Random Sorting Networks

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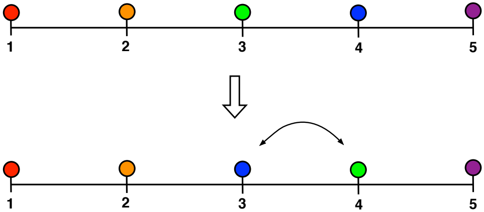
The interchange process on the line

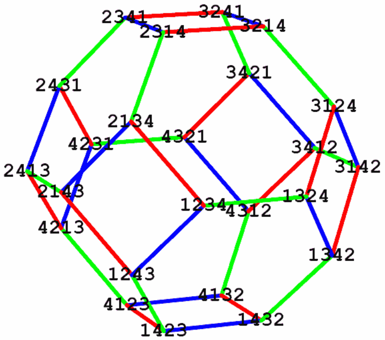


Exactly one particle sits on each site.

In each step two particles are swapped across an edge.

State space: all permutations of $1 \ldots n$.

The permutahedron



A random walk on the permutahedron.

Cayley graph of Sym_n.

Has diameter $\binom{n}{2}$. Farthest points: $1 \ldots n$ and $n \ldots 1$.

A shortest path between these points is called a **sorting network**.

How many sorting networks on $[n]$ are there?

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Theorem (Stanley (1984))

The number of sorting networks on $[n]$ *is*

$$\frac{\binom{n}{2}!}{1^n 3^{n-1} 5^{n-2} \cdots (2n-1)^1} =$$

$$\# \left\{ \textit{staircase shaped standard Young tableaux of size } \binom{n}{2} \right\}.$$

Bijection [Edelman-Greene (1987)]: Between sorting networks and staircase shaped standard Young tableaux.

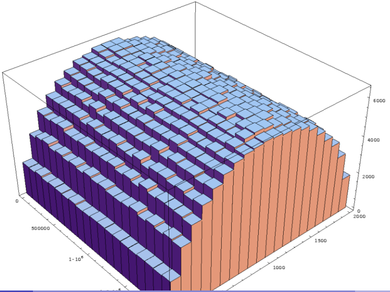
Angel, Holroyd, Romik and V, 2007

Let $(s_1, s_2, \ldots, s_N)$ be the swaps of a uniform random sorting network on $[n]$. Consider the scaled space-time empirical measure of swaps:

$$\eta = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{i}{N}, \frac{2s_i}{n} - 1\right).$$

Theorem (Angel et. al. 2007)

The measure η converges weakly: $\eta \to \frac{2}{\pi}\sqrt{1-y^2}\,dxdy$.



RSN: Many conjectures

$\mathbf{RSN}^n = (\sigma_t; 0 \le t \le N)$ a uniform random sorting network on $1 \ldots n$.

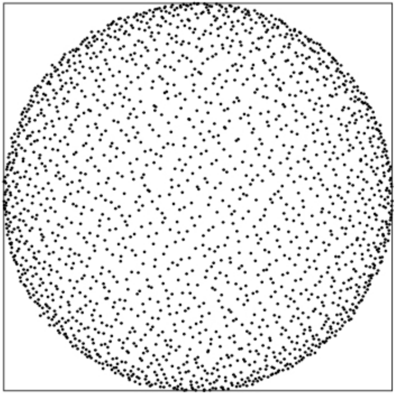
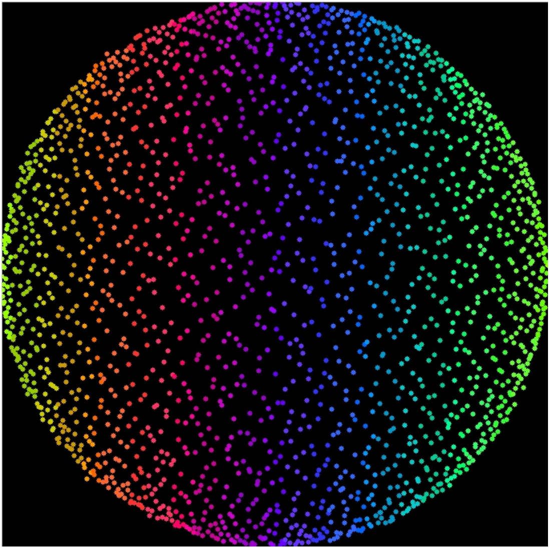
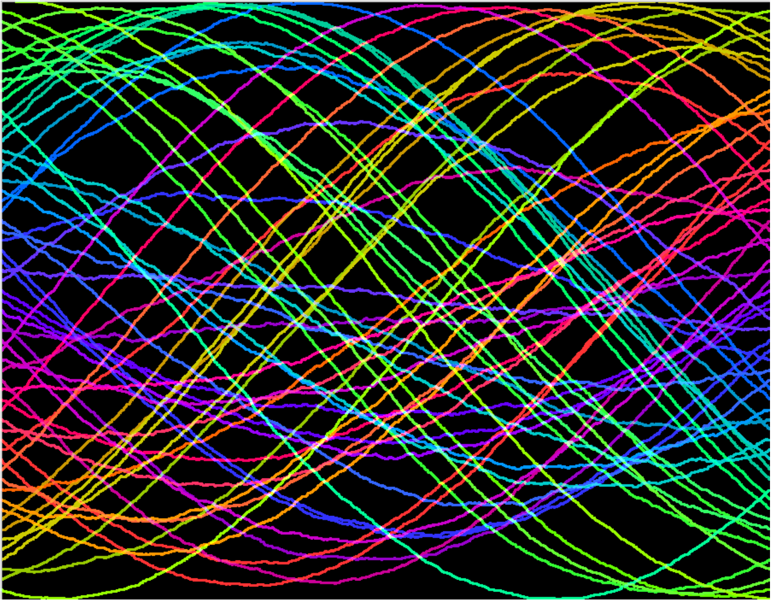


