A 2d growth model in the Anisotropic KPZ class

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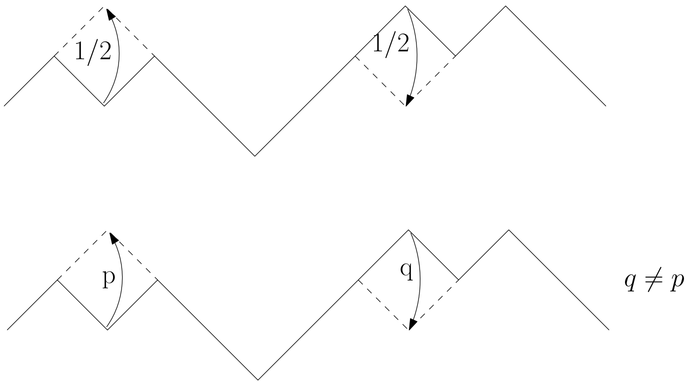
Framework: (2d)-stochastic growth models

Stochastic growth modeled by (irreversible) Markov chains with local update rules.
Typical questions:

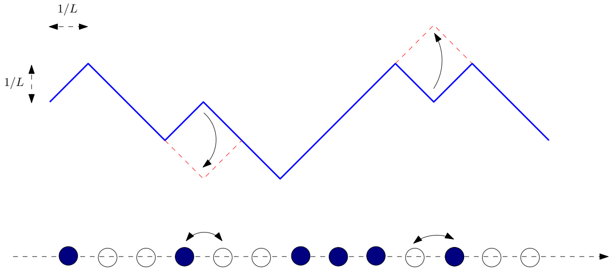
- stationary states (for *interface gradients*)
- space-time correlations of height fluctuations
- hydrodynamic limit
- formation of shocks
- ...

Main object of this talk: 2-dimensional models (lozenge tiling dynamics) where these questions can be (partly) answered

Symmetric vs. asymmetric random dynamics

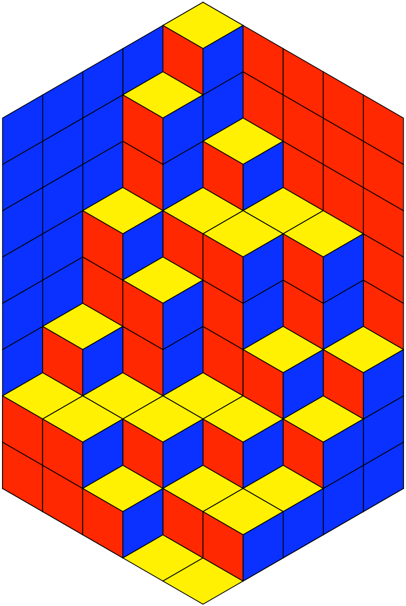


For $d = 1$: Symmetric vs. Asymmetric Simple Exclusion Process

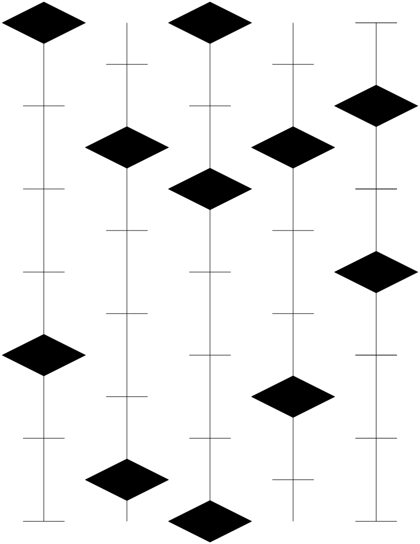


In both SSEP/ASEP, Bernoulli(ρ) are invariant.
For $p \neq q$, irreversibility (particle flux).

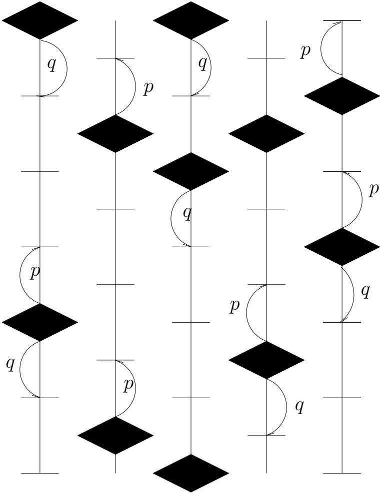
Generalization to $(2 + 1)$ dimensions?



