Isochronous centers of polynomial Hamiltonian systems and correction of vector fields

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(A joint work with Jacky Cresson)

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Introduction

We consider the **complex representation** of a *real planar vector field* with a **center** in 0 denoted by

$$X_{lin} = i(x\partial_x - y\partial_y)$$

with $x, y \in \mathbb{C}$ with $y = \bar{x}$.

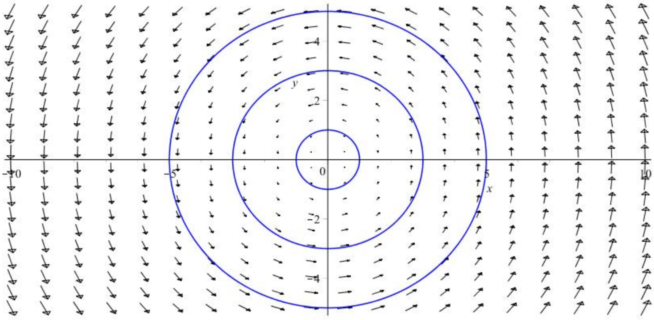

Figure – The equilibrium point 0 is a center.

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The problem of isochronous center

Which conditions on P and Q are necessary to preserve the isochronicity ?

If X is also Hamiltonian, we have the following conjecture :

Jarque-Villadelprat's conjecture (2002) [1]

Every center of a real planar polynomial Hamiltonian system of even degree is nonisochronous.

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- B.Schuman (2001) : true in the homogeneous case,
- Jarque-Villadelprat (2002) : true in the quartic case,
- Other cases : the conjecture is open !

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The isochronicity is equivalent to the linearizability.

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How to study the linearizability of a vector field ?

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Correction and mould calculus

- Formalism : **Mould calculus** introduced by J.Ecalle in 70's.

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Definition of Correction

- X analytic vector field and X_{lin}=linear part of X

Find a vector field Z of the following commuting problem :

- $X - Z$ formally conjugate to X_{lin},
- $[X_{lin}, Z] = 0$,

The solution Z is the **correction** of X.

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Criterion of linearizability [Ecalle,Vallet]

A vector field is linearizable if and only if its correction is zero.

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The interest of this formalism :

- An algorithmic and explicit way to compute the conditions of linearizability.
- To distinguish what depends on the coefficients of P and Q and what is universal for the linearizability.

Our results

We consider a polynomial perturbation as above :

$$X = X_{lin} + \sum_{r=k}^{l} X_r$$

with

- $X_r = P_r(x,y)\partial_x + Q_r(x,y)\partial_y$,
- $P_r(x,y) = \sum_{j=0}^{r} p_{r-j-1,j}x^{r-j}y^j, \quad Q_r(x,y) = \sum_{j=0}^{r} q_{r-j,j-1}x^{r-j}y^j.$
- $p_{r-j-1,j}, \; q_{r-j,j-1} \in \mathbb{C}$ with the following conditions :

Real system condition : $p_{j,k} = \bar{q}_{k,j}$ with $j+k = r-1$

Hamiltonian condition : $p_{j-1,r-j} = -\frac{r-j+1}{j} q_{j-1,r-j}$ with $j = 1, \ldots r$.

Theorem 1 [P.,Cresson]

Let X be a real Hamiltonian vector field of even degree 2n given by :

$$X = X_{lin} + \sum_{r=2}^{2n} X_r$$

If X satisfies one of the following conditions :

1. there exists $1 \leq k < n-1$ such that $p_{i,i} = 0$ for $i = 1, ..., k-1$ and $Im(p_{k,k}) > 0$,
2. $p_{i,i} = 0$ for $i = 1, ..., n-1$,

Then the vector field X is nonisochronous.

Theorem 2 [P.,Cresson]

A real Hamiltonian vector field of the form :

$$X = X_{lin} + X_k + ... + X_{2n},$$

for $k \geq 2$ and $n \leq k - 1$, is nonisochronous.

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- $X = X_{lin} + X_2 + X_3 + X_4 + X_5 + X_6$ with $Im(p_{1,1}) > 0$ or $p_{1,1} = 0$ and $Im(p_{2,2}) > 0$
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Proofs of the theorems

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Prepared form of a vector field and Mould expansion

We consider a vector field $X = X_{lin} + \sum X_r$. The *prepared form* of X is :

$$X = X_{lin} + \sum_{n \in A(X)} B_n,$$

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- **Alphabet** : $A(X) \subset \mathbb{Z}^2$,
- **Homogeneous differential operator** : B_n satisfying
$$B_n(x^{m^1} y^{m^2}) = \beta_n x^{m^1+n^1} y^{m^2+n^2} \text{ with } \beta_n \in \mathbb{C}$$

Example of decomposition

We consider the following vector field :

$$X = X_{lin} + X_2$$

where
$$X_2 = \left(p_{1,0}x^2 + p_{0,1}xy + p_{-1,2}y^2\right)\partial_x + \left(q_{-1,2}x^2 + q_{1,0}xy + q_{0,1}y^2\right)\partial_y,$$

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The alphabet and the operators are given by :

- $B_{(1,0)} = x(p_{1,0}x\partial_x + p_{0,1}y\partial_y)$,
- $B_{(0,1)} = y(p_{0,1}x\partial_x + p_{0,1}y\partial_y)$,
- $B_{(2,-1)} = p_{2,-1}x^2\partial_y$,
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This operation is called mould expansion.

