Variational formulas, Busemann functions, and fluctuation exponents for the corner growth model with exponential weights

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Contents

1. Introduction

These notes discuss variational formulas, Busemann functions, and fluctuation exponents for the exactly solvable corner growth model with i.i.d. exponential weights. This is a preliminary version of text for the proceedings of the 2017 American Mathematical Society Short Course on Random Growth Models, organized by Michael Damron, Firas Rassoul-Agha and T.S. and held January 2–3 in Atlanta. This version does not yet have all the intended results nor complete citations of relevant past work.

Notation, definitions and terminology. $\mathbb{Z}_{\geqslant 0} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$. The standard basis vectors of $\mathbb{R}^2$ are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For a point $x = (x_1, x_2) \in \mathbb{R}^2$ the ℓ^1-norm is $|x|_1 = |x_1| + |x_2|$ and integer parts are taken coordinatewise: $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor)$. We call the x-axis occasionally the e_1-axis, and similarly the y-axis and the e_2-axis are the same thing. C is a constant whose value can change from line to line. For $n \in \mathbb{Z}_{>0}$ the segment is $[n] = \{1, 2, \dots, n\}$.

$X \sim \text{Exp}(\lambda)$ for $0 < \lambda < \infty$ means that random variable X has exponential distribution with rate λ. This X is a positive random variable whose probability distribution satisfies $P(X > t) = e^{-\lambda t}$ for $t \geqslant 0$. It has mean $E(X) = \lambda^{-1}$ and variance $\text{Var}(X) = \lambda^{-2}$.

We write ω_x and $\omega(x)$ interchangeably for the weight attached to lattice point x. $\overline{X} = X - EX$ is the centered random variable X.

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2. Variational formulas for last-passage percolation shapes

2.1. **Directed last-passage percolation on $\mathbb{Z}^d$.** We consider here a general setting before specializing to the two-dimensional corner growth model. Let $(\Omega, \mathfrak{S}, \mathbb{P})$ be a Polish product probability space $\Omega = \Gamma^{\mathbb{Z}^d}$ of random environments $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \in \Omega$, with Borel σ-algebra $\mathfrak{S}$, and a product probability measure $\mathbb{P}$ under which the coordinates are i.i.d. random variables: for any distinct lattice points $x_1, \dots, x_n \in \mathbb{Z}^d$ and any Borel sets $B_1, \dots, B_n \subset \Gamma$,

$$\mathbb{P}\{\omega : \omega_{x_i} \in B_i \text{ for } i = 1, \dots, n\} = \prod_{i=1}^{n} \mathbb{P}\{\omega : \omega_0 \in B_i\}. \tag{2.1}$$

The group of translations or shifts $\{\theta_x\}_{x\in\mathbb{Z}^d}$ act on Ω by $(\theta_x\omega)_y = \omega_{x+y}$.

Let $\mathcal{R}$ be a finite subset of $\mathbb{Z}^d$. A lattice path $x_{0,n} = (x_k)_{k=0}^n \subset \mathbb{Z}^d$ is admissible if its steps satisfy $z_k = x_k - x_{k-1} \in \mathcal{R}$. Let $\mathcal{U} = \operatorname{co}\mathcal{R}$ be the convex hull of $\mathcal{R}$ in $\mathbb{R}^d$, and $\operatorname{ri}\mathcal{U}$ the relative interior of $\mathcal{U}$. We put ourselves in the directed setting by assuming that

$$0 \notin \mathcal{U}. \tag{2.2}$$

This implies the existence of a vector $\hat{u} \in \mathbb{R}^d$ and $\delta > 0$ such that $z \cdot \hat{u} \geqslant \delta$ for all $z \in \mathcal{R}$.

For convenience we also assume that $\mathbb{Z}^d$ is the smallest additive group that contains $\mathcal{R}$. Without this assumption we would carry along the group generated by $\mathcal{R}$ in the development.

The weights of admissible steps z are determined by a measurable function $V : \Omega \times \mathcal{R} \to \mathbb{R}$ about which we assume the following:

$$\forall z \in \mathcal{R},\ V(\omega, z) \text{ is a local function of } \omega \text{ and for some } p > d,\ V(\cdot\,, z) \in L^p(\mathbb{P}). \tag{2.3}$$

By definition, a local function of ω is one that depends on only finitely many coordinates of ω.

Example 2.1. The basic example to think about is the two-dimensional corner growth model, with real weights on the vertices: $\omega = (\omega_x)_{x\in\mathbb{Z}^2} \in \Omega = \mathbb{R}^{\mathbb{Z}^2}$. The set of admissible steps is $\mathcal{R} = \{e_1, e_2\}$, and potential given by the weight at the origin: $V(\omega, z) = \omega_0$. The set of possible limiting velocities of paths is the closed line segment $\mathcal{U} = [e_1, e_2]$, and its relative interior is the open line segment $\operatorname{ri}\mathcal{U} = (e_1, e_2)$. △

Example 2.2. The formulation covers also weights on directed edges. Let $\mathcal{R} = \{e_1, e_2, \dots, e_d\}$ and let $\vec{\mathcal{E}}_d = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : y - x \in \mathcal{R}\}$ be the set of directed nearest-neighbor edges on $\mathbb{Z}^d$. Let $\omega = (\omega(e))_{e\in\vec{\mathcal{E}}_d}$ be a configuration of weights on directed nearest-neighbor edges. The potential picks out the edge weight: $V(\theta_x\omega, z) = \omega(x, x+z)$ for $x \in \mathbb{Z}^d$ and $z \in \mathcal{R}$. △

The point-to-level last-passage percolation with external field or tilt $h \in \mathbb{R}^d$ is defined by

$$G_n(h) = \max_{x_{0,n}:\, x_0=0} \Big\{ \sum_{k=0}^{n-1} V(\theta_{x_k}\omega, z_{k+1}) + h \cdot x_n \Big\}, \qquad h \in \mathbb{R}^d. \tag{2.4}$$

The maximum is over admissible n-step paths $x_{0,n} = (x_k)_{k=0}^n$ that start at the origin $x_0 = 0$ and whose steps are denoted by $z_k = x_k - x_{k-1}$.

The point-to-point last-passage percolation with restricted path length is defined by

$$G_{x,(n),y} = \max_{x_{0,n}:\, x_0=x,\, x_n=y} \sum_{k=0}^{n-1} V(\theta_{x_k}\omega,\ z_{k+1}), \qquad x \in \mathbb{Z}^d. \tag{2.5}$$

The maximum is over admissible n-step paths $x_{0,n} = (x_k)_{k=0}^n$ that start at x and end at y. If y cannot be reached from x with an admissible n-step path then set $G_{x,(n),y} = -\infty$. Our convention is $G_{x,(0),x} = 0$.

Remark 2.3. The number of steps in an admissible path from x to y is determined uniquely by x and y for all pairs x, y iff 0 does not lie in the affine hull of $\mathcal{R}$. This is true for natural directed examples such as $\mathcal{R} = \{e_1, e_2, \ldots, e_d\}$. Then we can write $G_{x,y} = G_{x,(n),y}$ where n is the unique number of admissible steps from x to y. △

We take the existence of the limiting shape functions for granted, as stated in the next theorem.

THEOREM 2.4. *Let $\mathbb{P}$ be an i.i.d. product probability measure and assume* (2.2) *and* (2.3)*.*

(i) *There exists a finite, convex, Lipschitz function $g_{\text{pl}} : \mathbb{R}^d \to \mathbb{R}$ such that*

$$g_{\text{pl}}(h) = \lim_{n\to\infty} n^{-1} G_n(h) \quad \mathbb{P}\text{-a.s.} \tag{2.6}$$

(ii) *There exists a nonrandom finite, concave, continuous function $g_{\text{pp}} : \mathcal{U} \to \mathbb{R}$ such that*

$$g_{\text{pp}}(\xi) = \lim_{n\to\infty} n^{-1} G_{0,(n),[n\xi]}, \qquad \xi \in \mathcal{U} \tag{2.7}$$

where $[n\xi]$ is a point reachable in n steps and approximately $n\xi$. The limits satisfy

$$g_{\text{pl}}(h) = \sup_{\xi\in\mathcal{U}} \{g_{\text{pp}}(\xi) + h \cdot \xi\}. \tag{2.8}$$

The theorem above is a part of Theorem 2.4 in [5].

Sketch of the argument for the duality (2.8) between point-to-point and point-to-line.

$$\begin{aligned} \frac{1}{n} G_n(h) &= \max_{x_{0,n}:\, x_0=0} \frac{1}{n} \Big\{ \sum_{k=0}^{n-1} V(T_{x_k}\omega, z_{k+1}) + h \cdot x_n \Big\} \\ &= \max_x \frac{1}{n} \big\{ G_{0,(n),x} + h \cdot x \big\} = \sup_{\xi\in\mathcal{U}} \Big\{ \frac{1}{n} G_{0,(n),[n\xi]} + h \cdot \frac{[n\xi]}{n} \Big\} \\ &\longrightarrow \sup_{\xi\in\mathcal{U}} \{\, g_{\text{pp}}(\xi) + h \cdot \xi \,\} \end{aligned}$$

so

$$g_{\text{pl}}(h) \;=\; \sup_{\xi\in\mathcal{U}} \{\, g_{\text{pp}}(\xi) + h \cdot \xi \,\}.$$

By convex duality, equation (2.8) implies

$$g_{\text{pp}}(\xi) = \inf_{h\in\mathbb{R}^d} \{g_{\text{pl}}(h) - h \cdot \xi\}, \qquad \xi \in \mathcal{U}. \tag{2.9}$$

Let us say that $\xi \in \mathcal{U}$ and $h \in \mathbb{R}^d$ are *dual* if

$$g_{\text{pl}}(h) = g_{\text{pp}}(\xi) + h \cdot \xi.$$

LEMMA 2.5. *Every $\xi \in \operatorname{ri}\mathcal{U}$ has a dual $h \in \mathbb{R}^d$.*

The lemma is proved by arguing that the infimum in (2.9) can be restricted to a compact set. See Lemma 4.3 in [5].

In order to develop variational formulas for the limits g_{pp} and g_{pl}, we introduce a class of stationary processes we call cocycles and state an ergodic theorem for them.

2.2. **Stationary cocycles.**

DEFINITION 2.6 (Cocycles). *A measurable function $B : \Omega \times \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is a* stationary cocycle *if it satisfies these two conditions for $\mathbb{P}$-a.e. ω and all $x, y, z \in \mathbb{Z}^d$:*

$$\begin{aligned} B(\omega, z + x, z + y) &= B(\theta_z\omega, x, y) \qquad &(\textit{stationarity}) \\ B(\omega, x, y) + B(\omega, y, z) &= B(\omega, x, z) \qquad &(\textit{additivity}). \end{aligned}$$

$\mathcal{K}$ denotes the space of stationary cocycles B such that $\mathbb{E}|B(x, y)| < \infty$ $\forall x, y \in \mathbb{Z}^d$. $\mathcal{K}_0$ denotes the subspace of $F \in \mathcal{K}$ such that $\mathbb{E}[F(x, y)] = 0$ $\forall x, y \in \mathbb{Z}^d$.

A special class of cocycles is given by gradients $\nabla\varphi(\omega, x, y) = \varphi(\theta_y\omega) - \varphi(\theta_x\omega)$. $\mathcal{K}_0$ is the $L^1(\mathbb{P})$-closure of gradients of integrable functions.

The first lattice variable in our definition of a cocycle is superfluous: if we put $\widetilde{F}(\omega, y) = F(\omega, 0, y)$ then $F(\omega, x, y) = F(\theta_x\omega, 0, y - x) = \widetilde{F}(\theta_x\omega, y - x)$. Occasionally we may simplify by dropping the first lattice variable and write $F(\omega, x)$ for $F(\omega, 0, x)$.

Our convention for centering non-mean-zero cocycles is the following. For $B \in \mathcal{K}$ there exists a vector $h(B) \in \mathbb{R}^d$ such that

$$\mathbb{E}[B(0, x)] = -h(B) \cdot x \qquad \forall x \in \mathbb{Z}^d. \tag{2.10}$$

Existence of $h(B)$ follows because $c(x) = \mathbb{E}[B(0, x)]$ is an additive function on the group $\mathbb{Z}^d$. Then

$$F(\omega, x, y) = -h(B) \cdot (y - x) - B(\omega, x, y), \qquad x, y \in \mathbb{Z}^d \tag{2.11}$$

is a centered stationary $L^1(\mathbb{P})$ cocycle.

Consider this assumption on a given $F \in \mathcal{K}_0$.

$$\begin{aligned} &\exists\ \overline{F} : \Omega \times \mathcal{R} \to 0 \text{ such that the following properties hold for } z \in \mathcal{R}\backslash\{0\} \text{ and } \mathbb{P}\text{-a.s.:} \\ &\qquad F(\omega, 0, z) \leqslant \overline{F}(\omega, z) \\ &\text{and} \\ &\qquad \varlimsup_{\delta \searrow 0} \varlimsup_{n\to\infty} \max_{|x| \leqslant n} \frac{1}{n} \sum_{0 \leqslant i \leqslant n\delta} |\overline{F}(\theta_{x+iz}\omega, z)| = 0. \end{aligned} \tag{2.12}$$

A sufficient condition for the limit above is that the shifts of $\overline{F}$ are r_0-independent for some $r_0 < \infty$ and $\mathbb{E}|\overline{F}(\omega, z)|^{d+\varepsilon} < \infty$.

Theorem 2.7. *Let $F \in \mathcal{K}_0$. Under assumption (2.12) we have the following uniform ergodic theorem:*

$$\lim_{n\to\infty} \max_{|x| \leqslant n} \frac{|F(\omega, 0, x)|}{n} = 0 \qquad \mathbb{P}\text{-a.s.}$$

For a proof see Appendix A.3 of [6].

2.3. **Variational formulas.**

In this section we derive variational formulas for the restricted path length point-to-level and point-to-point last-passage values.

Theorem 2.8.

$$g_{\text{pl}}(h) = \inf_{F \in \mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \max_{z \in \mathcal{R}} \{V(\omega, z) + h \cdot z + F(\omega, 0, z)\}. \tag{2.13}$$

A minimizing $F \in \mathcal{K}_0$ exists for each $h \in \mathbb{R}^2$.

Abbreviate

$$K(F) = \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \max_{z \in \mathcal{R}} \{V(\omega, z) + h \cdot z + F(\omega, 0, z)\}. \tag{2.14}$$

Proof. **Upper bound.** Let $F \in \mathcal{K}_0$. Assume $K(F) < \infty$. Then

$$F(\omega, 0, z) \leqslant -V(\omega, z) - h \cdot z + K(F)$$

together with assumption (2.3) on V imply that F satisfies assumption (2.12) and therefore the uniform ergodic theorem (Theorem 2.7) applies.

$$
\begin{aligned}
g_{\text{pl}}(h) &= \lim_{n\to\infty} \max_{x_{0,n}} \frac{1}{n}\Big\{ \sum_{k=0}^{n-1} V(\theta_{x_k}\omega, z_{k+1}) + h\cdot x_n \Big\} \\
&= \lim_{n\to\infty} \max_{x_{0,n}} \frac{1}{n}\Big\{ \sum_{k=0}^{n-1} V(\theta_{x_k}\omega, z_{k+1}) + h\cdot x_n + F(\omega,0,x_n) \Big\} \\
&= \lim_{n\to\infty} \max_{x_{0,n}} \frac{1}{n} \sum_{k=0}^{n-1} \big[V(\theta_{x_k}\omega, z_{k+1}) + h\cdot z_{k+1} + F(\theta_{x_k}\omega, 0, z_{k+1}) \big] \\
&\leqslant K(F)
\end{aligned}
$$

because the last upper bound is valid $\mathbb{P}$-a.s. for each term. We have shown that

$$
g_{\text{pl}}(h) \leqslant \inf_{F\in\mathcal{K}_0} K(F).
$$

Lower bound. Let $\lambda > g_{\text{pl}}(h)$. Set $u_n(\omega) = e^{G_n(h)-n\lambda}$ with the interpretation that $u_0 = 1$. Since $n^{-1}G_n(h) \to g_{\text{pl}}(h)$ almost surely, $u_n(\omega) < e^{-n\varepsilon}$ for large n and a fixed $\varepsilon > 0$. Hence f below is a well-defined finite function:

$$
\begin{aligned}
f(\omega) \;&=\; \sum_{n=0}^{\infty} u_n(\omega) \;=\; 1 + \sum_{n=1}^{\infty} \exp\Big\{ \max_{x_{0,n}} \Big[V(\omega,x_1) + h\cdot x_1 - \lambda \\
&\qquad\qquad + \sum_{k=1}^{n-1} V(\theta_{x_k}, z_{k+1}) + h\cdot(x_n - x_1) - (n-1)\lambda \Big] \Big\} \\
&=\; 1 + \max_z e^{V(\omega,z)+h\cdot z-\lambda} \sum_{n=1}^{\infty} u_{n-1}(\theta_z\omega) \;\geqslant\; \max_z e^{V(\omega,z)+h\cdot z-\lambda} f(\theta_z\omega) \\
&=\; e^{-\lambda}\cdot e^{\max_z [V(\omega,z)+h\cdot z+\log f(\theta_z\omega)]}.
\end{aligned}
$$

Rearrange this to

$$
\lambda \geqslant \max_z \{ V(\omega,z) + h\cdot z + \log f(\theta_z\omega) - \log f(\omega) \} \qquad \text{a.s}
$$

from which

$$
\lambda \geqslant K(\nabla \log f) \geqslant \inf_{F\in\mathcal{K}_0} K(F)
$$

provided $\nabla \log f \in \mathcal{K}_0$ which is implied by the next lemma. Let $\lambda \searrow g_{\text{pl}}(h)$ to get

$$
g_{\text{pl}}(h) \geqslant \inf_{F\in\mathcal{K}_0} K(F).
$$

The existence of minimizer is proved by a weak convergence argument that we skip. It is given for positive temperature polymer models in Theorem 2.3 of [8]. ☐

LEMMA 2.9. *For a measurable function* $\varphi : \Omega \to \mathbb{R}$ *define* $\nabla\varphi(\omega,x,y) = \varphi(\theta_y\omega) - \varphi(\theta_x\omega)$. *Then* $K(\nabla\varphi) < \infty$ *implies* $\nabla\varphi(\cdot\,,z) \in L^1$ $\forall z \in \mathcal{R}$.

Proof. For each $z \in \mathcal{R}$,

$$
V(\omega,z) + h\cdot z + \nabla\varphi(\omega,0,z) \leqslant K(\nabla\varphi) < \infty \qquad \text{a.s.}
$$

and so $(\nabla\varphi(\cdot\,,z))^+ \in L^1(\mathbb{P})$. Suppose $(\nabla\varphi(\cdot\,,z))^- \notin L^1$. Then a contradiction arises as follows, where the first equality comes by the pointwise ergodic theorem:

$$\begin{aligned}-\infty &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\nabla\varphi(\theta_{kz}\omega,z) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}[\varphi(\theta_{(k+1)z}\omega)-\varphi(\theta_{kz}\omega)]\\ &= \lim_{n\to\infty}\frac{1}{n}[\varphi(\theta_{nz}\omega)-\varphi(\omega)] = 0 \qquad \text{in probability.}\end{aligned}$$ □

Theorem 2.10. *For each $\xi \in \mathrm{ri}\,\mathcal{U}$ we have this variational formula.*

$$g_{\mathrm{pp}}(\xi) \;=\; \inf_{B\in\mathcal{K}}\ \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \max_{z\in\mathcal{R}}\{V(\omega,z)-B(\omega,0,z)-h(B)\cdot\xi\}. \tag{2.15}$$

For each $\xi \in \mathrm{ri}\,\mathcal{U}$ there exists a maximizing $B \in \mathcal{K}$ such that $h(B)$ is dual to ξ.

Proof. From duality (2.9),

$$\begin{aligned}g_{\mathrm{pp}}(\xi) &= \inf_{h\in\mathbb{R}^d}\{g_{\mathrm{pl}}(h)-h\cdot\xi\}\\ &= \inf_{h\in\mathbb{R}^d}\inf_{F\in\mathcal{K}_0}\ \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \max_{z\in\mathcal{R}}\{V(\omega,z)+h\cdot z+F(\omega,0,z)-h\cdot\xi\}\\ &= \inf_{B\in\mathcal{K}}\ \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \max_{z\in\mathcal{R}}\{V(\omega,z)-B(\omega,0,z)-h(B)\cdot\xi\}\end{aligned}$$

where we let

$$B(\omega,x,y) = -h\cdot(y-x)-F(\omega,x,y)$$

define an element of $\mathcal{K}$ with

$$h(B)\cdot z = -\mathbb{E}[B(0,z)]\cdot z = h\cdot z \quad\Longrightarrow\quad h(B)=h.$$

The first infimum above is achieved at some h dual to ξ whose existence is given in Lemma 2.5, and for this h a minimizing F exists for $g_{\mathrm{pl}}(h)$. Thus the B defined above has $h(B)$ dual to ξ. □

2.4. **Cocycles adapted to the potential.** The cocycles and the potential V that defines the percolation both live on a general product space with some product probability measure $\mathbb{P}$. It is not evident how these two structures are connected. Next we identify a local condition that characterizes those cocycles that are relevant to the percolation problem in various ways.

Definition 2.11. *A cocycle $B \in \mathcal{K}$ is adapted to the potential V if*

$$\max_{z\in\mathcal{R}}[V(\omega,z)-B(\omega,0,z)] = 0 \quad \mathbb{P}\text{-}a.s. \tag{2.16}$$

This condition is linked to (i) minimizing cocycles, (ii) geodesics, (iii) Busemann functions and (iv) stationary percolation. We discuss briefly these four issues still in general, before moving to an exactly solvable model.