Figure: Permutation matrix of of $\mathbf{RSN}^{500}$ at half time (Courtesy A. Holroyd)

Halfway permutation for 2000



Selected trajectories



How do we study the geometry of large permutations?

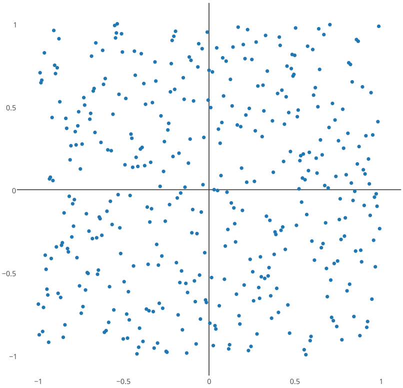


Figure: Scaled matrix of a uniform random permutation of 500 elements.

Permutation limit theory I

The empirical measure of a permutation σ of $[n]$ is

$$\mu^\sigma = \frac{1}{n} \sum_{i=1}^{n} \delta\left(\frac{2i}{n} - 1, \frac{2\sigma(i)}{n} - 1\right).$$

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It is the empirical joint distribution of $(i, \sigma i)$ scaled to live on $[-1, 1]^2$.

Defintion

Permutations σ_n of increasing length converge if μ^{σ_n} converges in distribution to a limiting measure μ.

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μ is a prob. measure on $[-1,1]^2$ with uniform marginals. A **permuton**.

Theorem (Hoppen et. al. 2013)

Any permuton is a limit of permutations.

Permutons are the same as **copulas**, used in statistics since 1981.

The halfway permutation

$\mathbf{RSN}^n = (\sigma_t; 0 \leq t \leq N)$ a uniform random sorting network on $[n]$.

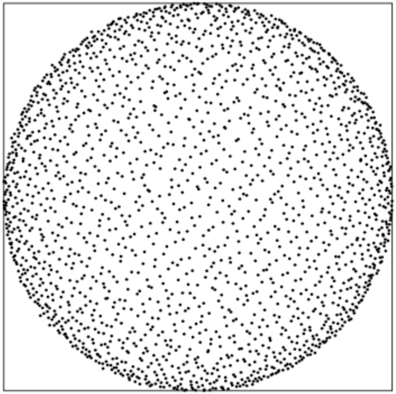


Figure: Empirical measure $\mu^{\sigma_{N/2}}$ of $\mathbf{RSN}^{500}$. (A. Holroyd)

The halfway permutation

Conjecture

Halfway permutation of $\mathbf{RSN}^n$ converges to the *Archimedean measure* $\mathscr{A}$.

- Unique probability measure on $\mathbb{R}^2$ with $\mathrm{Uni}[-1,1]$ projections along any line through the origin.
- Projection of normalized surface area measure of S^2 onto $\mathbb{R}^2$. Has density on the unit disk:

$$\frac{(2\pi)^{-1}}{\sqrt{1-x^2-y^2}}\,dxdy.$$

Permutation limit theory II

For the joint limit of **2 permutations** σ, τ:
the empirical distribution of $(i, \sigma i, \tau i)$ scaled to $[-1, 1]^3$.

