's talks | cp. Andrea's talk |

can be applied to determine the Stokes structure for some one-dimensional holonomic D-modules arising as

- the Fourier transform

of a less complicated module

in a topological way (without (multi-)summation)

given as

- linear Stokes data,
- Stokes multipliers

cp. Claude's talk on Monday

(on a joint work with C. Sabbah, Rend.Sem.Math.Univ.Padova 2015)

Situation:

- $\rho : u \mapsto t = u^p$ a ramification map,
- $\varphi(u) \in u^{-1}\mathbb{C}[u^{-1}]$ an exponential of pole order $q$,
- $R$ a regular singular connection at $u = 0$ with monodromy data $(V, T)$, extended to a free $\mathbb{C}[u, u^{-1}]$-module having regular singularity at 0 and $\infty$
- and

$$\mathcal{M} := \mathrm{El}(\rho, -\varphi, R) := \rho_+(\mathcal{E}^{-\varphi} \otimes R).$$

Consider the Fourier transform $\widehat{\mathcal{M}}$, which has a formal structure (Fan, Sabbah)

$$\widehat{\mathcal{M}}^{\wedge\infty} \simeq \mathrm{El}(\widehat{\rho}, \widehat{\varphi}, \widehat{R}),$$

where

- ramification order of $\widehat{\rho}$ is $p + q$,
- pole order of $\widehat{\varphi}$ is $q$ (in particular, have only 1 level assuming $p, q$ coprime),
- $\widehat{R}$ has monodromy $(V, (-1)^q T)$.

## Aim:

Determine the Stokes structure of $\widehat{\rho}^{+}\widehat{\mathcal{M}}$

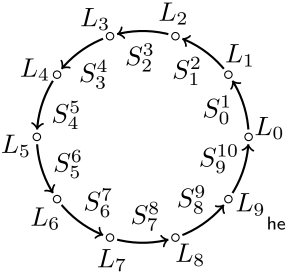
(i.e. avoiding ramification. To include it, this would amount to understand $\mu_{p+q}$-action)

in terms of linear Stokes data (recall Claude's talk on Monday):

- vector spaces and isomorphisms as in the picture,
- increasing filtration $F_\bullet L_{\text{even}}$,
- decreasing filtration $F^\bullet L_{\text{odd}}$,
- opposed to each other with respect to the isomorphisms $S_j^{j+1}$.

$$L_{2\mu} = \bigoplus_k F_k L_{2\mu} \cap S_{2\mu-1}^{2\mu}(F^k L_{2\mu-1})$$

$$L_{2\mu+1} = \bigoplus_k F^k L_{2\mu+1} \cap S_{2\mu}^{2\mu+1}(F_k L_{2\mu})$$




here: $q = 5$

## Geometric $\mathcal{D}$-module Fourier transform

In the notation

$$\begin{array}{ccc} & \mathbb{A}_t \times \mathbb{G}_{m,\eta} & \\ {}^{\pi}\swarrow & & \searrow^{\widehat{\pi}} \\ \mathbb{A}_t & & \mathbb{G}_{m,\eta} \end{array}$$

we have

$$(\widehat{\rho}^{+}\widehat{\mathcal{M}})_{\infty} = \mathbb{C}(\{\eta\}) \otimes_{\mathbb{C}[\eta,\eta^{-1}]} \widehat{\pi}_{+}(\pi^{+}M \otimes E^{-t/\widehat{\rho}(\eta)}).$$

Let $\widehat{\mathcal{L}}$ be the local system of $\widehat{\mathcal{M}}$ at the circle $S^1_\infty$ at infinity.