Interlaced particle configurations



The "single-flip dynamics"



“Analog” of Bernoulli measures: Ergodic Gibbs measures

- Choose $\rho = (\rho_1, \rho_2, \rho_3)$ with $\rho_i \in (0,1), \rho_1 + \rho_2 + \rho_3 = 1$. There exists a unique translation invariant, ergodic Gibbs measure π_ρ s.t. the density of horizontal, NW and NE lozenges are ρ_1, ρ_2, ρ_3.

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- Dimer-dimer correlations decay algebraically:

$$\pi_\rho(1_{e \in M}; 1_{e' \in M}) \approx |e - e'|^{-2}$$

"Free-fermion" measures.

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  “Free-fermion” measures.

- height function $\sim$ massless Gaussian field: if $\int_{\mathbb{R}^2} \varphi(x)dx = 0$,

$$\epsilon^2 \sum_x \varphi(\epsilon x) h_x \xrightarrow{\epsilon \to 0} \int \varphi(x) X(x) dx$$

  with $\langle X(x)X(y) \rangle = -\frac{1}{2\pi^2} \log |x - y|$.

What is known for single-flip dynamics, $p \neq q$?

- Stationary states: unknown. Presumably very different from π_ρ. Numerical simulations [Forrest-Tang-Wolf Phys Rev A 1992] show $t^{0.24\ldots}$ growth of height fluctuations.

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- totally asymmetric case ($p = 1, q = 0$): *non-explicit* hydrodynamic limit (hyperbolic rescaling)

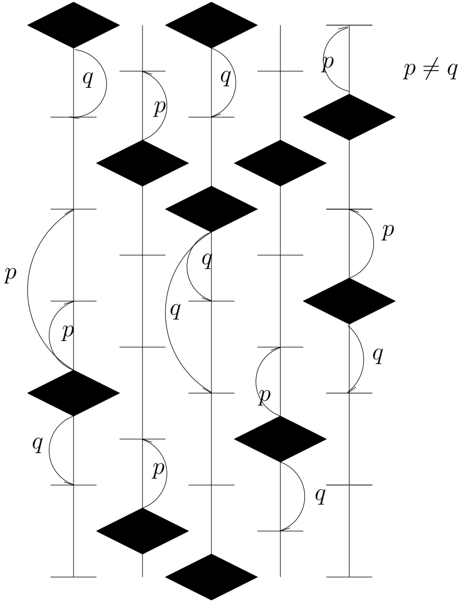
$$\lim_{L\to\infty} \frac{1}{L} h(xL, tL) = \phi(x,t) \quad \text{almost surely},$$

  where ϕ is Hopf-Lax solution of

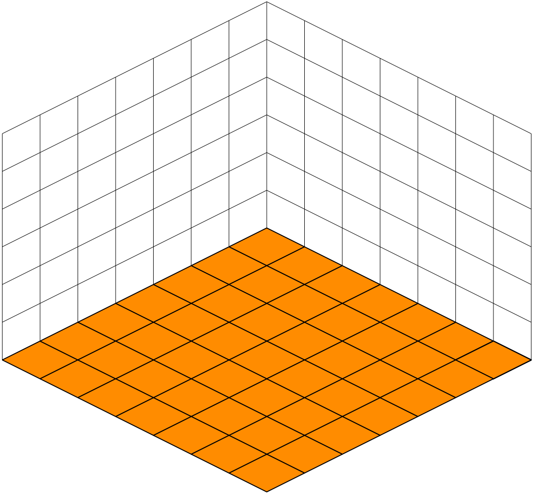
$$\partial_t \phi + V(\nabla\phi) = 0$$

  for some convex $V(\cdot)$.
  Super-additivity method [Seppäläinen, Rezakhanlou]

A growth process with longer jumps



Dynamics well defined?



Particles can leave to ∞ in infinitesimal time

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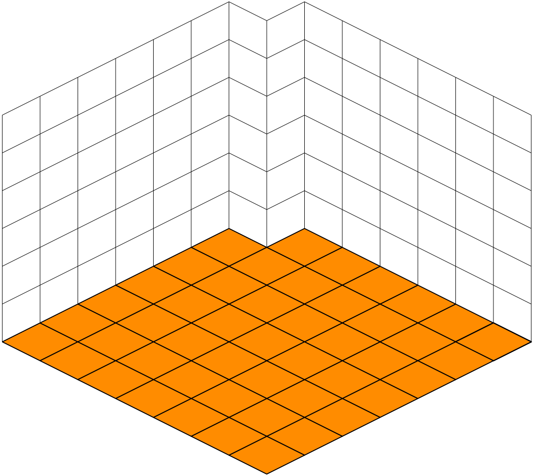