Minimizing cocycles. Suppose $B \in \mathcal{K}$ satisfies (2.16). Define a mean-zero cocycle $F \in \mathcal{K}_0$ by

$$F(\omega,x,y) = -h(B)\cdot(y-x)-B(\omega,x,y).$$

Then (2.16) becomes

$$0 = \max_{z\in\mathcal{R}}[V(\omega,z)+h(B)\cdot z+F(\omega,0,z)] \quad \mathbb{P}\text{-a.s.}$$

The one-sided bound $B(\omega,0,z) \geqslant V(\omega,z)$ is enough for the uniform ergodic theorem to work for F. Thus we can iterate the identity above and take a limit.

$$\begin{aligned}0 &= \max_{x_{0,n}}\Big\{\frac{1}{n}\sum_{k=0}^{n-1}V(\theta_{x_k}\omega,z_{k+1})+\frac{1}{n}h(B)\cdot x_n+\frac{1}{n}F(\omega,0,x_n)\Big\}\\ &= \frac{1}{n}G_n(h(B))+o(1) \quad\longrightarrow\quad g_{\mathrm{pl}}(h(B)).\end{aligned}$$

We can conclude that

$$g_{\rm pl}(h(B)) = 0 = \max_{z\in\mathcal{R}}\big[V(\omega,z) + h(B)\cdot z + F(\omega,0,z)\big] \quad \mathbb{P}\text{-a.s.}$$

The equality above shows that F minimizes in the variational formula

$$g_{\rm pl}(h) = \inf_{F\in\mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \max_{z\in\mathcal{R}}\{V(\omega,z) + h\cdot z + F(\omega,0,z)\}$$

for $h = h(B)$, even without the essential supremum over ω.

Furthermore: suppose $h(B)$ and ξ are dual. Then from above, almost surely,

$$g_{\rm pp}(\xi) = g_{\rm pl}(h(B)) - h(B)\cdot\xi = -h(B)\cdot\xi = \max_{z\in\mathcal{R}}[V(\omega,z) - B(\omega,0,z)] - h(B)\cdot\xi.$$

Thus B is a minimizer for $g_{\rm pp}(\xi)$.

In summary, we see that (2.16) is a criterion that finds cocycles that serve as minimizers in the variational formulas. For future use we record the outcome of the calculation above in the next lemma.

LEMMA 2.12. *Let $B \in \mathcal{K}$ be an integrable stationary cocycle adapted to the potential V as required by* (2.16) *and let $h(B)$ be the negative of the mean vector of B as defined in* (2.10)*. Then $g_{\rm pl}(h(B)) = 0$.*

Geodesics. Turns out that a cocycle B satisfying (2.16) is involved not only in optimization on the macroscopic level but also on the pathwise level. Suppose the path $(x_k)_{k=0}^n$ follows the maximal increments specified in (2.16), in other words, satisfies

$$V(\theta_{x_k}\omega, z_{k+1}) - B(\omega, x_k, x_{k+1}) = 0 \qquad \forall k = 0,1,\dots,n-1. \tag{2.17}$$

Then this path is a geodesic from x_0 to x_n. Here is the simple argument: consider any path $y_\bullet$ from $y_0 = x_0$ to $y_n = x_n$. Then, by (2.16), the stationarity and additivity of B, and (2.17),

$$\begin{aligned}\sum_{k=0}^{n-1} V(\theta_{y_k}\omega, y_{k+1} - y_k) &\leqslant \sum_{k=0}^{n-1} B(\theta_{y_k}\omega, 0, y_{k+1} - y_k) = \sum_{k=0}^{n-1} B(\omega, y_k, y_{k+1}) = B(\omega, x_0, x_n)\\ &= \sum_{k=0}^{n-1} B(\omega, x_k, x_{k+1}) = \sum_{k=0}^{n-1} V(\theta_{x_k}\omega, z_{k+1}).\end{aligned}$$

Busemann functions. Having seen the usefulness of condition (2.16), we must ask how cocycles that satisfy (2.16) arise? One way of obtaining such cocycles is through limits of local gradients of passage times, called *Busemann functions*.

Suppose that we are in a setting where admissible paths that connect two given points have a uniquely determined number of steps. Let $G_{x,y} = G_{x,(m),y}$ denote the point-to-point last-passage value where m is the unique number of steps from x to y. Fix a direction $\xi \in \mathrm{ri}\,\mathcal{U}$. Assume that for all sequences $\{v_n\} \subset \mathbb{Z}^d$ such that $v_n/|v_n| \to \xi$ we have this almost sure limit $\forall x, y \in \mathbb{Z}^d$:

$$B(\omega, x, y) = \lim_{n\to\infty}[G_{x,v_n} - G_{y,v_n}] \quad \mathbb{P}\text{-a.s.} \tag{2.18}$$

B is called a Busemann function in direction ξ. It is a stationary cocycle. Additivity is immediate from the limit (2.18). Stationarity comes from the fact that shifting v_n by a fixed amount does not alter its limiting direction ξ. Under additional assumptions (see for example Theorem 5.1 in [5]) this cocycle is integrable.

At this time we simply wish to observe that B satisfies (2.16):

$$\max_{z\in\mathcal{R}}[V(\omega,z) - B(\omega,0,z)] = \lim_{n\to\infty}\max_{z\in\mathcal{R}}[V(\omega,z) - G_{0,v_n} + G_{z,v_n}] = 0$$

where the last equality follows from

$$G_{0,v_n} = \max_{z\in\mathcal{R}}[V(\omega,z) + G_{z,v_n}]. \tag{2.19}$$

Proof of the limit in (2.18) is highly nontrivial. The route that we will take to finding cocycles that satisfy (2.16) will involve the last item below.

Stationary percolation. Once again we assume cocycle B satisfies (2.16) and develop this identity in a different direction. Let $v \in \mathbb{Z}^d$ be fixed.

$$\begin{aligned} 0 &= \max_{z\in\mathcal{R}}[V(\theta_x\omega, z) - B(\theta_x\omega, 0, z)] = \max_{z\in\mathcal{R}}[V(\theta_x\omega, z) - B(\omega, x, x+z)] \\ &= \max_{z\in\mathcal{R}}[V(\theta_x\omega, z) - B(\omega, x, v) + B(\omega, x+z, v)] \end{aligned}$$

from which we write

$$B(\omega, x, v) = \max_{z\in\mathcal{R}} [V(\theta_x\omega, z) + B(\omega, x+z, v)] \qquad \forall x, v \in \mathbb{Z}^d. \tag{2.20}$$

We iterate this. Fix also $u \in \mathbb{Z}^d$ and write $x_n = u + z_1 + \ldots + z_n$ for an admissible path from u with steps z_i.

$$\begin{aligned} B(\omega, u, v) &= \max_{z_1\in\mathcal{R}}[V(\theta_{x_0}\omega, z_1) + B(\omega, x_1, v)] \\ &= \max_{z_1,z_2\in\mathcal{R}}[V(\theta_{x_0}\omega, z_1) + V(\theta_{x_1}\omega, z_2) + B(\omega, x_2, v)] \\ &= \cdots = \max_{z_1,\ldots,z_n\in\mathcal{R}}\Big[\sum_{k=0}^{n-1} V(\theta_{x_k}\omega, z_{k+1}) + B(\omega, x_n, v)\Big] \\ &= \max_x [\, G_{u,(n),x} + B(\omega, x, v)]. \end{aligned} \tag{2.21}$$

We can turn this into a boundary value problem. Assume again that the number of steps on an admissible path is determined uniquely by the endpoints so that we can write

$$G_{x,y} = \begin{cases} G_{x,(n),y} & \text{if } y \text{ is reachable from } x \text{ along an admissible path of } n \text{ steps} \\ -\infty & \text{if } y \text{ is not reachable from } x \text{ along an admissible path.} \end{cases}$$

Let $\mathcal{H}$ and $\partial\mathcal{H}$ be finite subsets of $\mathbb{Z}^d$ with the property that any admissible path from $\mathcal{H}$ eventually intersects $\partial\mathcal{H}$. So in a sense $\partial\mathcal{H}$ is a "boundary" of $\mathcal{H}$.

For example, suppose we are in the directed case $\mathcal{R} = \{e_1, \ldots, e_d\}$. If $\mathcal{H}$ is the rectangle $\mathcal{H} = \prod_{i=1}^d \{0, 1, \ldots, N_i\}$, then $\partial\mathcal{H}$ could be its "northeast" boundary $\partial\mathcal{H} = \bigcup_{i=1}^d \{x \in \mathcal{H} : x_i = N_i\}$.

Lemma 2.13. *Assume that endpoints of paths determine uniquely the number of steps in the path. Assume that the stationary cocycle B satisfies (2.16). Fix $v \in \mathbb{Z}^d$ and finite subsets $\mathcal{H}$ and $\partial\mathcal{H}$ of $\mathbb{Z}^d$ such that every admissible path from $\mathcal{H}$ intersects $\partial\mathcal{H}$. Then*

$$B(\omega, u, v) = \max_{x\in\partial\mathcal{H}} [\, G_{u,x} + B(\omega, x, v)] \qquad \text{for all } u \in \mathcal{H}. \tag{2.22}$$

Proof. Fix $u \in \mathcal{H}$. Equation (2.21) gives

$$B(\omega, u, v) \geqslant G_{u,x} + B(\omega, x, v)$$

whenever there is an admissible path from u to x. This tells us that $\geqslant$ holds in (2.22). It also shows that if $u \in \mathcal{H} \cap \partial\mathcal{H}$ then the maximum in (2.22) is assumed at $x = u$ and the equality holds.

Next proof of $\leqslant$ in (2.22). By the directedness assumption (2.2) we can take n in (2.21) large enough so that every n-path from u intersects $\partial\mathcal{H}$. Fix a maximizing path $u = x_0, x_1, \ldots, x_n$ on the second last line of (2.21). Let $x_m \in \partial\mathcal{H}$. Then

$$\begin{aligned} B(\omega,u,v) &= \sum_{k=0}^{m-1} V(\theta_{x_k}\omega, z_{k+1}) + \sum_{k=m}^{n-1} V(\theta_{x_k}\omega, z_{k+1}) + B(\omega, x_n, v) \\ &\leqslant G_{u,x_m} + G_{x_m,x_n} + B(\omega,x_n,v) \leqslant G_{u,x_m} + B(\omega,x_m,v) \\ &\leqslant \text{right-hand side of (2.22)}. \end{aligned}$$

In the second-last inequality above we applied (2.21) with x_m in place of u and $n-m$ in place of n. ∎

Our interpretation is that equation (2.22) determines the values $\{B(u,v) : u \in \mathcal{H}\}$ from the last-passage percolation $G_{x,y}$ and given boundary values $\{B(x,v) : x \in \partial\mathcal{H}\}$. Then we can search for cases where the solution is tractable. In particular, we can look for distributional invariance. We can make this program work in exactly solvable cases.

The analogy the reader should have in mind is finding the invariant distribution of a Markov process such as an interacting particle system. The analogue of boundary values are the state variables at time zero. The analogue of the weights $V(\theta_x\omega, z)$ are the Poisson clocks or other random variables that govern the evolution of the particles.

3. Stationary exponential corner growth model in two dimensions

We restrict now to the two-dimensional corner growth model (CGM), with real weights on the vertices: $\omega = (\omega_x)_{\mathbb{Z}^2} \in \Omega = \mathbb{R}^{\mathbb{Z}^2}$. The set of admissible steps is $\mathcal{R} = \{e_1, e_2\}$, and potential given by the weight at the origin: $V(\omega, z) = \omega_0$. The set of possible limiting velocities of paths is the closed line segment $\mathcal{U} = [e_1, e_2]$, and its relative interior is the open line segment $\operatorname{ri}\mathcal{U} = (e_1, e_2)$.

In this setting we alter slightly the earlier definition (2.5) of the point-to-point last-passage time to include both endpoints of the path. This makes no difference to large-scale properties. Given an environment ω and two points $x, y \in \mathbb{Z}^2$ with $x \leqslant y$ coordinatewise, define

$$G_{x,y} = \max_{x_\bullet \in \Pi_{x,y}} \sum_{k=0}^{|y-x|_1} \omega_{x_k}. \tag{3.1}$$

$\Pi_{x,y}$ is the set of paths $x_\bullet = (x_k)_{k=0}^n$ that start at $x_0 = x$, end at $x_n = y$ with $n = |y-x|_1$, and have increments $x_{k+1} - x_k \in \{e_1, e_2\}$. Call such paths *admissible* or *up-right*. The zero-length path case is $G_{x,x} = \omega_x$. Our convention is that

$$G_{x,y} = -\infty \qquad \text{if } x \leqslant y \text{ fails.} \tag{3.2}$$

We work with the exponentially distributed weights, and so make the following assumption:

(3.3) the weights ω_x are independent rate 1 exponentially distributed random variables.

This means that $\mathbb{P}\{\omega_x > t\} = e^{-t}$ for $t \geqslant 0$. This is abbreviated as $\omega_x \sim \operatorname{Exp}(1)$.

By Theorem 2.4 we have the limiting point-to-point shape function defined by the almost sure limit

$$g_{\text{pp}}(\xi) = \lim_{N\to\infty} N^{-1} G_{0,\lfloor N\xi\rfloor} \qquad \text{for } \xi \in \mathbb{R}^2_{\geqslant 0}. \tag{3.4}$$

This function g_{pp} is concave, continuous and homogeneous [$g_{\text{pp}}(c\xi) = cg_{\text{pp}}(\xi)$ for $c \geqslant 0$]. A stronger result is also true: the *shape theorem* gives a uniform limit (Theorem 5.1(i) of [7]):

$$\lim_{n\to\infty} n^{-1} \max_{x \in \mathbb{Z}^2_{\geqslant 0} : |x|_1 = n} |G_{0,x} - g_{\text{pp}}(x)| = 0 \quad \mathbb{P}\text{-almost surely.} \tag{3.5}$$

Our first task is to construct a coupling of i.i.d. rate 1 exponential weights and a stationary integrable cocycle, for a given value of a parameter $0 < \rho < 1$, that together satisfy (2.16), essentially by solving the boundary value problem in (2.22). This construction will be performed on quadrants $u+\mathbb{Z}^2_{>0}$ with a specified origin $u \in \mathbb{Z}^2$.

Fix a parameter $0 < \rho < 1$ and an origin $u \in \mathbb{Z}^2$. Assume given a collection of mutually independent random variables

$$\{\omega_x,\, I_{u+ie_1},\, J_{u+je_2} : x \in u+\mathbb{Z}^2_{>0},\, i,j \in \mathbb{Z}_{>0}\} \tag{3.6}$$

with these marginal distributions:

$$\omega_x \sim \text{Exp}(1), \quad I_{u+ie_1} \sim \text{Exp}(1-\rho), \quad \text{and} \quad J_{u+je_2} \sim \text{Exp}(\rho). \tag{3.7}$$

The interpretation is that I_x is a weight for the edge $(x-e_1, x)$, J_x is a weight for edge $(x-e_2, x)$, and ω_x is a vertex weight. The edge weights are on the boundary of the quadrant $u+\mathbb{Z}^2_{\geqslant 0}$ and the vertex weights in the bulk.

In this setting define another last-passage process $G^{\rho}_{u,x}$ with origin fixed at u and that utilizes edge weights on the boundary and then bulk weights. First put $G^{\rho}_{u,u}=0$ and on the boundaries

$$G^{\rho}_{u,\,u+me_1} = \sum_{i=1}^{m} I_{u+ie_1} \quad \text{and} \quad G^{\rho}_{u,\,u+ne_2} = \sum_{j=1}^{n} J_{u+je_2}. \tag{3.8}$$

Then in the bulk for $x = u+(m,n) \in u+\mathbb{Z}^2_{>0}$,

$$G^{\rho}_{u,\,x} = \max_{1\leqslant k\leqslant m} \Big\{ \sum_{i=1}^{k} I_{u+ie_1} + G_{u+ke_1+e_2,\,x}\Big\} \bigvee \max_{1\leqslant \ell\leqslant n} \Big\{ \sum_{j=1}^{\ell} J_{u+je_2} + G_{u+\ell e_2+e_1,\,x}\Big\} \tag{3.9}$$

$G_{a,x}$ inside the braces is the last-passage value defined in (3.1). The superscript ρ in $G^{\rho}_{u,x}$ distinguishes this last-passage value from the one in (3.1) with i.i.d. bulk weights, and the first subscript u specifies that the I and J edge weights are placed on the axes $u+\mathbb{Z}_{>0}e_k$, $k=1,2$.

An equivalent definition of $G^{\rho}_{u,x}$ would be to give the boundary conditions (3.8) and the inductive equation

$$G^{\rho}_{u,x} = \omega_x + G^{\rho}_{u,\,x-e_1} \vee G^{\rho}_{u,\,x-e_2}\,, \qquad x \in u+\mathbb{Z}^2_{>0}. \tag{3.10}$$

From the given variables (3.6) we define further variables as follows, proceeding inductively to the north and east from the origin u: for all $x \in u+\mathbb{Z}^2_{>0}$,

$$\breve{\omega}_{x-e_1-e_2} = I_{x-e_2} \wedge J_{x-e_1} \tag{3.11}$$

$$I_x = \omega_x + (I_{x-e_2} - J_{x-e_1})^+ \tag{3.12}$$

$$J_x = \omega_x + (I_{x-e_2} - J_{x-e_1})^-. \tag{3.13}$$

The mapping above from $(\omega_x, I_{x-e_2}, J_{x-e_1})$ to $(\breve{\omega}_{x-e_1-e_2}, I_x, J_x)$ is illustrated in Figure 1. Note that (3.12)–(3.13) imply the symmetric counterpart of (3.11)

$$\omega_x = I_x \wedge J_x \tag{3.14}$$

and the additivity around the unit square:

$$I_{x-e_2} + J_x = J_{x-e_1} + I_x. \tag{3.15}$$

Utilizing (3.11)–(3.13) we extend (3.6) to the larger collection

$$\{\omega_x,\, I_{x-e_2},\, J_{x-e_1},\, \breve{\omega}_{x-e_1-e_2} : x \in u+\mathbb{Z}^2_{>0}\}. \tag{3.16}$$

This larger collection has an $\breve{\omega}_x$ variable for each vertex in the quadrant $u+\mathbb{Z}^2_{\geqslant 0}$, an I_x variable for each horizontal nearest-neighbor edge in the quadrant $u+\mathbb{Z}^2_{\geqslant 0}$, a J_x variable for each vertical nearest-neighbor edge in the quadrant $u+\mathbb{Z}^2_{\geqslant 0}$, and the originally given ω_x variables for points in the bulk $u+\mathbb{Z}^2_{>0}$.

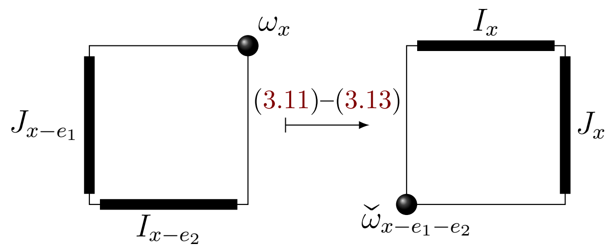

Figure 1. Mapping (3.11)–(3.13) on a single lattice square. The figure illustrates how southwest corners are flipped into northeast corners in the inductive construction of the increment variables.

The next theorem summarizes the properties of (3.16) and the connection with the process $G^\rho_{u,x}$. Point (iii) of the theorem uses the following definitions. A bi-infinite sequence $\mathcal{Y} = (y_k)_{k\in\mathbb{Z}}$ in $u+\mathbb{Z}^2_{\geqslant 0}$ is a *down-right path* if $y_k - y_{k-1} \in \{e_1, -e_2\}$ for all $k \in \mathbb{Z}$. $\mathcal{Y}$ decomposes the vertices of the quadrant into a disjoint union $u + \mathbb{Z}^2_{\geqslant 0} = \mathcal{G}_- \cup \mathcal{Y} \cup \mathcal{G}_+$ where

$$\mathcal{G}_- = \{x \in u + \mathbb{Z}^2_{\geqslant 0} : \exists j \in \mathbb{Z}_{>0} \text{ such that } x + (j,j) \in \mathcal{Y}\}$$

is the region strictly to the south and west of $\mathcal{Y}$ and

$$\mathcal{G}_+ = \{x \in u + \mathbb{Z}^2_{\geqslant 0} : \exists j \in \mathbb{Z}_{>0} \text{ such that } x - (j,j) \in \mathcal{Y}\}$$

is the region strictly to the north and east of $\mathcal{Y}$. Note that $\mathcal{G}_+$ is necessarily unbounded but $\mathcal{G}_-$ is finite iff all but finitely many of the points y_k lie on the axes $u + \mathbb{Z}_{>0}e_k$, $k = 1, 2$. In the extreme case $\mathcal{Y} = \{u + ie_1, u + je_2 : 0 \leqslant i, j < \infty\}$ consists of the axes, $\mathcal{G}_- = \varnothing$ and $\mathcal{G}_+ = u + \mathbb{Z}^2_{>0}$.

For an undirected nearest-neighbor edge e on $u + \mathbb{Z}^2_{\geqslant 0}$, we denote the weight by

$$t(e) = \begin{cases} I_x & \text{if } e = \{x - e_1, x\} \\ J_x & \text{if } e = \{x - e_2, x\}. \end{cases} \tag{3.17}$$

Theorem 3.1. *Fix $u \in \mathbb{Z}^2$ and $0 < \rho < 1$ and assume given independent variables (3.6) with marginal distributions (3.7). Then the variables in (3.16) have the following properties.*

(i) *For any down-right path $\mathcal{Y}$ in $u + \mathbb{Z}^2_{\geqslant 0}$, the random variables*

$$\{\,\breve\omega_z,\ t(\{y_{k-1}, y_k\}),\ \omega_x : z \in \mathcal{G}_-,\ k \in \mathbb{Z},\ x \in \mathcal{G}_+\} \tag{3.18}$$

are mutually independent with marginal distributions

$$\omega_x, \breve\omega_x \sim \text{Exp}(1), \quad I_x \sim \text{Exp}(1-\rho), \quad \textit{and} \quad J_x \sim \text{Exp}(\rho). \tag{3.19}$$

(ii) *The I_x and J_x variables are the increments of the $G^\rho_{u,x}$ last-passage process:*

$$\begin{aligned} I_x &= G^\rho_{u,x} - G^\rho_{u,x-e_1} \quad && \textit{for } x \in u + (\mathbb{Z}_{>0}) \times (\mathbb{Z}_{\geqslant 0}), \\ J_x &= G^\rho_{u,x} - G^\rho_{u,x-e_2} \quad && \textit{for } x \in u + (\mathbb{Z}_{\geqslant 0}) \times (\mathbb{Z}_{>0}). \end{aligned} \tag{3.20}$$

Theorem 3.1 rests on an inductive argument based on the next lemma, which describes the joint distribution preserved by the mapping in Figure 1.