Stokes filtration via moderate deRham complexes

$$\underbrace{\mathrm{DR}^{\mathrm{mod}\widehat{\infty}}(\widehat{\rho}^{+}\widehat{\mathcal{M}} \otimes \mathcal{E}^{-\widehat{\psi}(\eta)})}_{\simeq \widehat{\mathcal{L}}_{\leq \widehat{\psi}}}$$

Geometric $\mathcal{D}$-module Fourier transform

In the notation

$$\begin{array}{ccc} & \mathbb{A}_t \times \mathbb{G}_{\mathrm{m},\eta} & \\ \overset{\pi}{\swarrow} & & \overset{\widehat{\pi}}{\searrow} \\ \mathbb{A}_t & & \mathbb{G}_{\mathrm{m},\eta} \end{array}$$

we have

$$(\widehat{\rho}^+\widehat{\mathcal{M}})_\infty = \mathbb{C}(\{\eta\}) \otimes_{\mathbb{C}[\eta,\eta^{-1}]} \widehat{\pi}_+(\pi^+ M \otimes E^{-t/\widehat{\rho}(\eta)}).$$

Let $\widehat{\mathcal{L}}$ be the local system of $\widehat{\mathcal{M}}$ at the circle $S^1_\infty$ at infinity.

Stokes filtration via moderate deRham complexes

$$\underbrace{\mathrm{DR}^{\mathrm{mod}\widehat{\infty}}(\widehat{\rho}^+\widehat{\mathcal{M}} \otimes \mathcal{E}^{-\widehat{\psi}(\eta)})}_{\simeq \widehat{\mathcal{L}}_{\leq \widehat{\psi}}} \xrightarrow{\simeq} R\widetilde{\pi}_* \mathrm{DR}^{\mathrm{modD}}(\pi^+\mathcal{M} \otimes E^{-\widehat{\psi}(\eta) - t/\widehat{\rho}(\eta)})[1]$$

Isomorphism due to T. Mochizuki.

Universität Augsburg University

# Topological computations:

There is a problem here:

$$\pi^+\mathcal{M} \otimes E^{-\widehat{\psi}(\eta)-t/\widehat{\rho}(\eta)}$$

for the necessary choices $\widehat{\psi}(\eta) = \widehat{\varphi}(\zeta^j\eta)$ for $\zeta \in \mu_{p+q}$
(the exponentials of the Fourier transform)
contains exponentials

$$\varphi(u) - \widehat{\psi}(\eta) - \rho(u)/\widehat{\rho}(\eta)$$

with indeterminancies.

Consequently

$$\mathrm{DR}^{\mathrm{modD}}(\pi^+\mathcal{M} \otimes E^{\widehat{\psi}(\eta)-t/\widehat{\rho}(\eta)})$$

is not concentrated in one degree and therefore hard to understand (in particular its $R\widetilde{\pi}_*$).

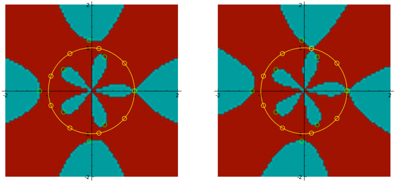
## Blowing-up

these indeterminancies gives rise to a good situation in Mochizuki's sense and hence the possibiliy to compute the $R\widetilde{\pi}_*$ of a sheaf, one can understand rather easily (and not a complex).

$\rightsquigarrow$ topological computation – even better: can define topological Fourier transform of Stokes-filtered local system compatible with the $\mathcal{D}$-module version.

### Example of such a sheaf:

In the case $p = 4, q = 5$, typical fibres w.r.t. $\widetilde{\pi}$ are



The sheaf restricted to this fibre, call it $\mathcal{G}$, is

- determined by the local system $\mathcal{L}$ inside the turquoise region and
- zero inside the red region.

$$(\widehat{\mathcal{L}}_{\leq \widehat{\psi}})_\theta = H^1_c(\text{fibre over } \theta; \mathcal{G}) \text{ for } \theta \in S^1_\infty.$$