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Resonant words

A word $\mathbf{n}$ is resonant if $\omega(\mathbf{n}) = 0$.

The correction and its mould

Theorem [Ecalle, Vallet]

The correction can be written :

$$Carr(X) = \sum_{\mathbf{n} \in A^*(X)} Carr^{\mathbf{n}} B_{\mathbf{n}} = \sum_{k \geq 1} \frac{1}{k} \sum_{\substack{\mathbf{n} \in A^*(X) \\ \ell(\mathbf{n}) = k}} Carr^{\mathbf{n}} [B_{\mathbf{n}}]$$

where :

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where :

- $[B_{\mathbf{n}}] = [B_{n_1 \cdot \ldots \cdot n_r}] = [\ldots[[B_{n_1}, B_{n_2}], B_{n_3}], \ldots], B_{n_r}]$,

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- $[B_{\mathbf{n}}] = [B_{n_1 \cdot \ldots \cdot n_r}] = [\ldots[[B_{n_1}, B_{n_2}], B_{n_3}], \ldots], B_{n_r}]$,
- $Carr^{\bullet}$ is the mould of the correction.

- The mould $Carr^\bullet$ is given for any word $\mathbf{n} = n_1 \cdot ... \cdot n_r$ by[4] :

4. It is not a trivial formula : related to the notion of variance of vector fields, see J.Ecalle and B.Vallet, "*Correction and linearization of resonant vector fields and diffeomorphisms*", Math. Z. 229, 249-318 (1998)

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For $\omega(\mathbf{n}) = 0$,

- If $\ell(\mathbf{n}) = 1$, $Carr^{\mathbf{n}} = 1$,
- If $\ell(\mathbf{n}) = 2$, $\mathbf{n} = n_1 \cdot n_2$, $Carr^{\mathbf{n}} = \frac{-1}{\omega(n_1)}$

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New writing of the Correction

We introduce the notion of depth :

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Correction via the depth

$$Carr(X) = \sum_{p \geq 1} Carr_p(X) \text{ with } Carr_p(X) = \sum_{\substack{\mathbf{n} \in A^*(X) \\ p(\mathbf{n}) = p}} \frac{1}{l(\mathbf{n})} Carr^{\mathbf{n}}[B_{\mathbf{n}}]$$

Linearizability and main property

Criterion of linearizability

A vector field X as above is linearizable if and only if $Carr_p(X) = 0$ for all $p \geq 1$.

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Property of $Carr_p(X)$

For any odd integer p, $Carr_p(X) = 0$.

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Vector field	X_{2l}	X_{2l+1}	...	X_{2n}
Depth	$2l-1$	$2l$	...	$2n-1$

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- Notation : $Carr_{p,\ell}(X_i)$ the contribution of X_i in depth p and ℓ the length of the corresponding words.

If $k = 2l + 1$:

- $Carr_{2l+2q}(X) = Carr_{2l+2q,1}(X_{2l+2q+1})$, for $0 \leq q \leq l - 1$,

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- $Carr_{4l}(X) = Carr_{4l,1}(X_{4l+1}) + Carr_{4l,2}(X_{2l})$

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General formulas

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where $ping(n) = ping(n^1, n^2) = (n^2, n^1)$.

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With the conditions for X to be real and Hamiltonian, we have :

$Carr_{2k}(X) = F \times (x\partial_x - y\partial_y)$ with :

$$F = p_{k,k} + i\left(\sum_{j=\lfloor\frac{2l+1}{2}\rfloor+1}^{2l} \frac{2l(2l+1)}{(2l-j+1)^2}|p_{j-1,2l-j}|^2 + \frac{2l}{2l+1}|p_{-1,2l}|^2\right)$$

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(C1) If $Im(p_{k,k}) > 0$, we have an obstruction to the isochronicity !
(C2) If $p_{k,k} = 0$, the sphere is trivial $\Rightarrow X_{2l} = 0$.

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2. If $p_{k,k} = 0$ for $1 \leq k \leq n-1$,
   $\Rightarrow$ by the condition **(C2)**, X is nonisochronous or X_r is trivial.

Proof of Theorem 2

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  We have : $2(k-1) \geq 2n > 2n-1$,
  $\Rightarrow$ No interaction between the length 1 and 2 in a same depth,
  $\Rightarrow$ each X_r is trivial or X is nonisochronous.

- If k is odd, we have an analogous result.

A last theorem [P.,Cresson]

Let X be a non trivial real polynomial Hamiltonian vector field on the form :

$$X = X_{lin} + X_k + \dots + X_{2l} + \sum_{n=1}^{m} \sum_{j=c_n}^{2(c_n-1)} X_j$$

where $k \geq 2$, $l \leq k-1$ and the sequence c_n is defined by : $c_1 = l$ and $\forall n \geq 2,\ c_n = 4(c_{n-1}-1)$. Then X is nonisochronous.

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Some examples :

- $X = X_{lin} + X_2 + X_4 + X_5 + X_6,$
- $X = X_{lin} + X_2 + X_4 + X_5 + X_6 + \sum_{j=12}^{22} X_j + \sum_{j=44}^{86} X_j + \sum_{j=172}^{342} X_j$

Perspectives

- To complete our Maple program,
- To try to generalize the Theorem 2 for $n > k - 1$,
- To extend our study to the isochronous complex Hamiltonian case.

Thank your for your attention !