Lemma 3.2. *Let $0 < \rho < 1$. Assume given independent variables $W \sim \text{Exp}(1)$, $I \sim \text{Exp}(1-\rho)$, and $J \sim \text{Exp}(\rho)$. Define*

$$\begin{aligned} W' &= I \wedge J \\ I' &= W + (I - J)^+ \\ J' &= W + (I - J)^-. \end{aligned} \tag{3.21}$$

Then the triple (W', I', J') *has the same distribution as* (W, I, J).

This lemma is proved by calculating a joint transform such as the Laplace transform or characteristic function, or by transforming the joint density.

Proof of Theorem 3.1. Part (i). This is proved inductively on $\mathcal{Y}$. The base case is $\mathcal{Y} = \{u + ie_1, u + je_2 : 0 \leqslant i, j < \infty\}$, in which case the claim simply amounts to the initial condition in (3.7).

Now assume given $\mathcal{Y} = \{y_k\}$ for which the claim in part (i) holds. We show that this claim continues to hold for any $\mathcal{Y}'$ obtained from $\mathcal{Y}$ by "flipping a southwest corner into a northeast corner". So pick any $x \in u + \mathbb{Z}^2_{>0}$ and $m \in \mathbb{Z}$ such that $(y_{m-1}, y_m, y_{m+1}) = (x - e_1, x - e_1 - e_2, x - e_2)$ are points along $\mathcal{Y}$. Define $\mathcal{Y}' = \{y'_k\}$ by setting

$$y'_k = y_k \qquad \text{for } k \neq m, \qquad \text{and} \qquad y'_m = y_m + e_1 + e_2 = x.$$

In other words, $\mathcal{Y}$ has a southwest corner at $x - e_1 - e_2$, and $\mathcal{Y}'$ has a northeast corner at x.

Transforming $\mathcal{Y}$ into $\mathcal{Y}'$ changes $\mathcal{G}_-$ to $\mathcal{G}'_- = \mathcal{G}_- \cup \{x - e_1 - e_2\}$ and $\mathcal{G}_+$ to $\mathcal{G}'_+ = \mathcal{G}_+ \backslash \{x\}$. Thus constructing the variables (3.18) for $\mathcal{Y}'$ involves transforming the triple $(\omega_x, I_{x-e_2}, J_{x-e_1})$ into $(\breve{\omega}_{x-e_1-e_2}, I_x, J_x)$ according to equations (3.11)–(3.13), and copying the remaining variables from (3.18) for $\mathcal{Y}$. The claim now follows for $\mathcal{Y}'$ by the induction assumption and Lemma 3.2. By the induction assumption, variables $(\omega_x, I_{x-e_2}, J_{x-e_1})$ have the independent exponential distributions required for the hypothesis of Lemma 3.2, and so by the lemma the triple $(\breve{\omega}_{x-e_1-e_2}, I_x, J_x)$ also has the independent exponential distributions required for (3.19).

Part (ii). The claim is true by construction for variables I_{u+ie_1} and J_{u+je_2} on the axes. Here is the inductive argument for I_x, assuming that the claim holds for I_{x-e_2} and J_{x-e_1} and utilizing (3.10):

$$\begin{aligned} I_x &= \omega_x + (I_{x-e_2} - J_{x-e_1})^+ = \omega_x + (G^\rho_{u,\,x-e_2} - G^\rho_{u,\,x-e_1})^+ \\ &= \omega_x + G^\rho_{u,\,x-e_1} \vee G^\rho_{u,\,x-e_2} - G^\rho_{u,\,x-e_1} \\ &= G^\rho_{u,\,x} - G^\rho_{u,\,x-e_1}. \end{aligned}$$

A similar argument works for J_x under the same inductive assumption. ∎

Let us observe some immediate and valuable consequences of Theorem 3.1.

By taking $\mathcal{Y}$ as the axes at a new origin $v \in u + \mathbb{Z}^2_{\geqslant 0}$, given by $y_k = v + ke_1$ and $y_{-k} = v + ke_2$ for $k \geqslant 0$, part (i) of the theorem implies that the process $\{\omega_{v+x}, I_{v+x-e_2}, J_{v+x-e_1} : x \in \mathbb{Z}^2_{>0}\}$ has the same distribution for all $v \in u + \mathbb{Z}^2_{\geqslant 0}$. Thus $B(x, y) = G^\rho_{u,y} - G^\rho_{u,x}$ is a stationary cocycle, restricted to the quadrant $x, y \in u + \mathbb{Z}^2_{\geqslant 0}$.

The variables $\{\breve{\omega}_x : x \in u + \mathbb{Z}^2_{\geqslant 0}\}$ are i.i.d. Exp(1) distributed.

We compute the limit shape functions for both last-passage percolation processes, the stationary one and the one with i.i.d. weights. For the stationary process define the function

$$g^\rho(s, t) = \frac{s}{1 - \rho} + \frac{t}{\rho}. \tag{3.22}$$

Proposition 3.3. *Fix* $0 < \rho < 1$. *The stationary corner growth model satisfies these properties:* $\mathbb{E}[G^\rho_{0,(m,n)}] = g^\rho(m, n)$ *for all* $m, n \in \mathbb{Z}_{\geqslant 0}$ *and the law of large numbers*

$$\lim_{N\to\infty} N^{-1} G^\rho_{0,(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} = g^\rho(s, t) \qquad \textit{almost surely and in } L^1 \textit{ for all } (s, t) \in \mathbb{R}^2_{\geqslant 0}. \tag{3.23}$$

Proof. Rewrite in terms of nearest-neighbor increments:

$$G^\rho_{0,(m,n)} = \sum_{i=1}^{m} I_{(i,0)} + \sum_{j=1}^{n} J_{(m,j)}. \tag{3.24}$$

Then use the translation invariance of the distributions which says that each nearest-neighbor increment has the exponential distribution imposed on the boundary variables in (3.7):

$$\mathbb{E}[G^{\rho}_{0,(m,n)}] = \sum_{i=1}^{m} \mathbb{E} I_{ie_1} + \sum_{j=1}^{n} \mathbb{E} J_{(m,j)} = \frac{m}{1-\rho} + \frac{n}{\rho}. \tag{3.25}$$

The limit of the stationary last-passage process is an application of the classical law of large numbers and some large deviation estimates, applied separately to the two sums: the limit below holds almost surely for any given $(s,t) \in \mathbb{R}^2_{\geqslant 0}$.

$$\begin{aligned} \lim_{N\to\infty} N^{-1} G^{\rho}_{0,(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} &= \lim_{N\to\infty} \Big\{ N^{-1} \sum_{i=1}^{\lfloor Ns\rfloor} I_{(i,0)} + N^{-1} \sum_{j=1}^{\lfloor Nt\rfloor} J_{(\lfloor Ns\rfloor,\, j)} \Big\} \\ &= s\mathbb{E}(I_{e_1}) + t\mathbb{E}(J_{e_2}) = \frac{s}{1-\rho} + \frac{t}{\rho}. \quad \square \end{aligned} \tag{3.26}$$

Next we take a limit in the coupling between the last-passage processes $G_{x,y}$ and $G^{\rho}_{0,x}$. Fix $s, t > 0$ and use (3.9) for $x = (\lfloor Ns\rfloor, \lfloor Nt\rfloor)$ to write

$$\begin{aligned} G^{\rho}_{0,(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} = \sup_{0\leqslant a\leqslant s} &\Big\{ \sum_{i=1}^{\lfloor Na\rfloor} I_{(i,0)} + G_{(\lfloor Na\rfloor,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} \Big\} \\ &\bigvee \sup_{0\leqslant b\leqslant t} \Big\{ \sum_{j=1}^{\lfloor Nb\rfloor} J_{(0,j)} + G_{(1,\lfloor Nb\rfloor),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} \Big\}. \end{aligned} \tag{3.27}$$

After letting $N \to \infty$, with some estimation on the right-hand side, utilizing limits (3.4) and (3.23), we have

$$\frac{s}{1-\rho} + \frac{t}{\rho} = \sup_{0\leqslant a\leqslant s} \Big\{ \frac{a}{1-\rho} + g_{\mathrm{pp}}(s-a,t) \Big\} \vee \sup_{0\leqslant b\leqslant t} \Big\{ \frac{b}{\rho} + g_{\mathrm{pp}}(s,t-b) \Big\}. \tag{3.28}$$

In the next theorem we take advantage of the connection above to find the shape function g_{pp} for the LPP process (3.1) with i.i.d. Exp(1) weights.

Theorem 3.4. *Assume (3.3). Then we have the following law of large numbers. For every $\xi \in \mathbb{R}^2_{\geqslant 0}$ the limit below holds with probability 1, with the shape function g_{pp} as given.*

$$\lim_{N\to\infty} N^{-1} G_{0,\lfloor N\xi\rfloor} = g_{\mathrm{pp}}(\xi) \equiv \big(\sqrt{\xi_1} + \sqrt{\xi_2}\,\big)^2. \tag{3.29}$$

Proof. By the general law of large numbers Theorem 2.4 for last-passage percolation, we know that the limit in (3.29) exists and that g_{pp} is finite, concave and continuous. Begin with (3.28) for $s = t$:

$$\frac{t}{1-\rho} + \frac{t}{\rho} = \sup_{0\leqslant a\leqslant t} \Big\{ \frac{a}{1-\rho} + g_{\mathrm{pp}}(t-a,t) \Big\} \vee \sup_{0\leqslant b\leqslant t} \Big\{ \frac{b}{\rho} + g_{\mathrm{pp}}(t,t-b) \Big\}.$$

Use the symmetry of g_{pp} and assume that $0 < \rho \leqslant 1/2$:

$$\begin{aligned} \frac{t}{1-\rho} + \frac{t}{\rho} &= \sup_{0\leqslant a\leqslant t} \Big\{ \frac{a}{1-\rho} + g_{\mathrm{pp}}(t-a,t) \Big\} \vee \sup_{0\leqslant b\leqslant t} \Big\{ \frac{b}{\rho} + g_{\mathrm{pp}}(t-b,t) \Big\} \\ &= \sup_{0\leqslant b\leqslant t} \Big\{ \frac{b}{\rho} + g_{\mathrm{pp}}(t-b,t) \Big\}. \end{aligned}$$

Let

$$f(b) = \begin{cases} -g_{\mathrm{pp}}(t-b,t), & 0 \leqslant b \leqslant t \\ \infty, & b < 0 \text{ or } b > t. \end{cases}$$

Then f is convex and lower semicontinuous. After a change of variable $x = 1/\rho \in [2, \infty)$, the equation above becomes

$$t\Big(x + 1 + \frac{1}{x-1}\Big) = \sup_{b\in\mathbb{R}}\{bx - f(b)\}, \qquad x \geqslant 2.$$

This is an instance of convex duality, so the convex conjugate f^* of f satisfies

$$f^*(x) = t\Big(x + 1 + \frac{1}{x-1}\Big) \qquad \text{for } x \geqslant 2.$$

The derivatives $(f^*)'(2+) = 0$ and $(f^*)'(\infty-) = t$ tell us that we can restrict the supremum in the double convex duality as below, for $0 \leqslant b \leqslant t$. Then find the supremum by calculus:

$$f(b) = f^{**}(b) = \sup_{x\geqslant 2}\{xb - f^*(x)\} = -\frac{b\sqrt{t(t-b)} + bt - t(\sqrt{t} + \sqrt{t-b}\,)^2}{\sqrt{t(t-b)}}.$$

Taking $b = t - s$ for $s \in [0, t]$ in the definition of f in terms of g gives

$$g_{\mathrm{pp}}(s, t) = (\sqrt{s} + \sqrt{t}\,)^2 \qquad \text{for } 0 \leqslant s \leqslant t.$$

Symmetry of g completes the proof. ☐

Next we use (3.28) to identify the characteristic direction $\xi(\rho) \in \mathcal{U}$ associated with parameter value $\rho \in (0, 1)$. By definition, $\xi(\rho)$ is the unique direction for which the optimal path for $G^{\rho}_{0,\lfloor N\xi(\rho)\rfloor}$ takes $o(N)$ steps on the coordinate axes as $N \to \infty$. It is the value of $\xi = (s, 1-s)$ for which the right-hand side of (3.28) with $(s, t) = (s, 1-s)$ is maximized at $a = b = 0$. This direction is given uniquely by

$$\xi(\rho) = \bigg(\frac{(1-\rho)^2}{(1-\rho)^2 + \rho^2}\,, \frac{\rho^2}{(1-\rho)^2 + \rho^2}\bigg). \tag{3.30}$$

An alternative characterization of the characteristic direction is by comparison of the stationary and i.i.d. limit shapes. In general $g_{\mathrm{pp}}(s, t) \leqslant g^{\rho}(s, t)$ for all $s, t \geqslant 0$, and

$$g_{\mathrm{pp}}(s, t) = g^{\rho}(s, t) \ \text{ if and only if } \ (s, t) = c\big((1-\rho)^2, \rho^2\big) \ \text{ for some } c \geqslant 0. \tag{3.31}$$

We can also record the limit of the point-to-line LPP process, defined by for $h \in \mathbb{R}^2$ by

$$G_n(h) = \max_{x_{0,n}:\, x_0 = 0}\Big\{\sum_{k=0}^{n-1} \omega_{x_k} + h \cdot x_n\Big\}.$$

Theorem 3.5. *Assume (3.3). Then for every $h \in \mathbb{R}^2$ the limit below holds with probability 1, with the limit function g_{pl} as given.*

$$\lim_{n\to\infty} n^{-1} G_n(h) = g_{\mathrm{pl}}(h) \equiv 1 + \frac{h_1 + h_2}{2} + \frac{1}{2}\sqrt{(h_1 - h_2)^2 + 4}\,. \tag{3.32}$$

Proof. The limit is given in Theorem 2.4. The formula for g_{pl} comes from the duality (2.8) with g_{pp} given in (3.29). ☐

4. Busemann functions for the exponential corner growth model

4.1. **Results.** In this section we prove the existence of the Busemann functions and show that they provide minimizers for the variational formulas.

We extend the constructions discussed in Section 3 to the full lattice $\mathbb{Z}^2$. As before a down-right path is a bi-infinite sequence $\mathcal{Y} = (y_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}^2$ such that $y_k - y_{k-1} \in \{e_1, -e_2\}$ for all $k \in \mathbb{Z}$. The lattice decomposes into a disjoint union $\mathbb{Z}^2 = \mathcal{G}_- \cup \mathcal{Y} \cup \mathcal{G}_+$ where the two regions are

$$\mathcal{G}_- = \{x \in \mathbb{Z}^2 : \exists j \in \mathbb{Z}_{>0} \text{ such that } x + j(e_1 + e_2) \in \mathcal{Y}\}$$

and

$$\mathcal{G}_+ = \{x \in \mathbb{Z}^2 : \exists j \in \mathbb{Z}_{>0} \text{ such that } x - j(e_1 + e_2) \in \mathcal{Y}\}.$$

It will be convenient to formalize the properties identified earlier in Theorem 3.1 in the following definition.

Definition 4.1. *Let $0 < \alpha < 1$. Let us say that a process*

$$\{\eta_x,\, I_x,\, J_x,\, \check{\eta}_x : x \in \mathbb{Z}^2\} \tag{4.1}$$

is an exponential-α last-passage percolation system if the following properties hold.

(a) *The process is stationary with marginal distributions*

$$\eta_x,\, \check{\eta}_x \sim \mathrm{Exp}(1), \quad I_x \sim \mathrm{Exp}(1-\alpha), \quad \textit{and} \quad J_x \sim \mathrm{Exp}(\alpha). \tag{4.2}$$

For any down-right path $\mathcal{Y} = (y_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}^2$, the random variables

$$\{\check{\eta}_z : z \in \mathcal{G}_-\}, \quad \{t(\{y_{k-1}, y_k\}) : k \in \mathbb{Z}\}, \quad \textit{and} \quad \{\eta_x : x \in \mathcal{G}_+\} \tag{4.3}$$

are all mutually independent, where the undirected edge variables $t(e)$ are defined as before in (3.17).

(b) *Equations (3.11)–(3.13) are in force at all $x \in \mathbb{Z}^2$.*

Recall in particular from (3.11) that the definition above implies the property

$$\check{\eta}_x = I_{x+e_1} \wedge J_{x+e_2}. \tag{4.4}$$

Theorem 4.2. *On the probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ of the i.i.d. $\mathrm{Exp}(1)$ weights $\omega = (\omega_x)_{x\in\mathbb{Z}^2}$ there exist for each $0 < \alpha < 1$ a stationary cocycle B^α and a family of random weights $\{X^\alpha_x\}_{x\in\mathbb{Z}^2}$ with the following properties.*

(i) *For each $0 < \alpha < 1$, process*

$$\{X^\alpha_x,\, B^\alpha_{x-e_1,x},\, B^\alpha_{x-e_2,x},\, \omega_x : x \in \mathbb{Z}^2\}$$

is an exponential-α last-passage system as described in Definition 4.1.

(ii) *There exists a single event Ω_2 of full probability such that for all $\omega \in \Omega_2$, all $x \in \mathbb{Z}^2$ and all $\lambda < \rho$ in $(0,1)$ we have the inequalities*

$$B^\lambda_{x,x+e_1}(\omega) \leqslant B^\rho_{x,x+e_1}(\omega) \quad \textit{and} \quad B^\lambda_{x,x+e_2}(\omega) \geqslant B^\rho_{x,x+e_2}(\omega). \tag{4.5}$$

(iii) *For each fixed $0 < \alpha < 1$ there exists an event $\Omega_2^{(\alpha)}$ of full probability such that the following holds: for each $\omega \in \Omega_2^{(\alpha)}$ and any sequence $v_n \in \mathbb{Z}^2$ such that $|v_n|_1 \to \infty$ and*

$$\lim_{n\to\infty} \frac{v_n}{|v_n|_1} = \xi(\alpha) = \left(\frac{(1-\alpha)^2}{(1-\alpha)^2 + \alpha^2}, \frac{\alpha^2}{(1-\alpha)^2 + \alpha^2}\right), \tag{4.6}$$

we have the limits

$$B^\alpha_{x,y}(\omega) = \lim_{n\to\infty} [G_{x,v_n} - G_{y,v_n}] \qquad \forall x, y \in \mathbb{Z}^2. \tag{4.7}$$

The next section is devoted to the proof of Theorem 4.2.

Remark 4.3. The process B^α has some regularity in α: it can be defined as left- or right-continuous (for each x, y, on an event of full probability). △

Part (i) of Theorem 4.2 together with (4.4) implies

$$\omega_x = B^\alpha_{x,x+e_1} \wedge B^\alpha_{x,x+e_2}, \tag{4.8}$$

in other words, cocycle B^α is adapted to the potential of the corner growth model in the sense of (2.16). (*Note that in this construction the given ω-weights are now playing the role of the $\breve{\omega}$-weights in Theorem 3.1, and the constructed weights X^α play the role of the ω-weights in Theorem 3.1.*)

Recall the variational formula for the point-to-point limit shape function, specialized to the two-dimensional corner growth model.

$$g_{\text{pp}}(\xi) \;=\; \inf_{B\in\mathcal{K}} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \max_{i=1,2} \{\omega_0 - B(\omega, 0, e_i) - h(B)\cdot\xi\}. \tag{4.9}$$

Above $h(B)$ is the negative of the mean vector:

$$h(B) = -\big(\mathbb{E}[B(0,e_1)], \mathbb{E}[B(0,e_2)]\big). \tag{4.10}$$

The next theorem shows that cocycle B^α from Theorem 4.2 is a minimizer in (4.9) for the characteristic direction $\xi(\alpha)$.

THEOREM 4.4. *Continue with the setting of Theorem 4.2. The following hold for each* $0 < \alpha < 1$.

(i) *The characteristic direction $\xi(\alpha)$ and the vector $h(B^\alpha)$ are dual to each other in the sense that*

$$g_{\text{pl}}(h(B^\alpha)) = g_{\text{pp}}(\xi(\alpha)) + h(B^\alpha)\cdot\xi(\alpha). \tag{4.11}$$

(ii) *B^α minimizes in the variational formula (4.9) in the characteristic direction:*

$$g_{\text{pp}}(\xi(\alpha)) \;=\; \max_{i=1,2}\big[\,\omega_0 - B^\alpha_{0,e_i}(\omega) - h(B^\alpha)\cdot\xi(\alpha)\,\big] \qquad \mathbb{P}\textit{-almost surely.} \tag{4.12}$$

Proof. Part (i). From the explicit formulas for the shape function (Theorem 3.4),

$$g_{\text{pp}}(s,t) = s + t + 2\sqrt{st}$$

and then

$$\nabla g_{\text{pp}}(s,t) = \big(1 + \sqrt{t/s}\,, 1 + \sqrt{s/t}\big).$$

From the explicit exponential distributions of B^α-increments (Theorem 4.2),

$$-h(B^\alpha) = \big(\,\mathbb{E}[B^\alpha_{0,e_1}]\,, \mathbb{E}[B^\alpha_{0,e_2}]\,\big) = \Big(\frac{1}{1-\alpha}\,, \frac{1}{\alpha}\Big) = \nabla g_{\text{pp}}(\xi(\alpha)).$$

By the strict concavity of g_{pp}, $\xi(\alpha)$ is the unique maximizer in the duality

$$g_{\text{pl}}(h(B^\alpha)) = \sup_{\xi\in\mathcal{U}}[\,g_{\text{pp}}(\xi) + h(B^\alpha)\cdot\xi\,].$$

Part (ii). By Lemma 2.12, $g_{\text{pl}}(h(B^\alpha)) = 0$. By the duality (4.11) and the adaptedness (4.8),

$$\begin{aligned} g_{\text{pp}}(\xi(\alpha)) &= g_{\text{pl}}(h(B^\alpha)) - h(B^\alpha)\cdot\xi(\alpha) = \omega_0 - B^\alpha_{0,e_1}\wedge B^\alpha_{0,e_2} - h(B^\alpha)\cdot\xi(\alpha) \\ &= \max_{i=1,2}\big[\,\omega_0 - B^\alpha_{0,e_i}(\omega) - h(B^\alpha)\cdot\xi(\alpha)\,\big]. \end{aligned}$$

□

4.2. **Proof of Theorem 4.2.** This technical section is devoted to proving the existence of the limiting Busemann functions. Before specializing to the processes we study, we state and prove general inequalities for planar last-passage increments. Early appearances of these types of inequalities in first-passage percolation can be found in [1, 2]. Let real weights $\{\widetilde{Y}_x\}_{x\in\mathbb{Z}^2}$ be given. Define last-passage times

$$\widetilde{G}_{x,y} = \max_{x_{0,n}} \sum_{k=0}^{n} \widetilde{Y}_{x_k} \tag{4.13}$$

where the maximum is over up-right paths from $x_0 = x$ to $x_n = y$ with $n = |y-x|_1$ and the admissible steps are as before $x_k - x_{k-1} \in \{e_1, e_2\}$. The convention is $\widetilde{G}_{v,v} = 0$. For $x \leqslant v - e_1$ and $y \leqslant v - e_2$ denote increments by

$$\widetilde{I}_{x,v} = \widetilde{G}_{x,v} - \widetilde{G}_{x+e_1,v} \qquad \text{and} \qquad \widetilde{J}_{y,v} = \widetilde{G}_{y,v} - \widetilde{G}_{y+e_2,v}\,.$$

For the precise statement of the lemma below it is important that the sum in (4.13) includes the first weight $\widetilde{Y}_x$, but the increments $\widetilde{I}$ and $\widetilde{J}$ are not sensitive to whether the last weight $\widetilde{Y}_y$ is included or excluded.