# Conclusion for $\mathcal{M} = \mathrm{El}(\rho, \varphi, R)$

- Can compute the linear Stokes data of $\widehat{\rho}^+ \widehat{\mathcal{M}}$ at $\infty$ purely topologically
  (direct images of $\mathbb{R}$-constructible sheaves, Leray-covering, ...)
- Example: $p = 4$, $q = 5$, $R = (V, T)$:

$$L_j := V^{\oplus p+q} =: \bigoplus_{k=0}^{p+q-1} V \otimes 1_k \text{ for all } j,$$

$$F_k L_{2\mu} = \bigoplus_{\nu \leq k} V \otimes 1_\nu$$

$$F^k L_{2\mu+1} = \bigoplus_{\nu \geq k} V \otimes 1_\nu$$

Universität Augsburg University

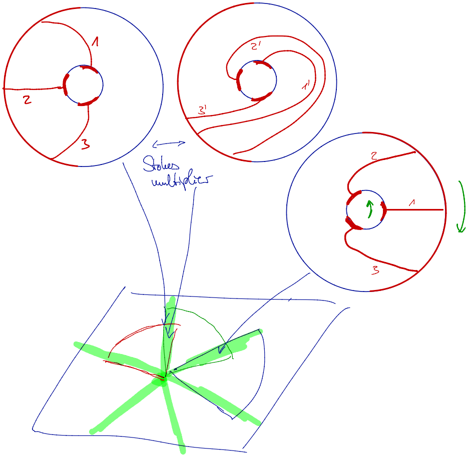
- Example continued:

$$S_{\text{odd}}^{\text{even}} = \begin{pmatrix} 1 & 1 & 0 & & & & & & \\ 0 & -1 & 0 & & & & & & \\ 0 & 1 & 1 & 1 & 0 & & & & \\ & & 0 & -1 & 0 & & & & \\ & & 0 & 1 & 1 & 1 & 0 & & \\ & & & & 0 & -1 & 0 & & \\ & & & & 0 & 1 & 1 & 1 & 0 \\ & & & & & & 0 & -1 & 0 \\ & & & & & & 0 & 1 & 1 \end{pmatrix}$$

$$S_{\text{odd}}^{\text{even}} = \begin{pmatrix} 1 & & & & & & & 0 & T \\ & 1 & 1 & 0 & & & & & \\ & 0 & -1 & 0 & & & & & \\ & 0 & 1 & 1 & 1 & 0 & & & \\ & & & 0 & -1 & 0 & & & \\ & & & 0 & 1 & 1 & 1 & 0 & \\ & & & & & 0 & -1 & 0 & \\ 0 & & & & & 0 & 1 & 1 & 1 \\ 0 & & & & & & & 0 & -1 \end{pmatrix}$$

all the same with exception of $S_9^0$ which includes $T$.

## Another example: Airy



## d'Agnolo-Kashiwara's R-H correspondence:

Let $X$ be a complex analytic manifold. Then we have:

$$D^b_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\mathcal{S}ol^E_X} E^b_{\mathbb{R}\text{-c}}(\mathbb{C}^{\mathrm{sub}}_X) \longrightarrow D^b(\mathcal{D}_X)$$

fully faithful — reconstruction functor

Important ingredients/constructions:

- bordered spaces
- convolution
- $\mathbb{R}$-constructibility
- the 'usual' functors like $Ef_{!!}, Ef^{-1}, \ldots$ and their compatibilities,
- $\mathcal{O}^E$ and hence $\mathcal{S}ol^E$

## Some definitions/facts:

- $\mathbb{C}_X^{\mathsf{E}} := \text{“}\varinjlim_{c\to\infty}\text{“}\, \mathbb{C}_{\{t\geq c\}}$
- Fully faithful embedding

$$e : D^b(\mathbb{C}_X^{\text{sub}}) \to E^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X^{\text{sub}}), F \mapsto \mathbb{C}_X^{\mathsf{E}} \otimes \pi^{-1}(F).$$

- $\mathcal{E}^{\varphi}_{U/X} := \mathcal{D}_X e^{\varphi}(*Y)$ for $U \subset X,\ Y = X \setminus U$.