For the joint limit of a **time-dependent permutations** $(\sigma_t, 0 \leq t \leq 1)$ with $\sigma_0 = id$
the empirical distribution of $(\sigma_t, 0 \leq t \leq 1)$ scaled to $[-1, 1]$.

This is the empirical distribution of the *particle trajectories*.

Defintion

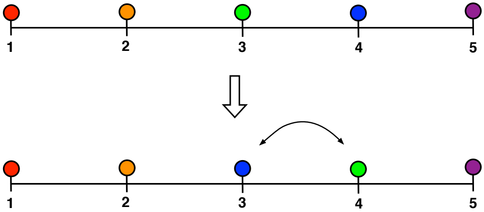
A **permuton process** is the law of a stochastic process $[0, 1] \to [-1, 1]$ with uniform marginals.

Natural weak topology with respect to sup norm convergence.

Proposition (Rahman, V, Vizer, 16+)

Any permution process is a limit of a **deterministic** sequence of time-dependent permutations.

Examples of permuton processes



Example

The interchange process on $\{1, \ldots, n\}$ run at speed n^3, rescaled to $[-1, 1]$ is concentrated at a single permuton process $\mathcal{B}$.

$\mathcal{B}$ is the law of stationary reflected Brownian motion on $[-1, 1]$.

Example: The Archimedean process

Defintion (The Archimedean process)

$$\mathscr{A}_t = \cos(\pi t)\,\mathbf{A}_x + \sin(\pi t)\,\mathbf{A}_y,$$

where $(\mathbf{A}_x, \mathbf{A}_y) \sim \mathscr{A}$.

Conjecture (Angel, Holroyd, Romik, V, 07; Rahman, V, 16+)

$\mathbf{RSN}^n$ converges after scaling to a deterministic limit, the *Archimedean process* $\mathscr{A}_t$.

The permutons $(\mathscr{A}_0, \mathscr{A}_{k/10})$ and random sorting networks

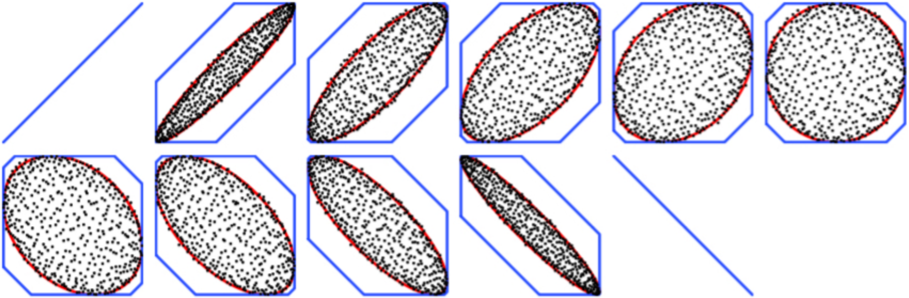


Figure: The path $\mu^{\sigma_{\lfloor tN \rfloor}}$ of $\mathbf{RSN}^{500}$ at times $t = k/10$. (Holroyd)

What's special about the Archimedean process?

The Dirichlet energy of a path γ in a compact metric space (K, d) is

$$\mathrm{En}\,[\gamma] = \sup_{0=t_0<t_1<\cdots<t_n=1} \sum_{i=1}^{n} \frac{d(\gamma_{t_{i-1}}, \gamma_{t_i})^2}{t_i - t_{i-1}},$$

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Theorem (Rahman, V, 16+)

$\mathscr{A}_t$ uniquely minimizes Dirichlet energy among permuton processes X with $X_0 = -X_1$.

What's special about the Archimedean process?

Theorem (Rahman, V, 16+)

$\mathscr{A}_t$ is the uniquely minimizes L^2-Dirichlet energy among permuton processes X with $X_0 = -X_1$.

Proof. X_0 and X_1 are opposite points of an L^2 ball of random variables with mean 0 and variance $2/3$.

Claim: $\mathscr{A}_t$ is the unique distribution of a half great circle with uniform marginals.

Proof.

If X is a great circle in any Hilbert space, then

$$X_t = X_0 \cos(\pi t) + X_{1/2} \sin(\pi t)$$

Since X_t are uniform, $(X_0, X_{1/2}) \sim \mathscr{A}$. Thus $(X_t) \sim (\mathscr{A}_t)$.