Lemma 4.5. *For $x \leqslant v - e_1$ and $y \leqslant v - e_2$*

$$\widetilde{I}_{x,v+e_2} \geqslant \widetilde{I}_{x,v} \geqslant \widetilde{I}_{x,v+e_1} \qquad \textit{and} \qquad \widetilde{J}_{y,v+e_2} \leqslant \widetilde{J}_{y,v} \leqslant \widetilde{J}_{y,v+e_1}\,. \tag{4.14}$$

Proof. Let $v = (m,n)$. The proof goes by an induction argument that starts from the north and east boundaries. On the north, for $x = (k,n)$ for some $k < m$,

$$\begin{aligned}\widetilde{I}_{(k,n),(m,n+1)} &= \widetilde{G}_{(k,n),(m,n+1)} - \widetilde{G}_{(k+1,n),(m,n+1)}\\ &= \widetilde{Y}_{k,n} + \widetilde{G}_{(k+1,n),(m,n+1)} \vee \widetilde{G}_{(k,n+1),(m,n+1)} - \widetilde{G}_{(k+1,n),(m,n+1)}\\ &\geqslant \widetilde{Y}_{k,n} = \widetilde{G}_{(k,n),(m,n)} - \widetilde{G}_{(k+1,n),(m,n)} = \widetilde{I}_{(k,n),(m,n)}\,.\end{aligned}$$

A similar argument (or the above inequality applied to transposed lattice points $(a',b') = (b,a)$) gives, for $y = (m,\ell)$ for some $\ell < n$,

$$\widetilde{J}_{(m,\ell),(m+1,n)} \geqslant \widetilde{J}_{(m,\ell),(m,n)}.$$

We also have the equalities, first for $y = (m,\ell)$ for some $\ell < n$

$$\begin{aligned}\widetilde{J}_{(m,\ell),(m,n+1)} &= \widetilde{G}_{(m,\ell),(m,n+1)} - \widetilde{G}_{(m,\ell+1),(m,n+1)}\\ &= \widetilde{Y}_{m,\ell} = \widetilde{G}_{(m,\ell),(m,n)} - \widetilde{G}_{(m,\ell+1),(m,n)} = \widetilde{J}_{(m,\ell),(m,n)}\end{aligned}$$

and similarly also

$$\widetilde{I}_{(k,n),(m+1,n)} = \widetilde{I}_{(k,n),(m,n)}.$$

These inequalities start the induction. Now let $u \leqslant v - e_1 - e_2$. Assume by induction that (4.14) holds for $x = u + e_2$ and $y = u + e_1$. We prove the first inequalities of (4.14) for $x = u$.

$$\begin{aligned}\widetilde{I}_{u,v+e_2} &= \widetilde{G}_{u,v+e_2} - \widetilde{G}_{u+e_1,v+e_2} = \widetilde{Y}_u + (\widetilde{G}_{u+e_2,v+e_2} - \widetilde{G}_{u+e_1,v+e_2})^+\\ &= \widetilde{Y}_u + (\widetilde{I}_{u+e_2,v+e_2} - \widetilde{J}_{u+e_1,v+e_2})^+\\ &\geqslant \widetilde{Y}_u + (\widetilde{I}_{u+e_2,v} - \widetilde{J}_{u+e_1,v})^+ = \widetilde{I}_{u,v}\,.\end{aligned}$$

The last equality comes by repeating the first three equalities with v instead of $v + e_2$.

Replacing the pair $(v+e_2, v)$ with $(v, v+e_1)$ in the argument above gives $\widetilde{I}_{u,v} \geqslant \widetilde{I}_{u,v+e_1}$. A symmetric argument works for the $\widetilde{J}$ inequalities. ☐

We introduce the following general notational device which we illustrate in the context of (4.13). If Λ is a subset of admissible paths from x to y, then

$$\widetilde{G}_{x,y}(\Lambda) = \max_{x_{0,n}\in\Lambda} \sum_{k=0}^{n} \widetilde{Y}_{x_k} \tag{4.15}$$

is the last-passage value obtained when the maximum is restricted to paths $x_{0,n}$ in Λ.

Return to the context of the CGM with exponential weights. Consider an origin u and a parameter $0 < \rho < 1$ fixed for the moment. Let the variables

$$\{\omega_x,\, I_{x-e_2},\, J_{x-e_1},\, \breve{\omega}_{x-e_1-e_2} : x \in u + \mathbb{Z}^2_{>0}\} \tag{4.16}$$

be defined via equations (3.11)–(3.13) from independent initial variables (3.6) with marginal distributions (3.7), so that the properties given in Theorem 3.1 are satisfied. Let $\xi = \xi(\rho)$ be the characteristic direction for ρ defined by (3.30).

Utilizing the edge weights I_x and J_x and the vertex weights $\breve{\omega}_x$ we define several last-passage percolation processes. First a process with i.i.d. weights:

$$\breve{G}_{x,y} = \max_{x_\bullet\in\Pi_{x,y}} \sum_{k=0}^{|y-x|_1} \breve{\omega}_{x_k} \qquad \text{for } y \geqslant x \geqslant u. \tag{4.17}$$

Its increments are defined by

$$\breve{I}_{x,v} = \breve{G}_{x,v} - \breve{G}_{x+e_1,v} \quad \text{for } u \leqslant x \leqslant v - e_1 \quad \text{and} \quad \breve{J}_{y,v} = \breve{G}_{y,v} - \breve{G}_{y+e_2,v} \quad \text{for } u \leqslant y \leqslant v - e_2.$$

Then we introduce an auxiliary increment-stationary last-passage process $\breve{G}^{NE}_{x,v}$ for $u \leqslant x \leqslant v$, with boundary edge weights on the north and east borders. First set $\breve{G}^{NE}_{v,v} = 0$. Then, on the north and east boundaries emanating from v in the negative directions,

$$\breve{G}^{NE}_{v-ke_1,v} = \sum_{i=0}^{k-1} I_{v-ie_1} \quad \text{and} \quad \breve{G}^{NE}_{v-\ell e_2,v} = \sum_{j=0}^{\ell-1} J_{v-je_2}. \tag{4.18}$$

In the bulk for $u \leqslant x \leqslant v - e_1 - e_2$ we define,

$$\begin{aligned} \breve{G}^{NE}_{x,v} &= \breve{\omega}_x + \breve{G}^{NE}_{x+e_1,v} \vee \breve{G}^{NE}_{x+e_2,v} \\ &= \max_{1\leqslant k\leqslant (v-x)\cdot e_1} \Big\{ \breve{G}_{x,v-ke_1-e_2} + \sum_{i=0}^{k-1} I_{v-ie_1} \Big\} \bigvee \max_{1\leqslant \ell\leqslant (v-x)\cdot e_2} \Big\{ \breve{G}_{x,v-e_1-\ell e_2} + \sum_{j=0}^{\ell-1} J_{v-je_2} \Big\} \end{aligned} \tag{4.19}$$

In the next lemma we check that the increments of the $\breve{G}^{NE}$ process are in fact the already given I and J variables.

Lemma 4.6. *For $u \leqslant x \leqslant v - e_1$ and $u \leqslant y \leqslant v - e_2$,*

$$I_{x+e_1} = \breve{G}^{NE}_{x,v} - \breve{G}^{NE}_{x+e_1,v} \qquad \textit{and} \qquad J_{y+e_2} = \breve{G}^{NE}_{y,v} - \breve{G}^{NE}_{y+e_2,v}. \tag{4.20}$$

Proof. The claim is true for $x = v - ke_1$ and $y = v - \ell e_2$ by definition (4.18). Here is the induction step for the edge $(x, x + e_1)$, assuming that (4.20) has been proved for edges $(x + e_2, x + e_1 + e_2)$ and $(x + e_1, x + e_1 + e_2)$.

$$\begin{aligned} \breve{G}^{NE}_{x,v} - \breve{G}^{NE}_{x+e_1,v} &= \breve{\omega}_x + \big(\breve{G}^{NE}_{x+e_2,v} - \breve{G}^{NE}_{x+e_1,v}\big)^+ = \breve{\omega}_x + \big(I_{x+e_1+e_2} - J_{x+e_1+e_2}\big)^+ \\ &= I_{x+e_1} \wedge J_{x+e_2} + \big(I_{x+e_1} - J_{x+e_2}\big)^+ = I_{x+e_1}. \end{aligned}$$

The second last equality used (3.11) and the cocycle property $I_{x+e_1} + J_{x+e_1+e_2} = J_{x+e_2} + I_{x+e_1+e_2}$. A similar argument extends (4.20) to the edge $(x, x + e_2)$. □

Next we consider restricted $\check{G}^{NE}$ last-passage values utilizing the notation introduced in (4.15). In particular, we consider last-passage values of the kind

$$\check{G}^{NE}_{0,\,v}(v-e_1\in x_\blacksquare) = \max_{1\leqslant k\leqslant (v-x)\cdot e_1} \Big\{ \check{G}_{x,v-ke_1-e_2} + \sum_{i=0}^{k-1} I_{v-ie_1} \Big\}$$

where the condition $v-e_1\in x_\blacksquare$ means that the path goes through the point $v-e_1$, which is equivalent to saying that the last step of the path goes from $v-e_1$ to v. We show that the asymptotics of the restricted $\check{G}^{NE}$ are the expected ones and calculate the limits.

Lemma 4.7. *Fix a point $a\in u+\mathbb{Z}^2_{\geqslant 0}$ and reals $0<s,t<\infty$. Let $v_n\in u+\mathbb{Z}^2_{\geqslant 0}$ be such that $|v_n|_1\to\infty$ and $v_n/|v_n|_1\to(s,t)/(s+t)$ as $n\to\infty$. Then we have the following almost sure limits.*

$$\begin{aligned} |v_n|_1^{-1}\,\check{G}^{NE}_{a,\,v_n}(v_n-e_1\in x_\blacksquare) &\underset{n\to\infty}{\longrightarrow} (s+t)^{-1}\sup_{0\leqslant\tau\leqslant s}\Big\{\frac{\tau}{1-\rho}+g_{\rm pp}(s-\tau,t)\Big\} \\ &= \begin{cases} g_{\rm pp}(s,t)=\big(\sqrt{s}+\sqrt{t}\big)^2, & \dfrac{s}{t}\leqslant\Big(\dfrac{1-\rho}{\rho}\Big)^2 \\ \dfrac{s}{1-\rho}+\dfrac{t}{\rho}\,, & \dfrac{s}{t}\geqslant\Big(\dfrac{1-\rho}{\rho}\Big)^2 \end{cases} \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} |v_n|_1^{-1}\,\check{G}^{NE}_{a,\,v_n}(v_n-e_2\in x_\blacksquare) &\underset{n\to\infty}{\longrightarrow} (s+t)^{-1}\sup_{0\leqslant\tau\leqslant t}\Big\{\frac{\tau}{\rho}+g_{\rm pp}(s,t-\tau)\Big\} \\ &= \begin{cases} g_{\rm pp}(s,t)=\big(\sqrt{s}+\sqrt{t}\big)^2, & \dfrac{s}{t}\geqslant\Big(\dfrac{1-\rho}{\rho}\Big)^2 \\ \dfrac{s}{1-\rho}+\dfrac{t}{\rho}\,, & \dfrac{s}{t}\leqslant\Big(\dfrac{1-\rho}{\rho}\Big)^2. \end{cases} \end{aligned} \tag{4.22}$$

Proof. We prove (4.21), the proof of (4.22) being entirely analogous. Fix $\varepsilon>0$, let $M=\lfloor\varepsilon^{-1}\rfloor$, and

$$q^n_j = j\Big\lfloor\frac{\varepsilon|v_n|_1 s}{s+t}\Big\rfloor \ \text{ for } 0\leqslant j\leqslant M-1, \text{ and } q^n_M=(v_n-a)\cdot e_1.$$

For large enough n it is the case that $q^n_M - C\varepsilon < q^n_{M-1} < q^n_M$.

Suppose a maximal path for $\check{G}^{NE}_{a,\,v_n}(v_n-e_1\in x_\blacksquare)$ enters the north boundary from the bulk at the point $v_n-(\ell,0)$ with $q^n_j<\ell\leqslant q^n_{j+1}$. By nonnegativity of the weights,

$$\begin{aligned} \check{G}^{NE}_{a,\,v_n}(v_n-e_1\in x_\blacksquare) &= \check{G}_{a,\,v_n-(\ell,1)} + \sum_{i=0}^{\ell-1} I_{v_n-ie_1} \\ &\leqslant \check{G}_{a,\,v_n-(q^n_j,1)} + \frac{q^n_j}{1-\rho} + \sum_{i=0}^{q^n_{j+1}-1}\big(I_{v_n-ie_1}-\tfrac{1}{1-\rho}\big) + \frac{q^n_{j+1}-q^n_j}{1-\rho}. \end{aligned}$$

Collect the bounds for all the intervals $(q^n_j, q^n_{j+1}]$:

$$\begin{aligned} \check{G}^{NE}_{a,\,v_n}(v_n-e_1\in x_\blacksquare) \ \leqslant \ \max_{0\leqslant j\leqslant M-1}\Big\{ & \check{G}_{a,\,v_n-(q^n_j,1)} + \frac{q^n_j}{1-\rho} \\ & + \sum_{i=0}^{q^n_{j+1}-1}\big(I_{v_n-ie_1}-\tfrac{1}{1-\rho}\big) + \frac{q^n_{j+1}-q^n_j}{1-\rho}\Big\} \end{aligned} \tag{4.23}$$

Divide through by $|v_n|_1$ and let $n \to \infty$. On the right-hand side the shape theorem (3.5) and the homogeneity of $g_{\rm pp}$ give

$$\frac{\breve G_{a,\,v_n-(q_j^n,1)}}{|v_n|_1} = g_{\rm pp}\biggl(\frac{v_n}{|v_n|_1} - \frac{q_j^n}{|v_n|_1}e_1 + \frac{O(1)}{|v_n|_1}\biggr) + o(1) \;\longrightarrow\; \frac{g_{\rm pp}(s-sj\varepsilon,t)}{s+t}.$$

The mean zero i.i.d. sum satisfies

$$\frac{1}{|v_n|_1}\sum_{i=0}^{q_{j+1}^n-1}\bigl(I_{v_n-ie_1} - \tfrac{1}{1-\rho}\bigr) \;\longrightarrow\; 0.$$

We get the upper bound

$$\begin{aligned}\varlimsup_{n\to\infty} |v_n|_1^{-1}\breve G^{NE}_{0,\,v_n}(v_n - e_1 \in x_\blacksquare) &\leqslant (s+t)^{-1}\max_{0\leqslant j\leqslant M-1}\Bigl[g_{\rm pp}(s-sj\varepsilon,t) + \frac{sj\varepsilon}{1-\rho} + C\varepsilon\Bigr]\\ &\leqslant (s+t)^{-1}\sup_{0\leqslant\tau\leqslant s}\Bigl[\frac{\tau}{1-\rho} + g_{\rm pp}(s-\tau,t)\Bigr] + C\varepsilon.\end{aligned}$$

Let $\varepsilon \searrow 0$ to complete the proof of the upper bound.

To get the matching lower bound let the supremum

$$\sup_{\tau\in[0,s]}\{\tau/(1-\rho) + g_{\rm pp}(s-\tau,t)\}$$

be attained at $\tau^* \in [0,s]$. With $m_n = |v_n|_1/(s+t)$ we have

$$\breve G^{NE}_{a,\,v_n}(v_n - e_1 \in x_\blacksquare) \geqslant \breve G_{a,v_n-(\lfloor m_n\tau^*\rfloor\vee 1,1)} \;+\; \sum_{i=0}^{(\lfloor m_n\tau^*\rfloor-1)^+} I_{v_n-ie_1}.$$

Let $n \to \infty$ to get

$$\varliminf_{n\to\infty} |v_n|_1^{-1}\breve G^{NE}_{0,\,v_n}(v_n - e_1 \in x_\blacksquare) \geqslant (s+t)^{-1}\Bigl[g_{\rm pp}(s-\tau^*,t) + \frac{\tau^*}{1-\rho}\Bigr].$$

This completes the proof of (4.21). ☐

As a consequence we record the expected asymptotics for the unrestricted process with edge weights on the north and east:

$$\begin{aligned}\lim_{n\to\infty}|v_n|_1^{-1}\,\breve G^{NE}_{a,\,v_n} &= \lim_{n\to\infty}|v_n|_1^{-1}\,\breve G^{NE}_{a,\,v_n}(v_n-e_1\in x_\blacksquare)\bigvee \breve G^{NE}_{a,\,v_n}(v_n-e_2\in x_\blacksquare)\\ &= (s+t)^{-1}\sup_{0\leqslant\tau\leqslant s}\Bigl\{\frac{\tau}{1-\rho} + g_{\rm pp}(s-\tau,t)\Bigr\}\bigvee\sup_{0\leqslant\tau\leqslant t}\Bigl\{\frac{\tau}{\rho} + g_{\rm pp}(s,t-\tau)\Bigr\}\\ &= (s+t)^{-1}\sup_{0\leqslant\tau\leqslant s}\Bigl\{\frac{\tau}{1-\rho} + \bigl(\sqrt{s-\tau}+\sqrt{t}\,\bigr)^2\Bigr\}\bigvee\sup_{0\leqslant\tau\leqslant t}\Bigl\{\frac{\tau}{\rho} + \bigl(\sqrt{s}+\sqrt{t-\tau}\,\bigr)^2\Bigr\}\\ &= \frac{s}{1-\rho} + \frac{t}{\rho}.\end{aligned} \tag{4.24}$$

In the next lemma we derive bounds on the limiting local gradients of the last-passage values $\breve G$ defined in terms of the i.i.d. $\breve\omega$ weights. Recall the definition (3.30) of the characteristic direction ξ associated to ρ.

Lemma 4.8. *Consider two sequences $\{v_n\}$ and $\{w_n\}$ in $a + \mathbb{Z}^2_{\geqslant 0}$ such that*

$$\lim_{n\to\infty}\frac{v_n}{|v_n|_1} = (s,1-s) \quad \textit{and} \quad \lim_{n\to\infty}\frac{w_n}{|w_n|_1} = (t,1-t).$$

Assume that

$$s < \frac{(1-\rho)^2}{(1-\rho)^2+\rho^2} < t.$$

(i) *Almost surely*

$$\check{G}^{NE}_{a,v_n}(v_n - e_2 \in x_\blacksquare) = \check{G}^{NE}_{a,v_n} \quad \textit{and} \quad \check{G}^{NE}_{a,w_n}(w_n - e_1 \in x_\blacksquare) = \check{G}^{NE}_{a,w_n} \tag{4.25}$$

for all large enough n.

(ii) *The following inequalities hold almost surely:*

$$\varlimsup_{n\to\infty}[\check{G}_{a,w_n} - \check{G}_{a+e_1,w_n}] \leqslant I_{a+e_1} \leqslant \varliminf_{n\to\infty}[\check{G}_{a,v_n} - \check{G}_{a+e_1,v_n}] \tag{4.26}$$

and

$$\varlimsup_{n\to\infty}[\check{G}_{a,v_n} - \check{G}_{a+e_2,v_n}] \leqslant J_{a+e_2} \leqslant \varliminf_{n\to\infty}[\check{G}_{a,w_n} - \check{G}_{a+e_2,w_n}]. \tag{4.27}$$

Proof. Part (i). We prove the second statement of (4.25). The maximizing path to w_n comes through either $w_n - e_1$ or $w_n - e_2$. So to get a contradiction we can assume that $\mathbb{P}(A) > 0$ for the event A on which $\check{G}^{NE}_{a,w_n}(w_n - e_2 \in x_\blacksquare) = \check{G}^{NE}_{a,w_n}$ happens for infinitely many n. On the event A we can take limits (4.22) and (4.24) to get

$$\sup_{0\leqslant\tau\leqslant 1-t}\Big\{\frac{\tau}{\rho} + \big(\sqrt{t} + \sqrt{1-t-\tau}\big)^2\Big\} = \frac{t}{1-\rho} + \frac{1-t}{\rho}.$$

But by (4.22), $\frac{t}{1-t} > \big(\frac{1-\rho}{\rho}\big)^2$ implies that the supremum on the left equals $\big(\sqrt{t} + \sqrt{1-t}\big)^2$ which is strictly less than the right-hand side by (3.31). Thus $\mathbb{P}(A) > 0$ is not possible.