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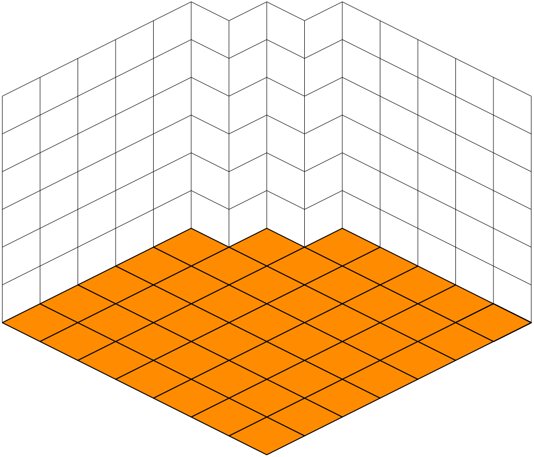
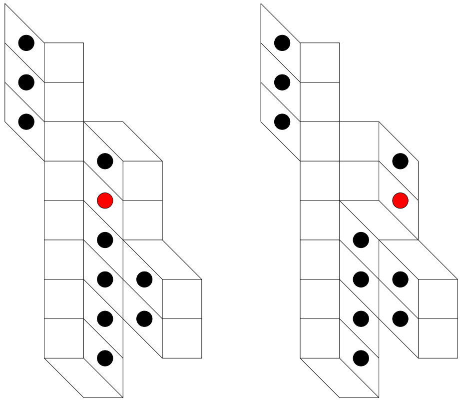


Particles can leave to ∞ in infinitesimal time

Some known results...

A. Borodin, P. L. Ferrari (CMP '14): $p = 1, q = 0$



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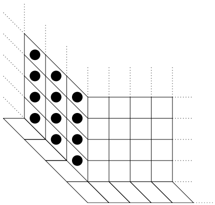

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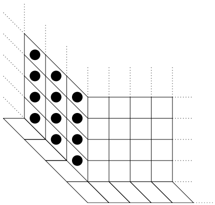

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$$\lim_{L\to\infty} \frac{1}{L} h(xL, \tau L) = \phi(x, \tau),$$

where

$$\partial_\tau \phi + v(\nabla \phi) = 0,$$

and

$$v(\nabla\phi) = -\frac{1}{\pi} \frac{\sin(\pi\partial_{x_1}\phi)\sin(\pi\partial_{x_2}\phi)}{\sin(\pi(1-\partial_{x_1}\phi-\partial_{x_2}\phi))}$$

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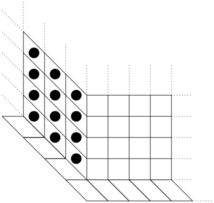

..., $\sqrt{\log t}$ Gaussian fluctuations:

$$\frac{1}{\sqrt{\log L}}[h(xL,\tau L) - \mathbb{E}h(xL,\tau L)] \Rightarrow \mathcal{N}(0, 1/(2\pi^2))$$

...and convergence of local statistics to those of a Gibbs measure.

...and some new ones

Theorem 1 [F. T., Ann. Probab. 2017+]

- Dynamics well defined if initial spacings grow sublinearly at infinity.

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- and fluctuations grow $\sqrt{\text{logarithmically}}$:
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(S. Chhita, P. L. Ferrari, F.T., in preparation)

...and some new ones

Theorem 2 [M. Legras, F. T., arXiv '17]
Totally asymmetric case: $p = 1, q = 0$.
• If the initial condition approximates a smooth profile:

$$\lim_L \frac{1}{L} h(xL) = \phi_0(x)$$

with $0 < \partial_{x_1}\phi_0 < 1, 0 < \partial_{x_2}\phi_0 < 1$ and $0 < (\partial_{x_1}\phi_0 + \partial_{x_2}\phi_0) < 1$, then

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where $\phi(x,0) = \phi_0(x)$ and $\partial_t \phi + v(\nabla\phi) = 0$.

• convergence to viscosity solution for $t > T_{shocks}$ if initial profile is convex.

Remarks on the hydrodynamic limit

- $v(\cdot)$ has singularities

$$\text{Recall: } v(\nabla\phi) = -\frac{1}{\pi}\frac{\sin(\pi\partial_{x_1}\phi)\sin(\pi\partial_{x_2}\phi)}{\sin(\pi(1-\partial_{x_1}\phi-\partial_{x_2}\phi))}$$

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- convex initial profile $\Rightarrow$ viscosity solution has variational expression (Hopf formula)
- Borodin-Ferrari initial condition: characteristics do not cross, classical solution for all times. General initial condition: singularities appear in finite time.
- in contrast with Borodin-Ferrari, we do not use "integrable probability" methods.