### Proposition

We have

$$\mathcal{S}ol^E_X(\mathcal{E}^{\varphi}_{U/X}) \simeq \underbrace{\mathbb{C}_X^E \overset{+}{\otimes} \overbrace{\mathbb{C}_{\{t+\mathrm{Re}\,\varphi(x)\geq 0\}}}^{=: e^{\varphi}_{U/X}}}_{=: E^{\varphi}_{U/X}} \in E^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X^{\text{sub}}).$$

Universität Augsburg University

$$
\begin{array}{ccccccc}
 & & & \mathbb{A}\times\mathbb{A}^*\times\mathbb{R} & & & \\
 & \swarrow^{\widetilde{p}} & & \downarrow^{\overline{p}} & & \searrow^{\widetilde{q}} & \\
\mathbb{A}\times\mathbb{R} & & & \mathbb{A} & & & \mathbb{A}^*\times\mathbb{R} \xrightarrow{\widetilde{k}} \mathbb{P}^*\times\mathbb{R}
\end{array}
$$

no $\mathbb{R}$, no $\widetilde{\ }, \overline{\ }$

- $\mathcal{D}$-module:

$$
\begin{aligned}
{}^\wedge : D^b_{\mathrm{hol}}(\mathcal{D}_{\mathbb{A},\infty}) &\to D^b_{\mathrm{hol}}(\mathcal{D}_{\mathbb{A},\infty}) \\
\mathcal{M} &\mapsto \widehat{\mathcal{M}} := Dq_*(\mathcal{E}^{zw}_{\mathbb{A}\times\mathbb{A}^*/\mathbb{P}\times\mathbb{P}^*} \otimes Dp^*\mathcal{M}).
\end{aligned}
$$

- Enhanced sheaves:

$$
\begin{aligned}
{}^\curlywedge : \widetilde{E}^b_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathbb{A}\times\mathbb{R},\infty}) &\to \widetilde{E}^b_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathbb{A}\times\mathbb{R},\infty}) \\
F &\mapsto F^\curlywedge := R\widetilde{q}_!(e^{zw}_{\mathbb{A}\times\mathbb{A}^*/\mathbb{P}\times\mathbb{P}^*} \overset{*}{\otimes} \widetilde{p}^{-1}F).
\end{aligned}
$$

- Enhanced ind-sheaves:

$$
\begin{aligned}
{}^\wedge : E^b_{\mathbb{R}\text{-c}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{A}}) &\to E^b_{\mathbb{R}\text{-c}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{A}}) \\
K &\mapsto K^\wedge := Eq_{!!}(E^{zw}_{\mathbb{A}\times\mathbb{A}^*/\mathbb{P}\times\mathbb{P}^*} \overset{+}{\otimes} p^{-1}F).
\end{aligned}
$$

(on a joint work with A. d'Agnolo, G. Morando and C. Sabbah)

The setting:

- $\mathcal{M}$ a regular singular $\mathcal{D}$-module on the affine line $\mathbb{A}$, localized at $\infty$ (notation $\mathrm{Mod}(\mathcal{D}_{\mathbb{A},\infty})$), with singularities $\Sigma \subset \mathbb{A}$,
- $F := \mathcal{S}ol_{\mathbb{P}}(\mathcal{M})|_{\mathbb{A}}$ the solutions complex, a perverse sheaf on $\mathbb{A}$.

## Known facts about the Fourier transform $\widehat{\mathcal{M}}$:

- regular singular at 0, irregular singular at $\infty$ and no other singular points,
- the Hukuhara-Levelt-Turritin formal decomposition has the form

$$\widehat{\mathcal{M}}^{\wedge\infty} \simeq \bigoplus_{c\in\Sigma} E^{\frac{c}{x}} \otimes R_c$$

in the coordinate $x$ of $\mathbb{P}^*$ centered at $\infty$.

Aim:
Determine the Stokes structure of the Fourier transform $\widehat{\mathcal{M}}$ at $\infty \in \mathbb{P}^*$.