Part (ii). We prove the statements for w_n. By a combination of developments from above, justified below, we derive the following sequence of inequalities and equalities that proves the first inequality of (4.26).

$$\begin{aligned}\check{G}_{a,w_n} - \check{G}_{a+e_1,w_n} &\leqslant \check{G}^{N}_{a,\,w_n+e_2} - \check{G}^{N}_{a+e_1,\,w_n+e_2}\\ &= \check{G}^{NE}_{a,\,w_n+e_1+e_2}(w_n+e_2 \in x_\blacksquare) - \check{G}^{NE}_{a+e_1,\,w_n+e_1+e_2}(w_n+e_2 \in x_\blacksquare)\\ &= \check{G}^{NE}_{a,\,w_n+e_1+e_2} - \check{G}^{NE}_{a+e_1,\,w_n+e_1+e_2}\\ &= I_{a+e_1}.\end{aligned} \tag{4.28}$$

The first inequality in (4.28) above is a special case of the first inequality of (4.14), applied to the situation where the weights in (4.13) are given by

$$\widetilde{Y}_x = \begin{cases} 0 & \text{if } x = w_n + e_2,\\ I_{x+e_1} & \text{if } x = w_n + e_2 - ie_1 \text{ for some } i \geqslant 1,\\ \check{\omega}_x & \text{if } x \leqslant w_n \end{cases}$$

The notation $\check{G}^{N}_{a,w_n+e_2}$ denotes a last-passage process in the rectangle $\{x : a \leqslant x \leqslant w_n + e_2\}$ that uses $\check{\omega}$ weights on the horizontal lines below the top one, and the I weights on the top vertical line $x \cdot e_2 = w_n \cdot e_2 + 1$ (north boundary), with an irrelevant zero weight assigned at the top right corner $w_n + e_2$.

In the first equality in (4.28) we move the upper right corner from $w_n + e_2$ one step to the right to $w_n + e_1 + e_2$ so that we can include the boundary weights both on the north and east boundaries. This is exactly the definition of $\check{G}^{NE}$ in (4.18) and (4.19). To preserve the equality we force the paths to go through $w_n + e_2$.

The second equality in (4.28) is valid almost surely for large enough n, by the already proved (4.25). The last equality comes from (4.20).

Similarly we reason for the e_2 increment:

$$\begin{aligned}\check G_{a,w_n}-\check G_{a+e_2,w_n} &\geqslant \check G^{N}_{a,\,w_n+e_2}-\check G^{N}_{a+e_2,\,w_n+e_2}\\ &=\check G^{NE}_{a,\,w_n+e_1+e_2}(w_n+e_2\in x_\blacksquare)-\check G^{NE}_{a+e_2,\,w_n+e_1+e_2}(w_n+e_2\in x_\blacksquare)\\ &=\check G^{NE}_{a,\,w_n+e_1+e_2}-\check G^{NE}_{a+e_2,\,w_n+e_1+e_2}\\ &=J_{a+e_2}.\end{aligned}\tag{4.29}$$

This proves the last inequality of (4.27). □

Next we use the estimates above to build an exponential-α last-passage system from limits of local gradients of last-passage values. Denote the $\mathrm{Exp}(\lambda)$ cumulative distribution function by

$$F_\lambda(s)=\begin{cases}0, & s<0\\ 1-e^{-\lambda s}, & s\geqslant 0.\end{cases}$$

Lemma 4.9. *Let i.i.d.* $\mathrm{Exp}(1)$ *weights* $\omega=(\omega_x)_{x\in\mathbb{Z}^2}$ *be given and define the point-to-point last-passage process* $\{G_{x,y}\}$ *by* (3.1). *Fix* $0<\alpha<1$ *and a sequence* $v_n\in\mathbb{Z}^2$ *such that* $|v_n|_1\to\infty$ *and*

$$\lim_{n\to\infty}\frac{v_n}{|v_n|_1}=\xi(\alpha)=\left(\frac{(1-\alpha)^2}{(1-\alpha)^2+\alpha^2},\frac{\alpha^2}{(1-\alpha)^2+\alpha^2}\right)\in \mathrm{ri}\,\mathcal{U}.\tag{4.30}$$

(i) *The limits*

$$B^\alpha_{x,y}=\lim_{n\to\infty}[G_{x,v_n}-G_{y,v_n}]\tag{4.31}$$

exist for $\mathbb{P}$*-almost every* ω *for all* $x,y\in\mathbb{Z}^2$ *and satisfy additivity* $B^\alpha_{x,y}+B^\alpha_{y,z}=B^\alpha_{x,z}$.

(ii) *Define*

$$X^\alpha_x=B^\alpha_{x-e_1,x}\wedge B^\alpha_{x-e_2,x}\qquad \text{for } x\in\mathbb{Z}^2.\tag{4.32}$$

Then the process

$$\{X^\alpha_x,\,B^\alpha_{x-e_1,x},\,B^\alpha_{x-e_2,x},\,\omega_x:x\in\mathbb{Z}^2\}$$

is an exponential-α *last-passage system as described in Definition 4.1.*

Proof. Part (i). Fix $a\in\mathbb{Z}^2$ and let

$$\overline{B}=\varlimsup_{n\to\infty}[G_{a,v_n}-G_{a+e_1,v_n}]\quad\text{and}\quad \underline{B}=\varliminf_{n\to\infty}[G_{a,v_n}-G_{a+e_1,v_n}].$$

To get control of the distributions of $\overline{B}$ and $\underline{B}$, we realize the processes $\{G_{a,y}\}$ on another probability space as instances of $\{\check G_{a,y}\}$. Then we can apply bounds (4.26)–(4.27).

Let $0<\lambda<\alpha<\rho<1$. This implies

$$\frac{(1-\rho)^2}{(1-\rho)^2+\rho^2}<\frac{(1-\alpha)^2}{(1-\alpha)^2+\alpha^2}<\frac{(1-\lambda)^2}{(1-\lambda)^2+\lambda^2}.\tag{4.33}$$

Take any lattice point $u\leqslant a$ as an origin. Suppose on some arbitrary probability space we have mutually independent variables $\sigma=(\sigma_x)_{x\in\mathbb{Z}^2}$, $(I^\lambda_{u+ie_1})_{i\geqslant1}$, $(J^\lambda_{u+je_2})_{j\geqslant1}$, $(I^\rho_{u+ie_1})_{i\geqslant1}$, and $(J^\rho_{u+je_2})_{j\geqslant1}$ with marginal distributions

$$\begin{aligned}&\sigma_x\sim\mathrm{Exp}(1),\quad I^\lambda_{u+ie_1}\sim\mathrm{Exp}(1-\lambda),\quad J^\lambda_{u+je_2}\sim\mathrm{Exp}(\lambda),\\ &\qquad\qquad\quad\;\, I^\rho_{u+ie_1}\sim\mathrm{Exp}(1-\rho),\quad\text{and}\quad J^\rho_{u+je_2}\sim\mathrm{Exp}(\rho).\end{aligned}\tag{4.34}$$

In other words, for parameters λ and ρ we have initial conditions of the kind described in (3.6) and (3.7). Iterating with equations (3.11)–(3.13), on the quadrant $u+\mathbb{Z}^2_{\geqslant0}$, construct two processes of the type (3.16): one denoted by

$$\{\sigma_x,\,I^\lambda_{x-e_2},\,J^\lambda_{x-e_1},\,\check\sigma^{[\lambda]}_{x-e_1-e_2}:x\in u+\mathbb{Z}^2_{>0}\}\tag{4.35}$$

with parameter λ, and the other denoted by

$$\{\sigma_x,\, I^{\rho}_{x-e_2},\, J^{\rho}_{x-e_1},\, \breve\sigma^{[\rho]}_{x-e_1-e_2} : x \in u + \mathbb{Z}^2_{>0}\} \tag{4.36}$$

with parameter ρ. By construction these processes have the properties given in Theorem 3.1, process (4.35) with parameter λ and $\breve\sigma^{[\lambda]}$ replacing $\breve\omega$, and process (4.36) with parameter ρ (as stated in Theorem 3.1) but with $\breve\sigma^{[\rho]}$ replacing $\breve\omega$. In particular, both $\{\breve\sigma^{[\lambda]}\}_{x\in u+\mathbb{Z}^2_{\geqslant 0}}$ and $\{\breve\sigma^{[\rho]}\}_{x\in u+\mathbb{Z}^2_{\geqslant 0}}$ are i.i.d. Exp(1) variables, and the superscripts $[\lambda]$ and $[\rho]$ simply remind us that these variables were constructed from edge variables with parameters λ and ρ, respectively.

We stipulated above that the initial edge weights on the axes $\{u + ie_k : i \geqslant 1, k = 1, 2\}$ for the λ and ρ systems were independent. In fact the coupling between processes (4.35) and (4.36) is immaterial because the two processes will not be used jointly.

The key point is that we can replace the weights ω on the right of (4.31) with $\breve\sigma^{[\lambda]}$ and $\breve\sigma^{[\rho]}$ without changing the distribution of the last-passage process. Let $\breve G^{[\lambda]}$ denote the last-passage process defined in (4.17) with i.i.d. Exp(1) weights $\breve\sigma^{[\lambda]}$, and similarly for $\breve G^{[\rho]}$.

We derive bounds for the distribution functions of $\overline{B}$ and $\underline{B}$.

$$\begin{aligned}\mathbb{P}\{\overline{B} \leqslant s\} &= \mathbb{P}\Big\{\varlimsup_{n\to\infty}(G_{a,v_n} - G_{a+e_1,v_n}) \leqslant s\Big\} = \mathbb{P}\Big\{\varlimsup_{n\to\infty}(\breve G^{[\rho]}_{a,v_n} - \breve G^{[\rho]}_{a+e_1,v_n}) \leqslant s\Big\}\\ &\geqslant \mathbb{P}\{I^{\rho}_{a+e_1} \leqslant s\} = F_{1-\rho}(s)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\{\underline{B} \leqslant s\} &= \mathbb{P}\Big\{\varliminf_{n\to\infty}(G_{a,v_n} - G_{a+e_1,v_n}) \leqslant s\Big\} = \mathbb{P}\Big\{\varliminf_{n\to\infty}(\breve G^{[\lambda]}_{a,v_n} - \breve G^{[\lambda]}_{a+e_1,v_n}) \leqslant s\Big\}\\ &\leqslant \mathbb{P}\{I^{\lambda}_{a+e_1} \leqslant s\} = F_{1-\lambda}(s).\end{aligned}$$

Above we first replaced the weights ω with $\breve\sigma^{[\lambda]}$ and $\breve\sigma^{[\rho]}$, respectively. Then we applied (4.26), as justified by (4.33). Last we used the known distributions of the I increment variables. Since $\overline{B} \geqslant \underline{B}$ always, we have deduced that

$$F_{1-\rho}(s) \leqslant \mathbb{P}\{\overline{B} \leqslant s\} \leqslant \mathbb{P}\{\underline{B} \leqslant s\} \leqslant F_{1-\lambda}(s) \qquad \text{for all } \lambda, \rho \text{ such that } \lambda < \alpha < \rho.$$

Letting $\lambda \nearrow \alpha$ and $\rho \searrow \alpha$ allows us to conclude that $\overline{B} = \underline{B} \sim \text{Exp}(1-\alpha)$. This proves the limit in (4.31) for $(x, y) = (a, a + e_1)$. Proof of the limit for $(x, y) = (a, a + e_2)$ proceeds analogously. Since $a \in \mathbb{Z}^2$ was arbitrary, we have the limit in (4.31) for all nearest-neighbor pairs x, y.

An arbitrary increment $y - x$ can be decomposed into a sum of nearest-neighbor increments, and then the limit follows for all pairs x, y by the additivity on the right-hand side of (4.31). Along the way one also derives the additivity $B^{\alpha}_{x,y} + B^{\alpha}_{y,z} = B^{\alpha}_{x,z}$.

Part (ii). We need to verify the properties in Definition 4.1. We begin with the joint distribution of B^{α}-increments along a down-right path.

Consider the joint distribution of $k + \ell$ nearest-neighbor increments $B^{\alpha}_{x_i,x_i+e_1}$ and $B^{\alpha}_{y_j,y_j+e_2}$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$. Fix an origin u such that all x_i and y_j lie in the quadrant $u + \mathbb{Z}^2_{\geqslant 0}$. We use again the auxiliary processes given in (4.35) and (4.36).

The processes $\{\breve G^{[\lambda]}_{x,y} : y \geqslant x \geqslant u\}$, $\{\breve G^{[\rho]}_{x,y} : y \geqslant x \geqslant u\}$, and $\{G_{x,y} : y \geqslant x \geqslant u\}$ all have the same distribution because they are defined the same way from i.i.d. Exp(1) weights. Hence from part (i) we can also conclude the existence of the almost sure limits

$$\breve B^{[\lambda]}_{x,y} = \lim_{n\to\infty}(G^{[\lambda]}_{x,v_n} - G^{[\lambda]}_{y,v_n}) \quad \text{and} \quad \breve B^{[\rho]}_{x,y} = \lim_{n\to\infty}(G^{[\rho]}_{x,v_n} - G^{[\rho]}_{y,v_n}) \quad \forall x, y \in u + \mathbb{Z}^2_{\geqslant 0}.$$

Then, first by distributional equality of processes,

$$
\begin{aligned}
&\mathbb{P}\big\{B^{\alpha}_{x_i,\,x_i+e_1} \leqslant s_i,\, B^{\alpha}_{y_j,\,y_j+e_2} > t_j \quad \forall i \in [k], j \in [\ell]\,\big\}\\
&= \mathbb{P}\big\{B^{[\lambda]}_{x_i,\,x_i+e_1} \leqslant s_i,\, B^{[\lambda]}_{y_j,\,y_j+e_2} > t_j \quad \forall i \in [k], j \in [\ell]\,\big\}\\
&\leqslant \mathbb{P}\big\{I^{\lambda}_{x_i+e_1} \leqslant s_i,\, J^{\lambda}_{y_j+e_2} > t_j \quad \forall i \in [k], j \in [\ell]\,\big\}
\end{aligned}
$$

The last step came from the inequalities (4.26) and (4.27) (the case of v_n of those inequalities is the right one to look at). Similarly, using the remaining two inequalities of (4.26) and (4.27), we deduce

$$
\begin{aligned}
&\mathbb{P}\big\{B^{\alpha}_{x_i,\,x_i+e_1} \leqslant s_i,\, B^{\alpha}_{y_j,\,y_j+e_2} > t_j \quad \forall i \in [k], j \in [\ell]\,\big\}\\
&= \mathbb{P}\big\{B^{[\rho]}_{x_i,\,x_i+e_1} \leqslant s_i,\, B^{[\rho]}_{y_j,\,y_j+e_2} > t_j \quad \forall i \in [k], j \in [\ell]\,\big\}\\
&\geqslant \mathbb{P}\big\{I^{\rho}_{x_i+e_1} \leqslant s_i,\, J^{\rho}_{y_j+e_2} > t_j \quad \forall i \in [k], j \in [\ell]\,\big\}.
\end{aligned}
$$

Assume in particular now that the edges $\{x_i, x_i+e_1\}$ and $\{y_j, y_j+e_2\}$ lie on a given down-right path. Then, part (i) of Theorem 3.1 applied to the processes (4.35) and (4.36) turns the bounds above into

$$
\begin{aligned}
\prod_{i\in[k]} F_{1-\rho}(s_i)\cdot \prod_{j\in[\ell]}(1-F_\rho(t_j)) \;\leqslant\; &\mathbb{P}\big\{B^{\alpha}_{x_i,\,x_i+e_1} \leqslant s_i,\, B^{\alpha}_{y_j,\,y_j+e_2} > t_j \quad \forall i \in [k], j \in [\ell]\,\big\}\\
&\leqslant \prod_{i\in[k]} F_{1-\lambda}(s_i)\cdot \prod_{j\in[\ell]}(1-F_\lambda(t_j)).
\end{aligned}
$$

Letting again $\lambda \nearrow \alpha$ and $\rho \searrow \alpha$ shows that along a down-right path, the variables $B^{\alpha}_{x,\,x+e_1}$ and $B^{\alpha}_{y,\,y+e_2}$ are independent with distributions $\mathrm{Exp}(1-\alpha)$ and $\mathrm{Exp}(\alpha)$, respectively, as required by part (a) of Definition 4.1.

Fix a down-right path $\mathcal{Y}$ in $\mathbb{Z}^2$. We verify the distributional properties on $\mathcal{G}_-$, $\mathcal{Y}$ and $\mathcal{G}_+$ inside an arbitrarily large rectangle.

Consider a large rectangle $\mathcal{D} = \{x : (M_0, N_0) \leqslant x \leqslant (M_1, N_1)\}$ whose lower left and upper right corners are (M_0, N_0) and (M_1, N_1). The e_1-edge variables $B^{\alpha}_{(i,N_1),(i+1,N_1)}$ for $M_0 \leqslant i \leqslant M_1 - 1$ on the north boundary, the e_2-edge variables $B^{\alpha}_{(M_1,j),(M_1,j+1)}$ for $N_0 \leqslant j \leqslant N_1 - 1$ on the east boundary, and the bulk variables ω_x for $(M_0, N_0) \leqslant x \leqslant (M_1 - 1, N_1 - 1)$ are mutually independent. (The ω-variables are independent of the B^α-variables to their north and east because limit (4.31) constructs a B^α-variable in terms of ω-weights to its north and east.)

In other words, the B^α-increments on the north and east boundaries and the ω-weights in the bulk of the rectangle satisfy the properties of the I, J boundary weights and $\breve{\omega}$-bulk weights in Theorem 3.1. Next we show by a south and westward induction that the joint distribution of (X^α, B, ω) is a correct one.

We claim that the variables satisfy the equations

$$
\begin{aligned}
X^{\alpha}_{x+e_1+e_2} &= B^{\alpha}_{x+e_2,\,x+e_1+e_2} \wedge B^{\alpha}_{x+e_1,\,x+e_1+e_2}\\
B^{\alpha}_{x,\,x+e_1} &= \omega_x + (B^{\alpha}_{x+e_2,\,x+e_1+e_2} - B^{\alpha}_{x+e_1,\,x+e_1+e_2})^{+}\\
B^{\alpha}_{x,\,x+e_2} &= \omega_x + (B^{\alpha}_{x+e_2,\,x+e_1+e_2} - B^{\alpha}_{x+e_1,\,x+e_1+e_2})^{-}.
\end{aligned}
\tag{4.37}
$$

In contrast with the iteration (3.11)–(3.13), the equations above proceed to the south and west.

The first equation in (4.37) is definition (4.32). The second and third come from the limits (4.31). For example,

$$\begin{aligned}B^{\alpha}_{x,\,x+e_1} &= \lim_{n\to\infty}[G_{x,v_n} - G_{x+e_1,\,v_n}] = \lim_{n\to\infty}[\omega_x + G_{x+e_1,\,v_n} \vee G_{x+e_2,\,v_n} - G_{x+e_1,\,v_n}] \\ &= \omega_x + \lim_{n\to\infty}[G_{x+e_2,\,v_n} - G_{x+e_1,\,v_n}]^+ \\ &= \omega_x + \lim_{n\to\infty}\big[(G_{x+e_2,\,v_n} - G_{x+e_1+e_2,\,v_n}) - (G_{x+e_1,\,v_n} - G_{x+e_1+e_2,\,v_n})\big]^+ \\ &= \omega_x + [B^{\alpha}_{x+e_2,\,x+e_1+e_2} - B^{\alpha}_{x+e_1,\,x+e_1+e_2}]^+.\end{aligned}$$

By Lemma 3.2 and induction, for any down-right path from the upper left corner (M_0, N_1) to the lower right corner (M_1, N_0), inside the rectangle $\mathcal{D}$, the B^α-increments on the path, the ω weights below and to the left of the path, and the X^α weights above and to the right of the path, are all independent with the correct marginal distributions stipulated in (4.2). Part (a) of Definition 4.1 has been verified.

It remains to check that equations (3.11)–(3.13) are satisfied.

$$\begin{aligned}&B^{\alpha}_{x,x+e_1} \wedge B^{\alpha}_{x,x+e_2} = \lim_{n\to\infty}[G_{x,v_n} - G_{x+e_1,v_n}] \wedge [G_{x,v_n} - G_{x+e_2,v_n}] \\ &= \lim_{n\to\infty}[\omega_x + G_{x+e_1,v_n} \vee G_{x+e_2,v_n} - G_{x+e_1,v_n}] \wedge [\omega_x + G_{x+e_1,v_n} \vee G_{x+e_2,v_n} - G_{x+e_2,v_n}] \\ &= \omega_x + [G_{x+e_2,v_n} - G_{x+e_1,v_n}]^+ \wedge [G_{x+e_1,v_n} - G_{x+e_2,v_n}]^+ \\ &= \omega_x.\end{aligned}$$

This verifies (3.11) (at x instead of at $x - e_1 - e_2$).