A heuristic link with 2D KPZ equation

One expects (in some sense) height fluctuations in stationary state π_ρ to be described by

$$\partial_t h(t,x) = \Delta h(t,x) + \nabla h(t,x) \cdot Q_\rho \nabla h(t,x) + \dot{W}(t,x)$$

with $\dot{W}$ a space-time white noise and Q_ρ the Hessian of $v(\rho)$.

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NB: very singular equation, unclear how to give math meaning

A heuristic link with 2D KPZ equation

Recall:

- for the single-flip dynamics, $v(\cdot)$ unknown but concave: signature of Q_ρ is $(-,-)$. "Isotropic KPZ equation"
- B-F dynamics. From explicit form of $v(\cdot)$, signature of Q_ρ is $(+,-)$. "Anisotropic KPZ equation"

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Wolf [PRL '91] predicted:

- Anisotropic case: non-linearity irrelevant, fluctuations grow $\sim \sqrt{\log t}$ as if $Q_\rho = 0$ (Stochastic Heat Equation).
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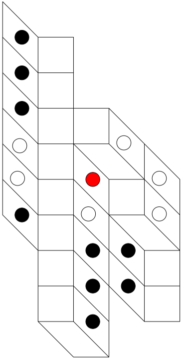
- Isotropic case: non-linearity relevant, fluctuations grow like t^ν, some non-trivial exponent $\nu > 0$.
  simulations: $\nu \approx 0.24$
- Joint work with A. Borodin and I. Corwin [CMP 2017+]: a variant of the $(2+1)$-d growth process in the AKPZ class for which convergence to the stochastic heat equation can be proven

Towards the Stochastic Heat Equation

One can generalize the model: rates depend on a parameter $r \in [0,1)$ and (in a special way) on the distances between a particle and its six neighbors



$$\text{rate} = \frac{(1-r^{B-1})(1-r^{D})}{1-r^{C+1}}$$

$r = 0$: back to Borodin $-$ Ferrari dyn.

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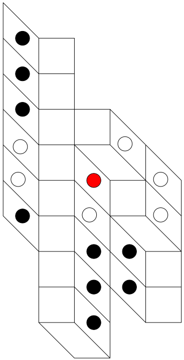

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$$r = 0 : \text{back to Borodin} - \text{Ferrari dyn.}$$

Theorem 3 [Corwin-Toninelli, ECP 2016]: explicit stationary measure of Gibbs type.

Towards the Stochastic Heat Equation

For $r = e^{-\varepsilon} \to 1$, with $1/\varepsilon$ rescaling of time and particle distances, particle positions z_p have Gaussian fluctuations.

Theorem 4 [Borodin-Corwin-Toninelli, CMP 2016+]:

$$\varepsilon(z_x(t/\varepsilon) - z_x(0)) \to Vt$$

and

$$\sqrt{\varepsilon}(z_x(t/\varepsilon) - z_x(0) - \varepsilon^{-1}Vt) \to \xi_{x,t}$$

and $\xi_{x,t}$ ($\Leftrightarrow$ height fluctuations w.r.t. deterministically growing profile) solve a linear system of SDEs.

Towards the Stochastic Heat Equation

In that limit, space-time correlations can be computed:

$$\mathbb{E}\left[\xi_{x,t}\ \xi_{y,s}\right] - \mathbb{E}\left[\xi_{x,t}\right]\mathbb{E}\left[\xi_{y,s}\right]$$

Towards the Stochastic Heat Equation

Along a special direction $U \in \mathbb{R}^2$ ("characteristics")

$$\mathbb{E}\left[\xi_{\frac{tU}{\delta}+\frac{x}{\sqrt{\delta}},\frac{t}{\delta}}\ \xi_{\frac{sU}{\delta}+\frac{y}{\sqrt{\delta}},\frac{s}{\delta}}\right]-\mathbb{E}\left[\xi_{\frac{tU}{\delta}+\frac{x}{\sqrt{\delta}},\frac{t}{\delta}}\right]\mathbb{E}\left[\xi_{\frac{sU}{\delta}+\frac{y}{\sqrt{\delta}},\frac{s}{\delta}}\right]$$

tends as $\delta \to 0$ to $C(s,t,x-y)$, the space-time correlation of the 2d SHE

$$\partial_t h = \Delta h + \dot{W}, \quad h(0,x) = 0.$$

For all other directions U', correlations ≈ 0 if $t - s \gg \sqrt{t}$.