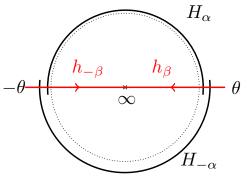
Universität Augsburg University

Aim:

Determine the Stokes structure of the Fourier transform $\widehat{\mathcal{M}}$ at $\infty \in \mathbb{P}^*$.

We know a priori:

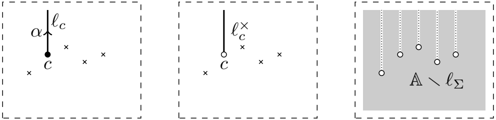
- exponential factors are $\{\varphi(x) = \frac{c}{x} \mid c \in \Sigma\}$ of slope 1,
- hence, after choosing a starting direction $\theta \in S^1_\infty$ of the real oriented blow-up of $\mathbb{P}^*$ at $\infty$, it suffices to consider two sectors:



## Linear algebra data associated to $F$, the quiver of $F$ :

After choice of a suitable pair $(\alpha, \beta) \in \mathbb{A} \times \mathbb{A}^*$ (fixing a prefered direction/orientation)

- vanishing cycles $\Phi_c(F) := R\Gamma_c(\mathbb{A}; \mathbb{C}_{\ell_c^\times} \otimes F)$
- (local) nearby cycles $\Psi_c(F) := R\Gamma_c(\mathbb{A}; \mathbb{C}_{\ell_c} \otimes F)$
- (global) nearby cycles $\Psi(F) := R\Gamma_c(\mathbb{A}; \mathbb{C}_{\mathbb{A} \smallsetminus \ell_\Sigma} \otimes F)$




and linear maps

$$\Psi(F) \underset{v_c}{\overset{u_c}{\rightleftarrows}} \Phi_c(F)$$

such that $1 - uv$ is invertible.

Consider the projections/inclusion

$$\begin{array}{ccccc} & & \mathbb{A} \times \mathbb{A}^* \times \mathbb{R} & & \\ & \swarrow^{\widetilde{p}} & \downarrow^{\overline{p}} & \searrow^{\widetilde{q}} & \\ \mathbb{A} \times \mathbb{R} & & \mathbb{A} & & \mathbb{A}^* \times \mathbb{R} \xrightarrow{\widetilde{k}} \mathbb{P}^* \times \mathbb{R} \end{array}$$

Corollary of to the functorialities in $E^b_{\mathbb{R}\text{-c}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{P}})$

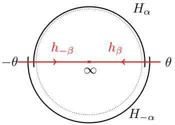
Let $\mathcal{M} \in D^b_{\mathrm{rs}}(\mathcal{D}_{\mathbb{A},\infty})$ and $F := \mathcal{S}ol_{\mathbb{P}}(\mathcal{M})|_{\mathbb{A}}$, then

$$\mathcal{S}ol^E_{\mathbb{P}^*}(\mathcal{M}^\wedge) \simeq \mathbb{C}^E_{\mathbb{P}^*} \overset{+}{\otimes} \underbrace{R\widetilde{k}_! \, R\widetilde{q}_! (\mathbb{C}_{\{t+\mathrm{Re}\,(zw)\geq 0\}} \otimes \overline{p}^{-1} F)[1]}_{\text{complex of usual sheaves on } \mathbb{P}^* \times \mathbb{R}} .$$

Define $K := R\widetilde{q}_! (\mathbb{C}_{\{t+\mathrm{Re}\,(zw)\geq 0\}} \otimes \overline{p}^{-1} F)$.

## Decomposition in sectors

Let $H_{\pm\alpha} := \{w \in \mathbb{A}^* \smallsetminus \{0\} \mid \pm \mathrm{Re}\, \alpha w \geq 0\}$ be the two (closed) sectors and $H_\alpha \cap H_{-\alpha} = h_\beta \cup h_{-\beta}$.