By definition (4.32) and the additivity of B^α,

$$\begin{aligned}B^{\alpha}_{x-e_1,\,x} &= B^{\alpha}_{x-e_1,\,x} \wedge B^{\alpha}_{x-e_2,\,x} + (B^{\alpha}_{x-e_1,\,x} - B^{\alpha}_{x-e_2,\,x})^+ \\ &= X^{\alpha}_x + (B^{\alpha}_{x-e_1-e_2,\,x-e_2} - B^{\alpha}_{x-e_1-e_2,\,x-e_1})^+.\end{aligned}$$

This is (3.12). Equation (3.13) is verified in a similar manner. ☐

LEMMA 4.10. *Fix a countable dense subset $D \subset (0,1)$. Then there exists an event Ω_0 of full probability such that the following holds for each $\omega \in \Omega_0$.*

(i) *For each $\rho \in D$ the process $\{B^{\rho}_{x,y}(\omega)\}_{x,y\in\mathbb{Z}^2}$ is well-defined by the limits in (4.31) for the specific sequence $v_n = \lfloor n\xi(\rho)\rfloor$. These processes satisfy the following inequalities:*

$$\begin{aligned}&B^{\lambda}_{x,x+e_1}(\omega) \leqslant B^{\rho}_{x,x+e_1}(\omega) \\ \text{and}\quad &B^{\lambda}_{x,x+e_2}(\omega) \geqslant B^{\rho}_{x,x+e_2}(\omega) \qquad \text{for all } x \in \mathbb{Z}^2 \text{ and } \lambda < \rho \text{ in } D.\end{aligned} \tag{4.38}$$

(ii) *For each $\alpha \in (0,1)$, and any sequence u_n in $\mathbb{Z}^2$ such that $|u_n|_1 \to \infty$ and*

$$\lim_{n\to\infty} \frac{u_n}{|u_n|_1} = \xi(\alpha) = \left(\frac{(1-\alpha)^2}{(1-\alpha)^2 + \alpha^2}, \frac{\alpha^2}{(1-\alpha)^2 + \alpha^2}\right), \tag{4.39}$$

we have these bounds:

$$\begin{aligned}\sup_{\lambda\in D:\lambda<\alpha} B^{\lambda}_{x,x+e_1} &\leqslant \varliminf_{n\to\infty}[G_{x,u_n} - G_{x+e_1,u_n}] \\ &\leqslant \varlimsup_{n\to\infty}[G_{x,u_n} - G_{x+e_1,u_n}] \leqslant \inf_{\rho\in D:\rho>\alpha} B^{\rho}_{x,x+e_1}\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}\sup_{\rho\in D:\rho>\alpha} B^{\alpha}_{x,x+e_2} &\leqslant \varliminf_{n\to\infty}[G_{x,u_n} - G_{x+e_2,u_n}] \\ &\leqslant \varlimsup_{n\to\infty}[G_{x,u_n} - G_{x+e_2,u_n}] \leqslant \inf_{\lambda\in D:\lambda<\alpha} B^{\alpha}_{x,x+e_2}.\end{aligned} \tag{4.41}$$

Proof. Define Ω_0 to be the event on which the limits in (4.31) hold for each $\rho \in D$, for the specific sequence $v_n = \lfloor n\xi(\rho) \rfloor$. Then fix $\omega \in \Omega_0$. Let $0 < \lambda < \alpha < \rho < 1$ be such that $\lambda, \rho \in D$. Let u_n be any sequence in $\mathbb{Z}^2$ such that $|u_n|_1 \to \infty$ and (4.39) holds. Let $b_n = |u_n|_1$. Then (4.33) implies that for large n,

$$\lfloor b_n\xi(\rho) \rfloor \cdot e_1 < u_n \cdot e_1 < \lfloor b_n\xi(\lambda) \rfloor \cdot e_1 \quad \text{and} \quad \lfloor b_n\xi(\lambda) \rfloor \cdot e_2 < u_n \cdot e_2 < \lfloor b_n\xi(\rho) \rfloor \cdot e_2. \tag{4.42}$$

Then Lemma 4.5 gives the bounds

$$G_{x,\lfloor b_n\xi(\lambda)\rfloor} - G_{x+e_1,\lfloor b_n\xi(\lambda)\rfloor} \leqslant G_{x,u_n} - G_{x+e_1,u_n} \leqslant G_{x,\lfloor b_n\xi(\rho)\rfloor} - G_{x+e_1,\lfloor b_n\xi(\rho)\rfloor}$$

and

$$G_{x,\lfloor b_n\xi(\rho)\rfloor} - G_{x+e_2,\lfloor b_n\xi(\rho)\rfloor} \leqslant G_{x,u_n} - G_{x+e_2,u_n} \leqslant G_{x,\lfloor b_n\xi(\lambda)\rfloor} - G_{x+e_2,\lfloor b_n\xi(\lambda)\rfloor}.$$

All the inequalities claimed follow by taking $n \to \infty$. ☐

Proof of Theorem 4.2. We start with a countable dense subset $D \subset (0,1)$, the processes B^λ for $\lambda \in D$ defined by the limits

$$B^\lambda_{x,y} = \lim_{n\to\infty} [G_{x,\lfloor n\xi(\lambda)\rfloor} - G_{y,\lfloor n\xi(\lambda)\rfloor}] \tag{4.43}$$

on the event Ω_0 of full probability given in Lemma 4.10. For each $\lambda \in D$, from Lemma 4.9 we know the additivity

$$B^\lambda_{x,y} + B^\lambda_{y,z} = B^\lambda_{x,z} \tag{4.44}$$

and that with $X^\lambda_x = B^\lambda_{x-e_1,x} \wedge B^\lambda_{x-e_2,x}$ the process $\{X^\lambda_x, B^\lambda_{x-e_1,x}, B^\lambda_{x-e_2,x}, \omega_x : x \in \mathbb{Z}^2\}$ is an exponential-λ last-passage system as described in Definition 4.1.

Let Ω_1 be the subset of Ω_0 on which

$$\begin{aligned} &\sup_{\lambda\in D:\lambda<\gamma} B^\lambda_{x,x+e_1} = B^\gamma_{x,x+e_1} = \inf_{\rho\in D:\rho>\gamma} B^\rho_{x,x+e_1} \\ \text{and} \quad &\sup_{\rho\in D:\rho>\gamma} B^\rho_{x,x+e_2} = B^\gamma_{x,x+e_2} = \inf_{\lambda\in D:\lambda<\gamma} B^\lambda_{x,x+e_2} \quad \text{for all } \gamma \in D. \end{aligned} \tag{4.45}$$

Event Ω_1 has full probability because of the monotonicity and control of distributions: for example, for the first equality in (4.45) reason as follows: by (4.38)

$$\sup_{\lambda\in D:\lambda<\gamma} B^\lambda_{x,x+e_1} = \lim_{D\ni\lambda\nearrow\gamma} B^\lambda_{x,x+e_1} \leqslant B^\gamma_{x,x+e_1},$$

but by Lemma 4.9(b) and the convergence of distributions, both $\lim_{D\ni\lambda\nearrow\gamma} B^\lambda_{x,x+e_1}$ and $B^\gamma_{x,x+e_1}$ have $\text{Exp}(1-\gamma)$ distribution. Hence they agree almost surely. By discarding another zero probability event we can assume that Ω_1 is invariant under translations.

In order to prove that B^γ is a stationary cocycle (Definition 2.6) for $\gamma \in D$, it remains to check the stationarity $B^\gamma_{x,y}(\theta_z\omega) = B^\gamma_{z+x,z+y}$. We apply bounds (4.40)–(4.41) to the sequence $u_n = \lfloor n\xi(\gamma)\rfloor + z$ that satisfies (4.39) with limit $\xi(\gamma)$. Together with (4.45) these bounds give (with an extension by additivity) almost sure equalities

$$B^\gamma_{x,y} = \lim_{n\to\infty} [G_{x,\,\lfloor n\xi(\gamma)\rfloor+z} - G_{y,\,\lfloor n\xi(\gamma)\rfloor+z}]$$

for any fixed z and all x, y. Consequently

$$\begin{aligned} B^\gamma_{x+z,y+z} &= \lim_{n\to\infty} [G_{x+z,\,\lfloor n\xi(\gamma)\rfloor+z} - G_{y+z,\,\lfloor n\xi(\gamma)\rfloor+z}] = \lim_{n\to\infty} [G_{x,\,\lfloor n\xi(\gamma)\rfloor} - G_{y,\,\lfloor n\xi(\gamma)\rfloor}] \circ \theta_z \\ &= B^\gamma_{x,y} \circ \theta_z. \end{aligned}$$

We have now checked that B^γ is a cocycle for each $\gamma \in D$.

We take the step to general $\alpha \in (0,1)$. For each $\alpha \in (0,1)$ define processes X^α and B^α by taking right limits from values in D: set for each $\omega \in \Omega_1$ and $x, y \in \mathbb{Z}$

$$(X^\alpha_x(\omega), B^\alpha_{x,y}(\omega)) = \lim_{D \ni \lambda \searrow \alpha} (X^\lambda_x(\omega), B^\lambda_{x,y}(\omega)). \tag{4.46}$$

These limits exist for nearest-neighbor pairs x, y by the monotonicity in (4.38), and extend to all pairs x, y by additivity on the right. The limit for $X^\alpha_x(\omega)$ comes along as a function by virtue of the definition $X^\lambda_x(\omega) = B^\lambda_{x-e_1,x}(\omega) \wedge B^\lambda_{x-e_2,x}(\omega)$ which is then also preserved to the limit.

Extend these functions in some arbitrary way outside Ω_1. By the arguments given above, we have not altered these functions on Ω_1 if α happens to lie in D. The properties of both Definitions 2.6 and 4.1 are preserved by the limits: B^α is a stationary cocycle and $\{X^\alpha_x, B^\alpha_{x-e_1,x}, B^\alpha_{x-e_2,x}, \omega_x : x \in \mathbb{Z}^2\}$ is an exponential-α last-passage system. We have verified part (i) of the theorem.

Inequalities (4.5) are valid on the event Ω_1 simultaneously for all $\lambda, \rho \in D$ and preserved by the limit in (4.46). Part (ii) is proved.

For part (iii), fix $0 < \alpha < 1$ and let $\Omega_2^{(\alpha)}$ be the intersection of the event Ω_1 above (which is contained in the event Ω_0 of Lemma 4.10) with the event on which

$$\sup_{\lambda \in D : \lambda < \alpha} B^\lambda_{x,x+e_1} = \inf_{\rho \in D : \rho > \alpha} B^\rho_{x,x+e_1} \quad \text{and} \quad \sup_{\rho \in D : \rho > \alpha} B^\alpha_{x,x+e_2} = \inf_{\lambda \in D : \lambda < \alpha} B^\alpha_{x,x+e_2}. \tag{4.47}$$

The equalities above hold with probability 1 by the argument used already above. First, by monotonicity inequality $\leqslant$ holds in both equalities above. Then the suprema and infima are limits, and the left- and right-hand sides of the equalities above are equal in distribution. Hence the left- and right-hand sides agree almost surely.

The coincidence of the lower and upper bounds in (4.40)–(4.41) imply that the claimed limit in (4.7) holds for nearest-neighbor pairs x, y. Extend to all x, y by additivity. This completes the proof of Theorem 4.2. ∎

5. Fluctuation exponent for the corner growth model with exponential weights

Return to the point-to-point last-passage process defined as before in (3.1) by

$$G_{x,y} = \max_{x_\bullet \in \Pi_{x,y}} \sum_{k=0}^{|y-x|_1} \omega_{x_k}$$

with i.i.d. Exp(1) weights (ω_x) and the maximum over up-right paths from x to y. In Theorem 3.4 we proved the law of large numbers

$$\lim_{N \to \infty} N^{-1} G_{0,\lfloor N\xi \rfloor} = g_{\text{pp}}(\xi) \equiv \big(\sqrt{\xi_1} + \sqrt{\xi_2}\,\big)^2.$$

for $\xi \in [0,\infty)^2$. The next result states that the fluctuation exponent of $G_{0,\lfloor N\xi \rfloor}$ is $1/3$, as predicted by Kardar-Parisi-Zhang (KPZ) universality.

Theorem 5.1. *Fix $\xi \in \mathbb{R}^2_{>0}$. There exists a constant $0 < C = C(\xi, p) < \infty$ such that for $N \geqslant 1$ and $1 \leqslant p < 3/2$,*

$$C^{-1} N^{p/3} \leqslant \mathbb{E}\big[\, |G_{0,\lfloor N\xi \rfloor} - N g(\xi)|^p \,\big] \leqslant C N^{p/3}. \tag{5.1}$$

Currently this theorem is not proved in these notes. It is a consequence of the fluctuation bounds for the increment-stationary last-passage process G^ρ to which we now turn.

We recall the setting from Section 3. The parameter $0 < \rho < 1$ of the boundary weights is fixed. We are given mutually independent random variables

$$\{\omega_x,\, I_{ie_1},\, J_{je_2} : x \in \mathbb{Z}^2_{>0},\; i, j \in \mathbb{Z}_{>0}\} \tag{5.2}$$

with marginal distributions

$$\omega_x \sim \text{Exp}(1), \quad I_{ie_1} \sim \text{Exp}(1-\rho), \quad \text{and} \quad J_{je_2} \sim \text{Exp}(\rho). \tag{5.3}$$

The last-passage process $G^\rho_{0,x}$ is defined for $x \geqslant 0$ by $G^\rho_{0,0} = 0$,

$$G^\rho_{0,\,me_1} = \sum_{i=1}^{m} I_{ie_1} \quad \text{and} \quad G^\rho_{0,\,ne_2} = \sum_{j=1}^{n} J_{je_2}, \tag{5.4}$$

and then for $x = (m,n) \in \mathbb{Z}^2_{>0}$,

$$G^\rho_{0,\,x} = \max_{1\leqslant k\leqslant m} \Big\{ \sum_{i=1}^{k} I_{ie_1} + G_{ke_1+e_2,\,x} \Big\} \bigvee \max_{1\leqslant \ell\leqslant n} \Big\{ \sum_{j=1}^{\ell} J_{je_2} + G_{\ell e_2+e_1,\,x} \Big\}. \tag{5.5}$$

$G_{a,x}$ inside the braces is the last-passage value defined in (3.1) for i.i.d. Exp(1) weights.

We do not have a closed form expression for $\mathbb{V}\text{ar}[G^\rho_{0,(m,n)}]$ but we can access it well enough to show that it obeys the fluctuation exponent 1/3 characteristic of the KPZ class. However, there in an extra twist. Notice in (5.3) that the boundary weights ω_{ie_1} and ω_{je_2} are larger on average than the bulk weights $\{\omega_x\}_{x\in\mathbb{Z}^2_{>0}}$. This implies that the boundaries are attractive to the maximizing path. It turns out that only when we take the point x to infinity in the characteristic direction $c((1-\rho)^2, \rho^2)$, the pull of the boundaries balance out and $G^\rho_{0,x}$ obeys KPZ fluctuations. Otherwise the boundaries swamp the effects of the percolation and $G^\rho_{0,x}$ obeys the classical central limit theorem.

Let N be a scaling parameter that increases to ∞. We consider the point-to-point last-passage percolation from 0 to a point $(m,n) = (m(N), n(N))$ that is taken to infinity as $N \to \infty$. Let κ_N denote the deviation of (m,n) from the characteristic direction:

$$\kappa_N = |\, m - N(1-\rho)^2 \,| + |\, n - N\rho^2 \,|. \tag{5.6}$$

THEOREM 5.2. *Assume weight distributions* (5.3) *and* $\kappa_N \leqslant a_0 N^{2/3}$ *for some constant* a_0. *Then* $\exists$ *constant* $0 < C = C(\rho, a_0) < \infty$ *such that*

$$C^{-1} N^{2/3} \leqslant \mathbb{V}\text{ar}[G^\rho_{0,(m,n)}] \leqslant C N^{2/3} \qquad \textit{for } N \geqslant 1. \tag{5.7}$$

We prove the upper bound in the theorem above completely and the lower bound for the case where κ_N is bounded by a constant.

As a fairly immediate corollary we obtain the behavior in off-characteristic directions. For concreteness, we state the result for the case where the horizontal direction is abnormally large.

COROLLARY 5.3. *Assume weight distributions* (5.3). *Suppose* $m, n \to \infty$. *Define parameter* N *by* $n = N\rho^2$, *and assume that*

$$N^{-\alpha}\big(m - N(1-\rho)^2\big) \to c_1 > 0 \quad \textit{as } m, n \to \infty$$

for some $\alpha > 2/3$. *Then as* $m, n \to \infty$,

$$N^{-\alpha/2}\big\{G^\rho_{0,(m,n)} - \mathbb{E}(G^\rho_{0,(m,n)})\big\}$$

converges in distribution to a centered normal distribution with variance $c_1(1-\rho)^{-2}$.

Proof. Recall that overline means centering of a random variable.

$$N^{-\alpha/2}\, \overline{G}^\rho_{0,(m,n)} = N^{-\alpha/2}\, \overline{G}^\rho_{0,(\lfloor N(1-\rho)^2\rfloor,\, \lfloor N\rho^2\rfloor)} \; + \; N^{-\alpha/2} \sum_{i=\lfloor N(1-\rho)^2\rfloor+1}^{m} \bar{I}_{(i,n)}$$

The mean square of the first term on the right is of order $N^{-\alpha} \cdot N^{2/3}$ and hence in the limit vanishes in L^2 and in probability. The second term is a sum of approximately $c_1 N^\alpha$ mean zero i.i.d. terms with variance $\mathbb{E}[\,\bar{I}^2_{(i,n)}] = (1-\rho)^{-2}$. This sum gives the normal limit, by the CLT. ∎

6. Proof of the fluctuation exponent for the stationary process

This proof was originally presented in article [3], which itself was based on the earlier work [4] on increasing sequences among planar Poisson points.

The first step towards the proof of Theorem 5.2 is an explicit formula that ties together $\mathbb{V}\text{ar}[G^{\rho}_{0,(m,n)}]$ and the amount of weight the maximizing path collects on the boundary.

For a given x, the last-passage problem (5.5) has an almost surely unique maximizing path $\bar{x}_{\scriptscriptstyle\blacksquare} = (\bar{x}_k)_{k=0}^{n}$ from $\bar{x}_0 = 0$ to $\bar{x}_n = x$ that satisfies $G^{\rho}_{0,x} = \sum_{k=1}^{n} \omega_{\bar{x}_k}$, where we utilized the notational convention on the axes that $\omega_{ke_1} = I_{ke_1}$ and $\omega_{\ell e_2} = J_{\ell e_2}$. For $r = 1, 2$, define the exit time (or exit point) of this path from the e_r axis by

$$\tau_r = \max\{k \geqslant 0 : \bar{x}_k \cdot e_{3-r} = 0\}, \qquad r = 1, 2. \tag{6.1}$$

If the first step of the path $\bar{x}_{\scriptscriptstyle\blacksquare}$ from the origin is e_r, then $1 \leqslant \tau_r \leqslant x \cdot e_r$ and $\tau_{3-r} = 0$. In other words, almost surely exactly one of τ_1 and τ_2 is positive (but which one is positive varies with the realization of the weights ω).

Further, introduce the sums of weights along the axes:

$$S_{1,k} = \sum_{i=1}^{k} I_{ie_1} \quad \text{and} \quad S_{2,\ell} = \sum_{j=1}^{\ell} J_{je_2}.$$

Then S_{r,τ_r} is the amount of weight that the maximizing path collects on the e_r-axis. Again, for each weight configuration ω, exactly one of S_{1,τ_1} and S_{2,τ_2} is positive and the other one zero. When necessary for distinguishing processes with different boundary weights (5.3), these variables will be adorned with superscripts, as in τ_r^{ρ} and $S^{\rho}_{r,k}$.

Next we state the variance formula for the last-passage value in the increment-stationary CGM.

Theorem 6.1. *Assume weight distributions* (5.3).

$$\begin{aligned} \mathbb{V}\text{ar}[G^{\rho}_{0,(m,n)}] &= -\frac{m}{(1-\rho)^2} + \frac{n}{\rho^2} + \frac{2}{1-\rho}\mathbb{E}[S_{1,\tau_1}] \\ &= \frac{m}{(1-\rho)^2} - \frac{n}{\rho^2} + \frac{2}{\rho}\mathbb{E}[S_{2,\tau_2}]. \end{aligned} \tag{6.2}$$

We skip the proof of this lemma for now. It involves explicit computations with exponential distributions and covariances.

This section proves Theorem 5.2. The section is divided into an upper bound proof and a lower bound proof.

6.1. Upper bound.

We couple the boundary variables for two different parameters $0 < \rho < \lambda < 1$ as follows:

$$I^{\lambda}_{ie_1} = \frac{1-\rho}{1-\lambda} I^{\rho}_{ie_1} > I^{\rho}_{ie_1} \qquad \text{and} \qquad J^{\lambda}_{je_2} = \frac{\rho}{\lambda} J^{\rho}_{je_2} < J^{\rho}_{je_2}. \tag{6.3}$$

From this follows for example that $\tau_1^{\lambda} \geqslant \tau_1^{\rho}$ and $\tau_2^{\lambda} \leqslant \tau_2^{\rho}$, and also

$$S^{\lambda}_{1,\ell} - S^{\rho}_{1,\ell} \leqslant S^{\lambda}_{1,k} - S^{\rho}_{1,k} \qquad \text{for } 0 \leqslant \ell \leqslant k. \tag{6.4}$$

We begin with auxiliary lemmas.

Lemma 6.2. *Let $0 < \varepsilon < 1$. Then there exists a constant $C = C(\varepsilon)$ such that, for $\varepsilon \leqslant \rho < \lambda \leqslant 1 - \varepsilon$,*

$$\mathbb{V}\text{ar}[G^{\lambda}_{0,(m,n)}] \leqslant \mathbb{V}\text{ar}[G^{\rho}_{0,(m,n)}] + Cm(\lambda - \rho).$$

Proof. From (6.3)

$$S^{\lambda}_{2,\tau_2} = \sum_{j=1}^{\tau_2^{\lambda}} J^{\lambda}_{je_2} \leqslant \sum_{j=1}^{\tau_2^{\rho}} J^{\lambda}_{je_2} = \frac{\rho}{\lambda}\sum_{j=1}^{\tau_2^{\rho}} J^{\rho}_{je_2} = \frac{\rho}{\lambda} S^{\rho}_{2,\tau_2}.$$

Using the second line of (6.2),

$$\begin{aligned}\mathbb{V}\mathrm{ar}[G^{\lambda}_{0,(m,n)}] &= \frac{m}{(1-\lambda)^2} - \frac{n}{\lambda^2} + \frac{2}{\lambda}\mathbb{E}[S^{\lambda}_{2,\tau_2}] \\ &\leqslant \frac{(1-\rho)^2}{(1-\lambda)^2}\cdot\frac{m}{(1-\rho)^2} - \frac{\rho^2}{\lambda^2}\cdot\frac{n}{\rho^2} + \frac{\rho^2}{\lambda^2}\cdot\frac{2}{\rho}\mathbb{E}[S^{\rho}_{2,\tau_2}] \\ &= \frac{\rho^2}{\lambda^2}\cdot\mathbb{V}\mathrm{ar}[G^{\rho}_{0,(m,n)}] + \frac{m}{(1-\rho)^2}\left(\frac{(1-\rho)^2}{(1-\lambda)^2} - \frac{\rho^2}{\lambda^2}\right) \\ &\leqslant \mathbb{V}\mathrm{ar}[G^{\rho}_{0,(m,n)}] + Cm(\lambda-\rho).\end{aligned}$$

□

LEMMA 6.3. *Let $0 < \varepsilon < 1$. Then there exists a constant $C = C(\varepsilon)$ such that, for $\varepsilon \leqslant \rho \leqslant 1-\varepsilon$,*

$$\mathbb{E}[S^{\rho}_{1,\tau_1}] \leqslant C\big(\mathbb{E}[\tau_1^{\rho}] + 1\big). \tag{6.5}$$

We skip the proof of the above lemma.