Remark: A similar behavior expected for growth models in the Anisotropic KPZ class. E.g. the Borodin-Ferrari dynamics.

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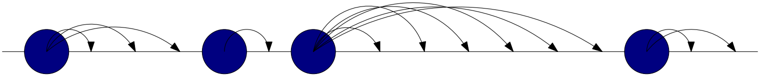
- ...but "natural" longer-jump versions can be analyzed in detail (some "integrable structure" behind)
- Caveat: long-jump and single-flip versions are in two different universality classes (AKPZ/KPZ)

Thanks!

Ideas I: Comparison with the Hammersley process (HP)



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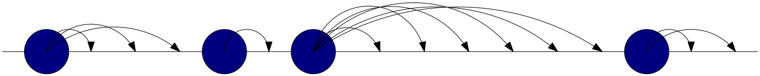

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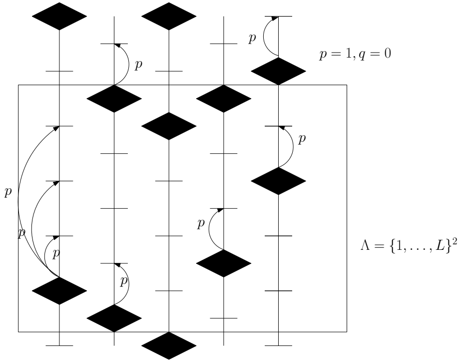
Lozenge dynamics $\sim$ infinite set of coupled Hammersley processes.
Comparison: lozenges move less than HP particles

Ideas II: Fluctuations



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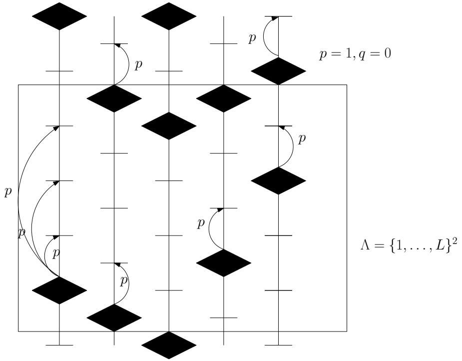

Let $Q_\Lambda(t) = \sum_{x \in \Lambda}(h_x(t) - h_x(0))$.

$$\frac{d}{dt}\langle Q_\Lambda(t)\rangle = \langle \sum_x |V(x, \uparrow) \cap \Lambda|\rangle, \quad \langle \cdot \rangle := \mathbb{E}_{\pi_\rho}.$$

Ideas II: Fluctuations

Similarly, one can prove

$$\frac{d}{dt}\langle (Q_\Lambda(t)-\langle Q_\Lambda(t)\rangle)^2\rangle \leq \sqrt{\langle (Q_\Lambda(t)-\langle Q_\Lambda(t)\rangle)^2\rangle} L\sqrt{\log L}+O(L^2)$$

so that

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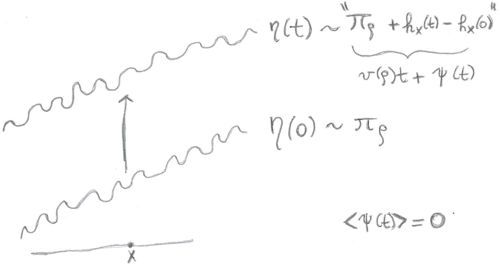

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Recall

$$\langle (Q_\Lambda(T) - \langle Q_\Lambda(T) \rangle)^2 \rangle = O(T^2 L^2 \log L).$$

If $L = 1$, we get the (useless) bound $\sqrt{\langle \psi(T)^2 \rangle} = O(T)$.
How to do better?

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If we choose $L = T$ we get then $\sqrt{\langle \psi(T)^2 \rangle} = O(\sqrt{\log T})$ as wished.

Ideas III: Invariance on the torus

For simplicity, $p = 1, q = 0$.

Stationary measure π_ρ^L: uniform measure with fraction ρ_i of lozenges of type $i = 1, 2, 3$.

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$$[\pi^L_\rho \mathcal{L}](\sigma) = \frac{1}{N^L_\rho}[\sum_n |I^+_n| - \sum_n |I^-_n|]$$

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Key fact:

Lemma: The probability of seeing an inter-particle gap $\geq \log R$ within distance R from the origin before time 1 is $O(R^{-K})$ for every K.