There are natural isomorphisms:

$$s_{\pm\alpha} : \mathcal{S}ol^E_{\mathbb{P}^*}(\widehat{\mathcal{M}})|_{H_{\pm\alpha}} \xrightarrow{\simeq} \bigoplus_{c\in\Sigma} (\Phi_c(F) \otimes E^{cw})$$

(where $\ldots|_Y := \pi^{-1}\mathbb{C}_Y \otimes \ldots$).

## Lemma, cp. [d'A-K, last section]

- If $S$ is a small sector such that $\mathrm{Re}\,(cw - dw) > 0$ on $S$, then

$$\mathrm{Hom}_{E^b_{\mathbb{R}\text{-c}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{P}^*})}(E^{cw}, E^{dw}) = 0.$$

- If $S$ contains exactly one Stokes line for each pair $(c, d)$ in $\Sigma$ with $c \neq d$ (e.g. $S = H_{\pm\alpha}$), then

$$\mathrm{End}_{E^b_{\mathbb{R}\text{-c}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{P}^*})}(\bigoplus_c \Phi_c \otimes E^{cw}|_S) \simeq \mathfrak{t},$$

where

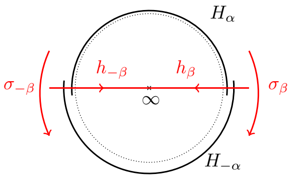
$$\mathfrak{t} := \bigoplus_{c\in\Sigma} \mathrm{End}(\Phi_c) \subset \mathrm{End}(\bigoplus_{c\in\Sigma} \Phi_c)$$

are the block diagonal matrices.

## Stokes glueing isomorphism

We have both isomorphisms $s_{\pm\alpha}$ on the common boundary half-lines of $H_{\pm\alpha}$ and $K$ is determined by the glueing isomorphisms

$$\sigma_{\pm\beta} := s_{-\alpha}|_{h_{\pm\beta}} \circ (s_{\alpha}|_{h_{\pm\beta}})^{-1} : \bigoplus_{c\in\Sigma}(\Phi_c(F)\otimes E^{cw}) \to \bigoplus_{c\in\Sigma}(\Phi_c(F)\otimes E^{cw}).$$



Consequences from Lemma 2 slides before:

- the decomposition isomorphisms $s_{\pm\alpha}$ are unique up to base change in t (block diagonal matrices),
- the glueing isomorphisms $\sigma_{\pm\beta}$ are upper/lower block triangular matrices, notation $\mathrm{End}^{\pm}$, with complex coefficients.

Note that $\alpha$ induces an ordering

$$c_1 <_\alpha c_2 <_\alpha \ldots <_\alpha c_n$$

by ordering of $\mathrm{Re}\,(\alpha \cdot c)$.

## Stokes multipliers

We obtain the glueing matrices, the Stokes multipliers

$$S_{\pm\beta} \in \mathrm{End}^{\pm}(\bigoplus_{c\in\Sigma} \Phi_c(F)).$$

The monodromy of the local system of solutions is

$$T = S_\beta^{-1} \cdot S_{-\beta}$$

up to conjugation.

# Topological computation

Recall:

$$\mathcal{S}ol^E_{\mathbb{P}^*}(\mathcal{M}^\wedge) \simeq \mathbb{C}^E_{\mathbb{P}^*} \overset{+}{\otimes} R\widetilde{k}_! \underbrace{R\widetilde{q}_!(\mathbb{C}_{\{t+\mathrm{Re}\,(zw)\geq 0\}} \otimes \overline{p}^{-1}F)}_{=:K}[1].$$

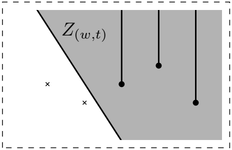
- Consider the stalk of $K$ at some $(w,t) \in \mathbb{A}^* \times \mathbb{R}$:

$$K_{(w,t)} \simeq R\Gamma_c(\mathbb{A}; \mathbb{C}_{Z_{(w,t)}} \otimes F),$$

$$Z_{(w,t)} = \{z \in \mathbb{A} \mid t + \mathrm{Re}\, zw \geq 0\} = -(t/|w|^2)\overline{w} + \{z \in \mathbb{A} \mid \mathrm{Re}\, z \geq 0\}\overline{w}$$

a closed half-space, $\overline{w} := w/|w|$.