The main estimate for the upper bound in (5.7) is contained in the next proposition.

PROPOSITION 6.4. *Consider the increment-stationary CGM $G^{\rho}_{0,(m,n)}$ with weight distributions (5.3) for a given $0 < \rho < 1$. Let κ_N be defined by (5.6). Three positive constants a_0, a_1 and N_0 are given and the assumption is that*

$$\kappa_N \leqslant a_0 N^{2/3} \quad \textit{and} \quad m \leqslant a_1 N \quad \textit{for} \quad N \geqslant N_0. \tag{6.6}$$

Then there exist constants $c_2, c_3 < \infty$ such that the following two bounds hold:

$$\mathbb{P}\{\tau_1^{\rho} \geqslant \ell\} \leqslant c_3\Big(\frac{N^2}{\ell^3} + (1+a_0)\frac{N^{8/3}}{\ell^4}\Big) \qquad \textit{for } N \geqslant N_0 \textit{ and } 1 \vee c_2\kappa_N \leqslant \ell \leqslant m \tag{6.7}$$

and

$$\mathbb{E}[(\tau_1^{\rho})^q] \leqslant \Big(c_2 a_0 + \frac{c_3}{3-q}\Big)N^{2q/3} \qquad \textit{for } N \geqslant N_0 \textit{ and } 1 \leqslant q < 3. \tag{6.8}$$

The functional dependencies of the constants c_2, c_3 on the parameters is as follows:

$$c_2 = c_2(\rho) \quad \textit{and} \quad c_3 = c_3(a_1, \rho). \tag{6.9}$$

Furthermore, c_2 and c_3 are locally bounded functions of their arguments.

The upper variance bound in (5.7) follows from a combination of (6.2), assumption (5.6), (6.5), and (6.8) for $q = 1$.

Proof of Proposition 6.4. Consider $N \geqslant N_0$ so that the assumptions are in force. Assume that, for some $0 < c_2 < \infty$, the integer ℓ satisfies

$$1 \vee c_2\kappa_N \leqslant \ell \leqslant m \leqslant a_1 N.$$

The proof will choose $c_2 = c_2(\rho)$ large enough. Let $0 < r < 1$ be a constant that will be set small enough in the proof. Let

$$\lambda = \rho + \frac{r\ell}{N}. \tag{6.10}$$

We take $r = r(a_1, \rho)$ at least small enough so that $ra_1 < \frac{1}{2}(1-\rho)$. This guarantees that for $N \geqslant 1$, $\lambda \in (\rho, \frac{1+\rho}{2})$ is also a legitimate parameter for an increment-stationary CGM.

In the first inequality below use $S^{\lambda}_{1,k} + G_{(k,1),(m,n)} \leqslant G^{\lambda}_{0,(m,n)}$. In the last inequality below use (6.4).

$$
\begin{aligned}
\mathbb{P}\{\tau_1^{\rho} \geqslant \ell\} &= \mathbb{P}\{\, \exists k \geqslant \ell :\, S^{\rho}_{1,k} + G_{(k,1),(m,n)} = G^{\rho}_{0,(m,n)} \,\} \\
&\leqslant \mathbb{P}\{\, \exists k \geqslant \ell :\, S^{\lambda}_{1,k} - S^{\rho}_{1,k} \leqslant G^{\lambda}_{0,(m,n)} - G^{\rho}_{0,(m,n)} \,\} \\
&\leqslant \mathbb{P}\{\, S^{\lambda}_{1,\ell} - S^{\rho}_{1,\ell} \leqslant G^{\lambda}_{0,(m,n)} - G^{\rho}_{0,(m,n)} \,\}
\end{aligned} \tag{6.11}
$$

Next we compute and bound the means of the random variables in the probability above. First

$$
\mathbb{E}[S^{\lambda}_{1,\ell} - S^{\rho}_{1,\ell}] = \ell\Big(\frac{1}{1-\lambda} - \frac{1}{1-\rho}\Big) = \frac{\ell}{(1-\lambda)(1-\rho)}(\lambda-\rho) = \frac{1}{(1-\lambda)(1-\rho)} \cdot \frac{r\ell^2}{N}
$$

Introduce the quantities

$$
\kappa^1_N = m - N(1-\rho)^2 \quad \text{and} \quad \kappa^2_N = n - N\rho^2 \tag{6.12}
$$

that satisfy (with κ_N as in (5.6))

$$
|\kappa^1_N| + |\kappa^2_N| \leqslant \kappa_N.
$$

Then the LPP values.

$$
\begin{aligned}
&\mathbb{E}[G^{\lambda}_{0,(m,n)} - G^{\rho}_{0,(m,n)}] = m\Big(\frac{1}{1-\lambda} - \frac{1}{1-\rho}\Big) + n\Big(\frac{1}{\lambda} - \frac{1}{\rho}\Big) \\
&\quad = \Big(\frac{m}{(1-\lambda)(1-\rho)} - \frac{n}{\lambda\rho}\Big)(\lambda-\rho) \\
&\quad = N\Big(\frac{1-\rho}{1-\lambda} - \frac{\rho}{\lambda}\Big)(\lambda-\rho) + \Big(\frac{\kappa^1_N}{(1-\lambda)(1-\rho)} - \frac{\kappa^2_N}{\lambda\rho}\Big)(\lambda-\rho) \\
&\quad = \frac{N}{\lambda(1-\lambda)}(\lambda-\rho)^2 + \Big(\frac{\kappa^1_N}{(1-\lambda)(1-\rho)} - \frac{\kappa^2_N}{\lambda\rho}\Big)(\lambda-\rho) \\
&\quad = \frac{r^2\ell^2}{\lambda(1-\lambda)N} + \Big(\frac{\kappa^1_N}{(1-\lambda)(1-\rho)} - \frac{\kappa^2_N}{\lambda\rho}\Big)\frac{r\ell}{N} \\
&\quad \leqslant \frac{r^2\ell^2}{\lambda(1-\lambda)N} + \frac{1}{(1-\lambda)(1-\rho)\wedge\lambda\rho} \cdot \frac{r\ell^2}{c_2 N}
\end{aligned} \tag{6.13}
$$

The last inequality came from $\kappa_N \leqslant \ell/c_2$.

Comparison of the means shows that if we choose c_2 large enough and then r small enough, both as functions of (λ,ρ), then for a large enough constant $c_3 = c_3(\lambda,\rho)$,

$$
\mathbb{E}[S^{\lambda}_{1,\ell} - S^{\rho}_{1,\ell}] > \mathbb{E}[G^{\lambda}_{0,(m,n)} - G^{\rho}_{0,(m,n)}] + \frac{r\ell^2}{c_3 N}.
$$

Since the range $\lambda \in (\rho, \frac{1+\rho}{2})$ is determined by ρ, the dependence on λ can be dropped and we have $c_2 = c_2(\rho)$, $r = r(a_1,\rho)$ and $c_3 = c_3(\rho)$.

We continue from line (6.11). Below we subsume r, a_1, ρ, λ dependent factors into a constant $C = C(a_1,\rho)$. Along the way we use Lemma 6.2, Theorem 6.1, (6.12), $\kappa_N \leqslant c_2^{-1}\ell$, $m \leqslant a_1 N$, and

Lemma 6.3.

$$
\begin{aligned}
\mathbb{P}\{\tau_1^\rho \geqslant \ell\} &\leqslant \mathbb{P}\Big\{ \overline{S^\lambda_{1,\ell} - S^\rho_{1,\ell}} \leqslant \overline{G^\lambda_{0,(m,n)} - G^\rho_{0,(m,n)}} - \frac{r\ell^2}{c_3 N} \Big\} \\
&\leqslant \mathbb{P}\Big\{ \overline{S^\lambda_{1,\ell} - S^\rho_{1,\ell}} \leqslant -\frac{r\ell^2}{2c_3 N} \Big\} + \mathbb{P}\Big\{ \overline{G^\lambda_{0,(m,n)} - G^\rho_{0,(m,n)}} \geqslant \frac{r\ell^2}{2c_3 N} \Big\} \\
&\leqslant \frac{CN^2}{\ell^4}\mathbb{V}\mathrm{ar}[S^\lambda_{1,\ell} - S^\rho_{1,\ell}] + \frac{CN^2}{\ell^4}\mathbb{V}\mathrm{ar}[G^\lambda_{0,(m,n)} - G^\rho_{0,(m,n)}] \\
&\leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\big(\mathbb{V}\mathrm{ar}[G^\lambda_{0,(m,n)}] + \mathbb{V}\mathrm{ar}[G^\rho_{0,(m,n)}]\big) \\
&\leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\big(\mathbb{V}\mathrm{ar}[G^\rho_{0,(m,n)}] + m(\lambda-\rho)\big) \\
&= \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\Big(-\frac{m}{(1-\rho)^2} + \frac{n}{\rho^2} + \frac{2}{1-\rho}\mathbb{E}[S^\rho_{1,\tau_1}] + a_1 N\cdot\frac{r\ell}{N}\Big) \\
&\leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\big(\mathbb{E}[\tau_1^\rho] + \ell\big) \leqslant \frac{CN^2}{\ell^3} + \frac{CN^2}{\ell^4}\mathbb{E}[\tau_1^\rho]. \qquad (6.14)
\end{aligned}
$$

Now use the assumption $\kappa_N \leqslant a_0 N^{2/3}$. Let $b = c_2 a_0 + 1 + C$, with C as above. This ensures $bN^{2/3} \geqslant c_2\kappa_N$ which lets us use the bound above for integers $\ell \geqslant bN^{2/3}$. By adjusting the constant C in the front we can apply the bound to all real $\ell \geqslant bN^{2/3}$.

$$
\begin{aligned}
\mathbb{E}[\tau_1^\rho] = \int_0^\infty \mathbb{P}(\tau_1^\rho \geqslant s)\,ds \;&\leqslant\; bN^{2/3} + C\int_{bN^{2/3}}^\infty \Big(\frac{N^2}{s^3} + \frac{N^2}{s^4}\mathbb{E}[\tau_1^\rho]\Big)\,ds \\
&= bN^{2/3} + \frac{CN^{2/3}}{2b^2} + \frac{C}{3b^3}\mathbb{E}[\tau_1^\rho] \;\leqslant\; bN^{2/3} + \tfrac12 N^{2/3} + \tfrac13\mathbb{E}[\tau_1^\rho].
\end{aligned}
$$

From this we obtain the bound

$$\mathbb{E}[\tau_1^\rho] \leqslant \big(c_2(\rho)a_0 + C_1(a_1,\rho)\big)N^{2/3}$$

and thereby (6.8) has been proved for $q=1$. Substituting this bound back into line (6.14) gives

$$\mathbb{P}\{\tau_1^\rho \geqslant \ell\} \leqslant C_2\Big(\frac{N^2}{\ell^3} + (1+a_0)\frac{N^{8/3}}{\ell^4}\Big)$$

for a constant $C_2 = C_2(a_1,\rho)$, verifying (6.7). Another integration with $b = c_2(\rho)a_0 + 1 + C_2$ proves (6.8) for $1 < q < 3$:

$$
\begin{aligned}
\mathbb{E}[(\tau_1^\rho)^q] = \int_0^\infty \mathbb{P}(\tau_1^\rho \geqslant s)\,qs^{q-1}\,ds \;&\leqslant\; bN^{2/3} + C_2\int_{bN^{2/3}}^\infty \big(N^2 s^{q-4} + (1+a_0)N^{8/3}s^{q-5}\big)\,ds \\
&= bN^{2/3} + \frac{C_2 b^{q-3}}{3-q}N^{\frac23 q} + \frac{C_2}{b^{4-q}}(1+a_0)N^{\frac23 q} \leqslant \Big(c_2(\rho)a_0 + \frac{C_3}{3-q}\Big)N^{2q/3}
\end{aligned}
$$

where we summarized the (a_1,ρ)-dependent constants into $C_3 = C_3(a_1,\rho)$. This completes the proof of Proposition 6.4, with c_3 defined as the constant that appears in front of the right-hand sides of (6.7)–(6.8). □

6.2. **Lower bound.** The parameter $0 < \rho < 1$ of the increment-stationary LPP process is fixed. Let N be the scaling parameter that is sent to infinity, and define the endpoint of the point-to-point LPP process by

$$(m,n) = \big(\lfloor N(1-\rho)^2\rfloor,\, \lfloor N\rho^2\rfloor\big)$$

going in the characteristic direction for ρ. We prove the lower bounds on the right and left tail stated in the theorem below.

THEOREM 6.5. *There exist constants $1 < a_1(\rho), a_2(\rho), N_0(\rho, s) < \infty$ such that, for $s \geqslant a_2(\rho)$ and $N \geqslant N_0(\rho, s)$,*

$$\mathbb{P}\{\omega : G^{\rho}_{0,(m,n)} \geqslant \mathbb{E}[G^{\rho}_{0,(m,n)}] + sN^{1/3}\} \geqslant e^{-a_1(\rho)s^{3/2}}. \tag{6.15}$$

Furthermore, there exist constants $0 < a_3(\rho), a_4(\rho), N_1(\rho, t) < \infty$ such that, for $0 < t \leqslant a_3(\rho)$ and $N \geqslant N_1(\rho, t)$,

$$\mathbb{P}\{\omega : G^{\rho}_{0,(m,n)} \leqslant \mathbb{E}[G^{\rho}_{0,(m,n)}] - tN^{1/3}\} \geqslant a_4(\rho)t^2. \tag{6.16}$$

This gives the lower variance bound in (5.7) for the case when κ_N is bounded:

$$\mathbb{V}\mathrm{ar}[G^{\rho}_{0,(m,n)}] = \mathbb{E}\big[\big(G_{0,(m,n)} - \mathbb{E}^{\rho}[G_{0,(m,n)}]\big)^2\,\big] \geqslant s^2N^{2/3} \cdot e^{-a_1(\rho)s^{3/2}}.$$

In this proof also we perturb the parameter of the boundary weights. Introduce a quantity $r > 0$ which, in the end, will be a constant multiple of $s^{1/2}$. Define another parameter for the increment-stationary CGM by

$$\lambda = \rho + \frac{r}{N^{1/3}}.$$

To guarantee that $\lambda \in (\rho, \frac{1+\rho}{2})$ we assume that

$$N \geqslant N_0 = N_0(\rho, r) = 8\big(\tfrac{r}{1-\rho}\big)^3.$$

N_0 will be increased along the proof, but remains a function of ρ and r.

Notational comment. In this section we find it convenient to attach the parameters ρ and λ to the measure $\mathbb{P}$ and the expectation $\mathbb{E}$ and variance $\mathbb{V}\mathrm{ar}$ to indicate which distribution is placed on the boundary variables. We denote all the weights now by ω_x and the last-passage value $G_{0,x}$ is defined by

$$G_{0,x} = \max_{x_{0,n} \in \Pi_{0,x}} \sum_{k=1}^{n} \omega_{x_k}$$

with maximum over paths that satisfy $x_0 = 0$ and $x_n = x$ with $n = |x|_1$. Under $\mathbb{P}^{\rho}$ the distributions are as in (5.3) but without the I and J notation, namely

$$\omega_x \sim \mathrm{Exp}(1) \text{ for bulk vertices } x \in \mathbb{Z}^2_{>0},\ \ \omega_{ie_1} \sim \mathrm{Exp}(1-\rho),\ \text{ and }\ \omega_{je_2} \sim \mathrm{Exp}(\rho). \tag{6.17}$$

$\mathbb{P}^{\lambda}_{0,(m,n)}$ is the probability distribution of the weights on the rectangle $[0, (m, n)]$. △

For $N \in \mathbb{Z}_{>0}$ and $r > 0$ define the event

$$A_{N,r} = \{(1-\rho)rN^{2/3} \leqslant \tau_1 \leqslant 4\rho^{-1}rN^{2/3}\}. \tag{6.18}$$

Variable τ_1 is the exit time from the e_1-axis of the maximizing path from 0 to (m, n), defined by (6.1). We develop a lower bound for the probability of $A_{N,r}$ under $\mathbb{P}^{\lambda}_{0,(m,n)}$, that is, for the increment-stationary process with parameter λ, restricted to the rectangle $[0, (m, n)]$. Note that this rectangle is *not* of the characteristic shape for λ, and we take advantage of this in the proof.

LEMMA 6.6. *There exists a constant $C_1 = C_1(\rho)$ such that the bound below holds for $r \geqslant 1$ and $N \geqslant N_0$:*

$$\mathbb{P}^{\lambda}_{0,(m,n)}(A_{N,r}) \geqslant 1 - C_1r^{-3} \tag{6.19}$$

Proof. We derive first an upper bound for $\mathbb{P}^{\lambda}_{0,(m,n)}\{\tau_1 > 4\rho^{-1}rN^{2/3}\}$. Define

$$\widetilde{m} = \lfloor N\rho^2\lambda^{-2}(1-\lambda)^2 \rfloor$$

so that $(\widetilde{m}, n)$ points in the characteristic direction for λ, up to an $O(1)$ error κ_N coming from integer parts. Furthermore,

$$\begin{aligned} m - \widetilde{m} &\leqslant N\big((1-\rho)^2 - \rho^2\lambda^{-2}(1-\lambda)^2\big) + 1 = N\frac{\lambda+\rho-2\lambda\rho}{\lambda^2}(\lambda-\rho) + 1 \\ &\leqslant 2\rho^{-1}rN^{2/3} \end{aligned}$$

for $N \geqslant N_0(\rho, r)$, for a suitably chosen $N_0(\rho, r)$. By Lemma A.2 in the appendix, and then by the upper bound (6.7),

$$\begin{aligned} \mathbb{P}^{\lambda}_{0,(m,n)}\{\tau_1 > 4\rho^{-1}rN^{2/3}\} &= \mathbb{P}^{\lambda}_{0,(\widetilde{m},n)}\{\tau_1 > 4\rho^{-1}rN^{2/3} - (m-\widetilde{m})\} \\ &\leqslant \mathbb{P}^{\lambda}_{0,(\widetilde{m},n)}\{\tau_1 > 2\rho^{-1}rN^{2/3}\} \leqslant \frac{c_4}{r^3} \end{aligned}$$

where $c_4 = c_4(\rho)$ contains c_3 from (6.7).

Next we derive an upper bound for $\mathbb{P}^{\lambda}_{0,(m,n)}\{\tau_1 < (1-\rho)rN^{2/3}\}$. Let

$$(\bar{m}, \bar{n}) = \big(\lfloor N(1-\lambda)^2\rfloor, \lfloor N\lambda^2\rfloor\big) \tag{6.20}$$

point in the characteristic direction λ. Bound these differences:

$$\begin{aligned} m - \bar{m} &\geqslant N((1-\rho)^2 - (1-\lambda)^2) - 1 = N(\lambda-\rho)(2-\rho-\lambda) - 1 = (2-\rho-\lambda)rN^{2/3} - 1 \\ &\geqslant (1-\rho)rN^{2/3} \end{aligned}$$

and

$$\bar{n} - n \geqslant N(\lambda^2 - \rho^2) - 1 = N(\lambda-\rho)(\rho+\lambda) - 1 \geqslant \rho r N^{2/3},$$

again for large enough N relative to (ρ, r). By Lemma A.3 in the appendix, and then by the upper bound (6.7),

$$\begin{aligned} \mathbb{P}^{\lambda}_{0,(m,n)}\big\{\tau_1 < (1-\rho)rN^{2/3}\big\} &\leqslant \mathbb{P}^{\lambda}_{0,(m,n)}\big\{\tau_1 < m - \bar{m}\big\} = \mathbb{P}^{\lambda}_{0,(\bar{m},\bar{n})}\big\{\tau_2 > \bar{n} - n\big\} \\ &\leqslant \mathbb{P}^{\lambda}_{0,(\bar{m},\bar{n})}\big\{\tau_2 > \rho r N^{2/3}\big\} \leqslant \frac{c_5}{r^3} \end{aligned}$$

where $c_5 = c_5(\rho)$ contains c_3 from (6.7).

Combine the bounds:

$$\begin{aligned} &\mathbb{P}^{\lambda}_{0,(m,n)}\big\{(1-\rho)rN^{2/3} \leqslant \tau_1 \leqslant 4\rho^{-1}rN^{2/3}\big\} \\ &\quad = 1 - \mathbb{P}^{\lambda}_{0,(m,n)}\{\tau_1 > 4\rho^{-1}rN^{2/3}\} - \mathbb{P}^{\lambda}_{0,(m,n)}\{\tau_1 < (1-\rho)rN^{2/3}\} \\ &\quad \geqslant 1 - C_1 r^{-3}. \end{aligned}$$

□

Computing as in (6.13),

$$\begin{aligned} \mathbb{E}^{\lambda}[G_{0,(m,n)}] - \mathbb{E}^{\rho}[G_{0,(m,n)}] &= m\Big(\frac{1}{1-\lambda} - \frac{1}{1-\rho}\Big) + n\Big(\frac{1}{\lambda} - \frac{1}{\rho}\Big) \\ &= \frac{N}{\lambda(1-\lambda)}(\lambda-\rho)^2 + \bigg\{\frac{\lfloor N(1-\rho)^2\rfloor - N(1-\rho)^2}{(1-\lambda)(1-\rho)} - \frac{\lfloor N\rho^2\rfloor - N\rho^2}{\lambda\rho}\bigg\}(\lambda-\rho) \\ &= \frac{r^2N^{1/3}}{\lambda(1-\lambda)} + O(1)\cdot\frac{r}{N^{1/3}} \geqslant c_6 r^2 N^{1/3} \end{aligned} \tag{6.21}$$

where $c_6 = c_6(\rho) > 0$ is a constant chosen small enough to satisfy the inequality above for all $N \geqslant N_0$ and $r \geqslant 1$.