Questions about permutons

The **length** of a permuton (X, Y) is the infimum of the energy of a permuton processes with $(X_0, X_1) = (X, Y)$.

Conjecture (Rahman, V, 16+)

$(X, -X)$ is the unique longest permuton.

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Proved by D. Dauvergne for the case when (X, Y) has a density. In that case, X_t is determined by (X_0, X_1). (False for $\mathscr{A}$!).

Application: lazy sorting networks

Fix $\alpha \in (0,1)$.

Defintion

A **lazy sorting network** is law of interchange process on the n-path conditioned to be $o(1)$-close to the reverse permutation at time $n^{2+\alpha}/2$.

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Corollary. The number of lazy sorting network paths is

$$\exp\left(\frac{1}{2}n^{2+\alpha}\log n - (\pi^2/6 + o(1))n^{2-\alpha}\right)$$

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But Stanley's formula for the number of sorting networks, $\alpha = 0$:

$$\exp\left(\frac{1}{2}n^{2}\log n - (1/4 - \log 2 + o(1))n^{2}\right)$$

The Edelman-Greene bijection

Lazy sorting networks still behave like random walks locally.

Sorting networks are different. (Particles can only swap once!)

EG: A bijection betwen random sorting network on $[n]$ and staircase-shaped young tableaux of volume $\binom{n}{2}$.

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Properties.

- bijection, so uniform measure is preserved
- early steps in the network are determined by values close to diagonal of the YT.
- similar to the RSK correspondence.
- The longest increasing subsequence of swaps is easy to read off and control.

Limits of sorting networks

Theorem (Angel, Holroyd, Dauvergne, V; Gorin, Rahman)

Consider the time scaled swap process

$$U_n(x,t) = \sigma_{\lfloor nt \rfloor}(x),$$

where σ is a uniform random sorting network. Then

$$U_n \xrightarrow[n\to\infty]{d} U.$$

Limits of sorting networks

Theorem (Angel, Holroyd, Dauvergne, V; Gorin, Rahman)

There exists a swap process U so that the following holds. For any $u \in (-1,1)$, and sequence k_n with $k_n/n \to (1+u)/2$. Consider the shifted, and time scaled swap process

$$U_n(x,t) = \sigma_{\lfloor nt/\sqrt{1-u^2} \rfloor}(k_n + x) - k_n,$$

where σ is a uniformly random sorting network. Then

$$U_n \xrightarrow[n\to\infty]{d} U.$$

Properties of the limit

- U is stationary and mixing of all orders in space.
- U and has stationary increments in time.
- For every t, the permutation $U(\cdot, t)$ is finitary.
- Particles in U have random speed, $S(x) = \lim_{t\to\infty} U(x,t)/t$ exists.
- $S(x)$ is a stationary and ergodic.
- (Rahman, V, 16+). Proving $ES(x)^2 \leq \mathcal{E}\mathscr{A}_t$ would imply the Archimedean process conjecture.

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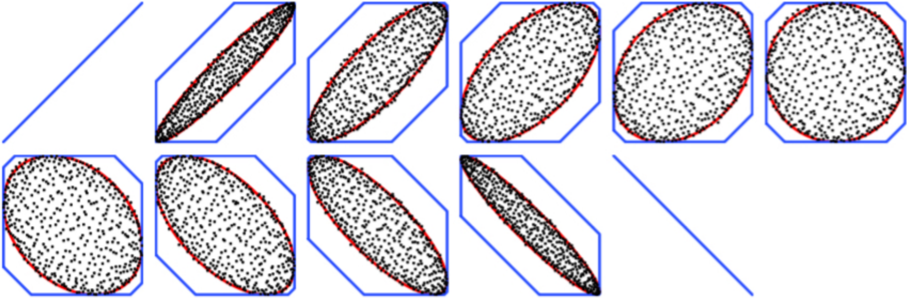
- Angel, Holroyd, Dauvergne, Virag: Hook formula, probabilistic arguments, monotonicity. First principles.
- Gorin, Rahman: Petrov's contour integral formula, an exact formula for plain partitions. Integrable probability, random matrix correlations in the limit.

Application of the local limit



Theorem (Dauvergne, V, 17+)

In any subsequential permuton process limit X of RSN,

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- Suppose the contrary.
- Then in large enough boxes we see particles travelling diagonally at a higher speed with high probability.
- Tiling the global sorting network with these boxes, we find an increasing subsequence that is too long before time 1.

Thank You