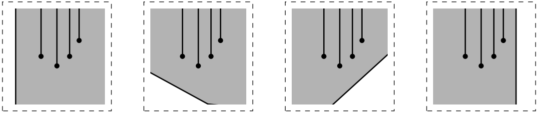
  - $Z_{(w,t)} \supset Z_{(w,s)}$ for $s < t$.
  - $\ell_c \subset Z_{(w,t)} \iff c \in Z_{(w,t)} \iff t + \mathrm{Re}\, cw \geq 0$.



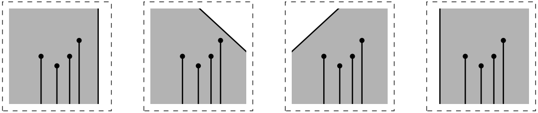
- for $|w| \gg 0$ and $t \gg 0$, we have $\ell_\Sigma \subset Z_{(w,t)}$, then

$$K_{(w,t)} \simeq R\Gamma_c(\mathbb{A}; \mathbb{C}_{Z_{(w,t)}} \otimes F) \simeq \bigoplus_{c\in\Sigma} R\Gamma_c(\mathbb{A}; \mathbb{C}_{\ell_c} \otimes F) = \bigoplus_{c\in\Sigma} \Phi_c(F).$$

- Can be globalized to obtain the decomposition isomorphisms $s_{\pm\alpha}$.
- The Stokes phenomenon is associated to the following easy observation: rotating $w$ with $|w| \gg 0$, we can use
  - $\ell_c$ as above for the direction $\alpha$ for $w$ inside one half-space = sector,



  - $\ell_c^-$ using the direction $-\alpha$ for $w$ inside the other half-space = sector.



Result - d'Agnolo, H , Morando, Sabbah

For $F \in \mathrm{Perv}_\Sigma(\mathbb{C}_\mathbb{A})$ with quiver $(\Psi, \Phi_i, u_i, v_i)_{c_i \in \Sigma}$, there is a topological way to compute the Stokes multipliers of its enhanced Fourier-Sato transform and the result is

$$S_\beta = \begin{pmatrix} 1 & u_1 v_2 & u_1 v_3 & \cdots & u_1 v_n \\ & 1 & u_2 v_3 & \cdots & u_2 v_n \\ & & \ddots & & \vdots \\ & & & & 1 \end{pmatrix},$$

$$S_{-\beta} = \begin{pmatrix} \mathbb{T}_1 & & & & \\ -u_2 v_1 & \mathbb{T}_2 & & & \\ -u_3 v_1 & -u_3 v_2 & \ddots & & \\ \vdots & \vdots & & \ddots & \\ -u_n v_1 & -u_n v_2 & \cdots & -u_n v_{n-1} & \mathbb{T}_n \end{pmatrix}.$$

Remark: Cp. the above with

- a general procedure by T. Mochizuki using rapid decay cycles,
- Malgrange's book, chapter XII.

Universität Augsburg University

## Monodromy

The monodromy of the Fourier transform $\widehat{M}$ around $\infty$ is

$$S_\beta^{-1} S_{-\beta} = 1 - \begin{pmatrix} u_1\, \mathrm{T}_2\, \mathrm{T}_3 \cdots \mathrm{T}_n\, v_1 & u_1\, \mathrm{T}_2\, \mathrm{T}_3 \cdots \mathrm{T}_n\, v_2 & \dots & u_1\, \mathrm{T}_2\, \mathrm{T}_3 \cdots \mathrm{T}_n\, v_n \\ u_2\, \mathrm{T}_3 \cdots \mathrm{T}_n\, v_1 & u_2\, \mathrm{T}_3 \cdots \mathrm{T}_n\, v_2 & \dots & u_2\, \mathrm{T}_3 \cdots \mathrm{T}_n\, v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1}\, T_n v_1 & u_{n-1}\, T_n v_2 & \dots & u_{n-1}\, T_n v_n \\ u_n v_1 & u_n v_2 & \dots & u_n v_n \end{pmatrix}$$

(This can also be obtained by determining the quiver of $\mathcal{S}ol(\widehat{M})$ at 0).