Let Λ_N denote the set of paths $x_{\bullet} \in \Pi_{0,(m,n)}$ that satisfy $x_1 = e_1$ and $x_k \cdot e_2 > 0$ for $k > \lfloor 4\rho^{-1}rN^{2/3}\rfloor$. In other words, the path stays on the x-axis for a while after leaving the origin, but does not stay on the x-axis beyond the point $\lfloor 4\rho^{-1} r N^{2/3}\rfloor e_1$. For any given weights $\{\omega_x\}$ on the rectangle $\{0,\dots,m\}\times\{0,\dots,n\}$, let

$$G_{0,(m,n)}(\Lambda_N) = \max_{x_\bullet\in\Lambda_N} \sum_{k=1}^{m+n} \omega_{x_k} \tag{6.22}$$

denote the LPP value whose maximum is restricted to the paths in Λ_N. Observe that $G_{0,(m,n)}(\Lambda_N) = G_{0,(m,n)}$ if event $A_{N,r}$ of (6.18) occurs for weights $\{\omega_x\}$. (This would be true even if the lower bound in $A_{N,r}$ would be relaxed to $\tau_1 \geqslant 1$ instead of $\tau_1 \geqslant (1-\rho) r N^{2/3}$.)

We derive our second probability bound. Define the event

$$B_{N,r} = \bigl\{\omega : G_{0,(m,n)}(\Lambda_N) \geqslant \mathbb{E}^\rho[G_{0,(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3}\bigr\}. \tag{6.23}$$

Lemma 6.7. *There exists a constant $C_2 = C_2(\rho)$ such that the bound below holds for $r \geqslant 1$ and $N \geqslant N_0$:*

$$\mathbb{P}^\lambda_{0,(m,n)}(B_{N,r}) \geqslant 1 - C_2 r^{-3}. \tag{6.24}$$

Proof. Since by (6.21)

$$\mathbb{E}^\rho[G_{0,(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3} \leqslant \mathbb{E}^\lambda[G_{0,(m,n)}] - \tfrac{1}{2}c_6 r^2 N^{1/3},$$

we can bound the complementary probability as follows. Constant C changes from line to line. Below we use Lemma 6.2.

$$\begin{aligned}
\mathbb{P}^\lambda_{0,(m,n)}(B^c_{N,r}) &= \mathbb{P}^\lambda_{0,(m,n)}\bigl\{G_{0,(m,n)}(\Lambda_N) < \mathbb{E}^\rho[G_{0,(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3}\bigr\} \\
&\leqslant \mathbb{P}^\lambda_{0,(m,n)}\bigl\{G_{0,(m,n)}(\Lambda_N) < \mathbb{E}^\lambda[G_{0,(m,n)}] - \tfrac{1}{2}c_6 r^2 N^{1/3}\bigr\} \\
&\leqslant \mathbb{P}^\lambda_{0,(m,n)}\bigl\{G_{0,(m,n)} < \mathbb{E}^\lambda[G_{0,(m,n)}] - \tfrac{1}{2}c_6 r^2 N^{1/3}\bigr\} + \mathbb{P}^\lambda_{0,(m,n)}(A^c_{N,r}) \\
&\leqslant \frac{C}{r^4 N^{2/3}} \mathbb{V}\mathrm{ar}^\lambda[G_{0,(m,n)}] + \frac{C_1}{r^3} \\
&\leqslant \frac{C}{r^4 N^{2/3}} \bigl(\mathbb{V}\mathrm{ar}^\rho[G_{0,(m,n)}] + m(\lambda-\rho)\bigr) + \frac{C_1}{r^3} \\
&\leqslant \frac{C}{r^4} + \frac{C_1}{r^3} \leqslant \frac{C_2}{r^3}.
\end{aligned}$$

∎

With the preliminary work done, we turn to the proof of Theorem 6.5.

Proof of Theorem 6.5. We construct a coupling of three environments. Let ω^ρ and ω^λ denote environments as described in (6.17) with parameters ρ and λ. We assume that these environments are coupled so that in the bulk, for $x \in \mathbb{Z}^2_{>0}$, $\omega^\rho_x = \omega^\lambda_x = \omega_x$, while the boundary variables $\{\omega^\rho_{ie_1}, \omega^\rho_{je_2}, \omega^\lambda_{ie_1}, \omega^\lambda_{je_2} : i,j \in \mathbb{Z}_{>0}\}$ are mutually independent.

Construct a mixed environment $\widehat{\omega}$ as follows:

$$\begin{aligned}
\widehat{\omega}_{ie_1} &= \omega^\lambda_{ie_1} \quad \text{for} \quad 1 \leqslant i \leqslant \lfloor 4\rho^{-1} r N^{2/3}\rfloor \\
\text{and} \qquad \widehat{\omega}_x &= \omega^\rho_x \quad \text{for} \quad x \notin \bigl\{ie_1 : 1 \leqslant i \leqslant \lfloor 4\rho^{-1} r N^{2/3}\rfloor\bigr\}.
\end{aligned}$$

Thus in the bulk all weights agree and are i.i.d. Exp(1): for $x \in \mathbb{Z}^2_{>0}$, $\widehat{\omega}_x = \omega^\rho_x = \omega^\lambda_x = \omega_x$. On the boundary $\widehat{\omega}$ follows ω^λ on the segment that is relevant for the event $B_{N,r}$ and elsewhere $\widehat{\omega}$ follows ω^ρ. Note that $\omega^\lambda \in B_{N,r}$ iff $\widehat{\omega} \in B_{N,r}$.

Let the distributions of the three environments ω^ρ, ω^λ and $\widehat{\omega}$, restricted to the rectangle $\{0,\dots,m\}\times\{0,\dots,n\}$, be denoted by $\mathbb{P}^\rho_{0,(m,n)}$, $\mathbb{P}^\lambda_{0,(m,n)}$ and $\widehat{\mathbb{P}}_{0,(m,n)}$, respectively. These are all probability measures on the product space $\mathbb{R}_+^{\{0,\dots,m\}\times\{0,\dots,n\}}$. The Radon-Nikodym derivative

$$f_N(\omega) = \frac{d\widehat{\mathbb{P}}_{0,(m,n)}}{d\mathbb{P}^\rho_{0,(m,n)}}(\omega) = \prod_{i=1}^{\lfloor 4\rho^{-1} r N^{2/3}\rfloor} \frac{\lambda}{\rho}\, e^{-(\lambda-\rho)\omega_{ie_1}}$$

is a product of the Radon-Nikodym derivatives of the exponential single weight marginal distributions on that segment of the boundary where ω^ρ and $\widehat{\omega}$ differ. Computation of the mean square gives

$$\begin{aligned}
\mathbb{E}^\rho_{0,(m,n)}[f_N^2] &= \left(\frac{\lambda^2}{\rho^2}\int_0^\infty e^{-2(\lambda-\rho)s}\rho e^{-\rho s}\,ds\right)^{\lfloor 4\rho^{-1} r N^{2/3}\rfloor} = \left(\frac{\lambda^2}{\rho(2\lambda-\rho)}\right)^{\lfloor 4\rho^{-1} r N^{2/3}\rfloor}\\
&= \exp\left\{\lfloor 4\rho^{-1} r N^{2/3}\rfloor\left[2\log\left(1+\frac{r}{\rho N^{1/3}}\right) - \log\left(1+\frac{2r}{\rho N^{1/3}}\right)\right]\right\}\\
&\leqslant e^{4r^3\rho^{-3}}.
\end{aligned}$$

Now fix $r \geqslant 1$ large enough relative to $C_2 = C_2(\rho)$ from (6.24) so that $C_2 r^{-3} < 1$.

$$\begin{aligned}
1 - C_2 r^{-3} &\leqslant \mathbb{P}^\lambda_{0,(m,n)}(B_{N,r}) = \widehat{\mathbb{P}}_{0,(m,n)}(B_{N,r}) = \mathbb{E}^\rho_{0,(m,n)}[\mathbb{1}_{B_{N,r}} f_N]\\
&\leqslant \left\{\mathbb{P}^\rho_{0,(m,n)}(B_{N,r})\right\}^{1/2}\left\{\mathbb{E}^\rho_{0,(m,n)}[f_N^2]\right\}^{1/2}\\
&\leqslant \left\{\mathbb{P}^\rho_{0,(m,n)}(B_{N,r})\right\}^{1/2} e^{2r^3\rho^{-3}}.
\end{aligned}$$

Since $G_{0,(m,n)} \geqslant G_{0,(m,n)}(\Lambda_N)$, from this comes the lower bound

$$\begin{aligned}
&\mathbb{P}^\rho_{0,(m,n)}\left\{\omega : G_{0,(m,n)} \geqslant \mathbb{E}^\rho[G_{0,(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3}\right\}\\
&\geqslant \mathbb{P}^\rho_{0,(m,n)}\left\{\omega : G_{0,(m,n)}(\Lambda_N) \geqslant \mathbb{E}^\rho[G_{0,(m,n)}] + \tfrac{1}{2}c_6 r^2 N^{1/3}\right\}\\
&\geqslant e^{-4r^3\rho^{-3}}\left(1 - C_2 r^{-3}\right)^2.
\end{aligned} \tag{6.25}$$

To complete the proof of inequality (6.15) of Theorem 6.5, set $s = \frac{1}{2}c_6 r^2$ and let $a_1(\rho), a_2(\rho)$ be suitable functions of ρ, c_6 and C_2.

To prove the second inequality (6.16), abbreviate temporarily $X = G_{0,(m,n)} - \mathbb{E}^\rho[G_{0,(m,n)}]$ and first derive this estimate from inequality (6.15):

$$0 = \mathbb{E}^\rho[X] = \mathbb{E}^\rho[X^+] - \mathbb{E}^\rho[X^-] \geqslant sN^{1/3}e^{-a_1(\rho)s^{3/2}} - \mathbb{E}^\rho[X^-] = 2tN^{1/3} - \mathbb{E}^\rho[X^-]$$

where we set

$$t = \tfrac{1}{2}s\, e^{-a_1(\rho)s^{3/2}}.$$

Note that for $s \geqslant a_2(\rho)$ as in the statement of Theorem 6.5, t is bounded above by some constant $a_3(\rho)$ but can be arbitrarily small. Next,

$$\begin{aligned}
2tN^{1/3} \leqslant \mathbb{E}^\rho[X^-] &= \mathbb{E}^\rho[X^-,\, X^- < tN^{1/3}] + \mathbb{E}^\rho[X^-,\, X^- \geqslant tN^{1/3}]\\
&\leqslant tN^{1/3} + \left(\mathbb{E}^\rho[(X^-)^2]\right)^{1/2}\left(\mathbb{P}^\rho\{X^- \geqslant tN^{1/3}\}\right)^{1/2}\\
&\leqslant tN^{1/3} + \left(\mathbb{V}\mathrm{ar}^\rho[G_{0,(m,n)}]\right)^{1/2}\left(\mathbb{P}^\rho\{X^- \geqslant tN^{1/3}\}\right)^{1/2}
\end{aligned}$$

From which we deduce, together with the upper variance bound from Theorem 5.2, for some constant $a_4(\rho)$,

$$\mathbb{P}^\rho\{X^- \geqslant tN^{1/3}\} \geqslant a_4(\rho)t^2.$$

This inequality is the same as (6.16). This completes the proof of Theorem 6.5. □

Appendix A. Coupling LPP processes

We first prove a lemma for deterministic weights. Fix a point $a \in \mathbb{Z}^2$. Suppose boundary weights $\{\omega_{a+ke_r} : k \in \mathbb{Z}_{>0}, r \in \{1,2\}\}$ on the south and west boundaries of $a + \mathbb{Z}^2_{\geqslant 0}$ and bulk weights $\{\omega_x\}_{x \in a+\mathbb{Z}^2_{>0}}$ are given. Put an irrelevant weight $\omega_a = 0$ in the corner a. Let $G_{a,x}$ denote the LPP value for points $x \in a + \mathbb{Z}^2_{\geqslant 0}$ and let $\pi^{a,x}_{\scriptscriptstyle\blacksquare}$ be a maximizing path from a to x. (If it is not unique, make an arbitrary choice.)

Let $b \geqslant a$ on $\mathbb{Z}^2$. On the lattice $b + \mathbb{Z}^2_{\geqslant 0}$, put a corner weight $\eta_b = 0$ and define boundary weights

$$\eta_{b+ke_r} = G_{a,\,b+ke_r} - G_{a,\,b+(k-1)e_r} \qquad \text{for } k \in \mathbb{Z}_{>0} \text{ and } r \in \{1,2\}. \tag{A.1}$$

In the bulk use $\eta_x = \omega_x$ for $x \in b + \mathbb{Z}^2_{>0}$. Denote the LPP process in $b + \mathbb{Z}^2_{\geqslant 0}$ that uses weights $\{\eta_x\}_{x \in b+\mathbb{Z}^2_{\geqslant 0}}$ by

$$\widetilde{G}_{b,x} = \max_{x_{\scriptscriptstyle\blacksquare} \in \Pi_{b,x}} \sum_{i=0}^{|x-b|_1} \eta_{x_i}, \qquad x \in b + \mathbb{Z}^2_{\geqslant 0}. \tag{A.2}$$

Lemma A.1. *Let $a \leqslant b \leqslant v$ in $\mathbb{Z}^2$. Then $G_{a,v} = G_{a,b} + \widetilde{G}_{b,v}$. The restriction of any maximizing path for $G_{a,v}$ to $b + \mathbb{Z}^2_{\geqslant 0}$ is part of a maximizing path for $\widetilde{G}_{b,v}$. The edges in the interior of $b + \mathbb{Z}^2_{\geqslant 0}$ of any maximizing path for $\widetilde{G}_{b,v}$ extend to a maximizing path for $G_{a,b}$.*

Proof. If $v = b + ke_r$ (that is, v is on the boundary of $b + \mathbb{Z}^2_{\geqslant 0}$) the situation is straightforward. Suppose $v > b$ coordinatewise. Suppose a maximal path from a to v enters $b + \mathbb{Z}^2_{>0}$ by taking the step from $x = b + ke_r$ to $y = b + ke_r + e_{3-r}$. Suppose a maximal path for $\widetilde{G}_{b,v}$ enters $b + \mathbb{Z}^2_{>0}$ by taking the step from $\widetilde{x} = b + \ell e_s$ to $\widetilde{y} = b + \ell e_s + e_{3-s}$. Then

$$\begin{aligned} G_{a,v} &= G_{a,x} + G_{y,v} = G_{a,b} + \sum_{i=1}^{k} \eta_{b+ie_r} + G_{y,v} \\ &\leqslant G_{a,b} + \widetilde{G}_{b,v} = G_{a,b} + \sum_{i=1}^{\ell} \eta_{b+ie_s} + G_{\widetilde{y},v} \\ &= G_{a,\widetilde{x}} + G_{\widetilde{y},v} \leqslant G_{a,v}. \end{aligned}$$

Thus the inequalities above are in fact equalities. ☐

Write $\mathbb{P}_{0,v}$ for the probability measure of the LPP process in the rectangle $[0, v]$ with boundary and bulk weights (5.3).

Lemma A.2. *Let $1 \leqslant k < k + \ell \leqslant m$. Then $\mathbb{P}_{0,(m,n)}(\tau_1 \geqslant k + \ell) = \mathbb{P}_{0,(m-k,n)}(\tau_1 \geqslant \ell)$.*

Proof. Take $a = 0$, $b = (k, 0)$ and $v = (m, n)$ in Lemma A.1. Then, under $\mathbb{P}_{0,(m,n)}$, the LPP process $\widetilde{G}_{b,x}$ in $[b, v]$ has the same distribution, modulo the translation of the origin to b, as an LPP process under $\mathbb{P}_{0,(m-k,n)}$. By Lemma A.1 the maximizing paths from a and b to v agree in their portions inside $[k + 1, m] \times [0, n]$. ☐

Lemma A.3. *Let $1 \leqslant \bar{m} < m$ and $1 \leqslant n < \bar{n}$. Then $\mathbb{P}_{0,(m,n)}(\tau_1 < m - \bar{m}) = \mathbb{P}_{0,(\bar{m},\bar{n})}(\tau_2 > \bar{n} - n)$.*

Proof. We couple these LPP processes as follows. Let

$$a = (\bar{m} - m, 0), \quad a' = (0, n - \bar{n}) \quad \text{and} \quad v = (\bar{m}, n).$$

The origin 0 takes the role of b in Lemma A.1.

Let i.i.d. Exp(1) weights $\{\omega_x\}_{x \in \mathbb{Z}^2}$ be given. Then place independent boundary edge weights with distributions dictated by (5.3) on the south and west boundaries of the lattice region $(a + \mathbb{Z}^2_{\geqslant 0}) \cup (a' + \mathbb{Z}^2_{\geqslant 0})$:

(a) On horizontal boundary edges put $\mathrm{Exp}(1-\rho)$ weights $\sigma_{(i-1)e_1,ie_1}$ for $\bar{m}-m+1 \leqslant i \leqslant 0$ and $\sigma_{a'+(i-1)e_1,\,a'+ie_1}$ for $i \in \mathbb{Z}_{>0}$.

(b) On vertical boundary edges put $\mathrm{Exp}(\rho)$ weights $\sigma_{(j-1)e_2,\,je_2}$ for $n-\bar{n}+1 \leqslant j \leqslant 0$ and $\sigma_{a+(j-1)e_2,\,a+je_2}$ for $j \in \mathbb{Z}_{>0}$.

Next consider two LPP processes that emanate from a and a' and use the boundary weights described above in (a) and (b): $G_{a,y}$ for points y on the y-axis, and $G_{a',x}$ for points x on the x-axis. (The restriction put on y implies that $G_{a,y}$ does not need boundary weights on the x-axis beyond the interval $[a,0]$, and similarly $G_{a',x}$ does not need boundary weights on the y-axis beyond the interval $[a',0]$.) Let these processes define boundary weights on $\mathbb{Z}^2_{\geqslant 0}$: $\eta_{(i-1)e_1,\,ie_1} = G_{a',\,ie_1} - G_{a',(i-1)e_1}$ and $\eta_{(j-1)e_2,\,je_2} = G_{a,\,je_2} - G_{a,(j-1)e_2}$ for $i,j \in \mathbb{Z}_{>0}$.

Now consider three LPP processes with lower left corners a, 0 and a':

(i) $\widetilde{G}_{a,x}$ uses boundary weights $\sigma_{(i-1)e_1,\,ie_1}$ for $\bar{m}-m+1 \leqslant i \leqslant 0$ and $\eta_{(i-1)e_1,\,ie_1}$ for $i \in \mathbb{Z}_{>0}$ on the horizontal axis emanating from a and $\sigma_{a+(j-1)e_2,\,a+je_2}$ for $j \in \mathbb{Z}_{>0}$ on the vertical axis emanating from a.

(ii) $\widetilde{G}_{0,x}$ uses boundary weights $\eta_{(i-1)e_1,\,ie_1}$ and $\eta_{(j-1)e_2,\,je_2}$ on the standard axes emanating from 0.

(iii) $\widetilde{G}_{a',x}$ uses boundary weights $\sigma_{a'+(i-1)e_1,\,a'+ie_1}$ for $i \in \mathbb{Z}_{>0}$ on the horizontal axis emanating from a' and weights $\sigma_{(j-1)e_2,\,je_2}$ for $n-\bar{n}+1 \leqslant j \leqslant 0$ and $\eta_{(j-1)e_2,\,je_2}$ for $j \in \mathbb{Z}_{>0}$ on the vertical axis emanating from a'.

Let $\widetilde{\mathbb{P}}$ denote the probability measure under which this coupling has been constructed, that is, the probability measure of the independent weights ω_x and $\sigma_{x,\,x+e_k}$.

Let A be the event that the (a.s. unique) maximal path for $\widetilde{G}_{a,v}$ does not go through the origin. Let B the event that the (a.s. unique) maximal path for $\widetilde{G}_{a',v}$ goes through the point e_2. Lemma A.1 applies to the pair $\widetilde{G}_{a,v}$ and $\widetilde{G}_{0,v}$, and also to the pair $\widetilde{G}_{a',v}$ and $\widetilde{G}_{0,v}$. Thus the maximizing paths for $\widetilde{G}_{a,v}$ and $\widetilde{G}_{a',v}$ agree from that point onwards at which they exit the y-axis. Both A and B are equivalent to the statement that this point is strictly above the origin on the y-axis. Hence $A = B$.

On the other hand, LPP processes $\{\widetilde{G}_{a,\,a+x}\}_{x\in\mathbb{Z}^2_{\geqslant 0}}$ and $\{\widetilde{G}_{a',\,a'+x}\}_{x\in\mathbb{Z}^2_{\geqslant 0}}$ both have the same distribution as the LPP process $\{G^\rho_x\}_{x\in\mathbb{Z}^2_{\geqslant 0}}$ with stationary increments. Event A is equivalent to the condition that the maximizing path for $\widetilde{G}_{a,v}$ takes at most $m-\bar{m}-1$ consecutive e_1-steps from a, which is the same as $\tau_1 < m-\bar{m}$ for $G^\rho_{0,(m,n)}$. Similarly, event B says that the maximizing path for $\widetilde{G}_{a',v}$ takes at least $\bar{n}-n+1$ consecutive e_2-steps from a', which for $G^\rho_{(\bar{m},\bar{n})}$ is the same as $\tau_2 > \bar{n}-n$. Thus

$$\mathbb{P}_{0,(m,n)}(\tau_1 < m-\bar{m}) = \widetilde{\mathbb{P}}(A) = \widetilde{\mathbb{P}}(B) = \mathbb{P}_{0,(\bar{m},\bar{n})}(\tau_2 > \bar{n}-n). \qquad \square$$

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