# Airy function

G.G. Stokes observed the Stokes phenomenon by studying the Airy function, an entire solution to

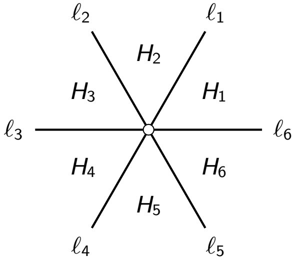
$$(\partial_y^2 - y)u(y) = 0.$$

We have

$$\mathcal{A} := \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(\partial_y^2 - y) \simeq (\mathcal{E}^{x^3/3})^{\wedge}$$

Let us study the Stokes structure of the Airy equation as a Fourier transform.

- De-ramify via $r : \mathbb{C}_v \to \mathbb{C}_y$, $v \mapsto y = v^2$, i.e. consider $r^{-1}\mathcal{A}$.
- Consider the sectors and their intersections in $\mathbb{C}_v$ at $v = \infty$:



- coordinate change $\mathbb{C}_u \times \mathbb{C}_v^\times \xrightarrow{\simeq} \mathbb{C}_x \times \mathbb{C}_v^\times$ given by

$$\begin{cases} x = \sqrt{-1}\, uv, \\ v = v, \end{cases} \tag{1}$$

and

$$f\colon \mathbb{C}_u \to \mathbb{C}_z = \mathbb{A}, \quad u \mapsto u^3 - 3u,$$
$$g\colon \mathbb{C}_v \to \mathbb{C}_w = \mathbb{A}^*, \quad v \mapsto \sqrt{-1}\, v^3/3,$$

so that $xy + x^3/3 = xv^2 + x^3/3 = f(u)g(v) = zw$.

- Set

$$F := Rf_! \mathbb{C}_{\mathbb{C}_u}[1] \in D^b(\mathbb{C}_{\mathbb{A}}).$$

Recall

$$F := Rf_!\mathbb{C}_{\mathbb{C}_u}[1] \in D^b(\mathbb{C}_{\mathbb{A}}).$$

Then $F \in \mathrm{Perv}_\Sigma(\mathbb{A})$ for $\Sigma = \{-2, 2\}$, and the quiver of $F$ is

$$\begin{array}{ccc}
\Phi_2(F) & & \mathbb{C} \\
u_2 \uparrow\downarrow v_2 & & \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \uparrow\downarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
\Psi(F) & \simeq & \mathbb{C}^3 \\
u_{-2} \downarrow\uparrow v_{-2} & & \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \downarrow\uparrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
\Phi_{-2}(F) & & \mathbb{C}.
\end{array} .$$

## Key observation

$$Er^{-1}A|_{\mathbb{C}_v^\times} \simeq Eg^{-1}((eF)^\curlywedge)|_{\mathbb{C}_v^\times}.$$

- Exponential components of $(eF)^{\curlywedge}$ at $\infty$ are $E^{\pm 2w}$,
- Stokes multipliers are

$$S_\beta = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad S_{-\beta} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- $Eg^{-1}E^{\pm 2w} \simeq E^{\pm \frac{2}{3}\sqrt{-1}\, v^3}$,
- $g^{-1}H_\alpha = \bigcup_k H_{2k-1}, \quad g^{-1}H_{-\alpha} = \bigcup_k H_{2k}$ and
- hence

$$S_{2k} = S_\beta, \quad S_{2k-1} = S_{-\beta}^{-1}.$$