KPZ wandering exponent for random walk in i.i.d. dynamic Beta random environment

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Joint work with Márton Balázs (Bristol)
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Coin tosses and random walk

Toss a coin: Heads with probability p*, Tails with probability* $1 - p$.

Tails$=\uparrow$, *Heads*$=\longrightarrow$

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Repeated tosses $\longleftrightarrow$ up-right path (Random Walk on $\mathbb{Z}^2$):

HHTHTTTHHTHT $\longleftrightarrow$ 

Classical results: LLN, CLT, LDP

$X_0 = 0$, X_n=position on up-right path after n tosses/steps.

$X_n \cdot e_1 = \#H$, $X_n \cdot e_2 = n - X_n \cdot e_1 = \#T$ (up to toss n).

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Large Deviation Principle (LDP): P(proportion of H $\geq s > p$) $\approx e^{-nH(s)}$

$$-n^{-1} \log P\{X_n \cdot e_1 \geq sn\} \to H(s) = s \log \frac{s}{p} + (1-s) \log \frac{1-s}{1-p}$$

$H(s)$ = entropy of coin s relative to coin p.

Conditioned random walk

Large deviations also tell us that $X_{0,n} = (X_0, \dots, X_n)$ conditioned on

$X_n/n \approx \zeta = se_1 + (1-s)e_2$

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New random walk still has CLT fluctuations (of size $\sqrt{n}$).

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$P\{X_{n+1} = x + e_1 \,|\, X_n = x\} = p_x$

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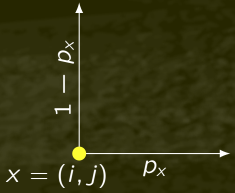

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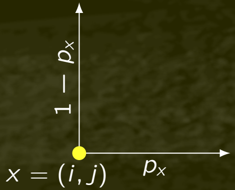

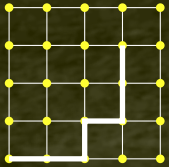

HHTHTT has probability

$p_{0,0}p_{1,0}(1-p_{2,0})p_{2,1}(1-p_{3,1})(1-p_{3,2})$

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Averaged CLT: if the environment is averaged out, then

$$\frac{X_n - n\xi}{\sqrt{\bar{p}(1-\bar{p})n}} \to Ze_1 - Ze_2 \text{ (in distribution)}, Z \text{ Standard Normal.}$$

LLN, CLT, LDP

Also, Quenched CLT (R-A, Seppäläinen '05): for almost every environment $\{p_x : x \in \mathbb{Z}_+^2\}$

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Note: once environment is fixed, X_n is no longer a random walk with i.i.d. increments.

LLN, CLT, LDP

Averaged LDP: when environment is averaged out and $s > \bar{p}$

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H_q is deterministic but in general does not have an explicit expression (though some variational formulas are available).

$H_q(s) > H_a(s)$ unless $s = \bar{p}$, in which case both $= 0$.

Solvable model

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Can also compute the quenched rate $H_q(s)$ explicitly (later).

KPZ fluctuation exponent

Barraquand and Corwin '15 observed a connection to KPZ:

Theorem. For the Beta(α, β) case

$$\frac{\log P^\omega\{X_n \cdot e_1 \geq sn\} + nH_q(s)}{\sigma(s) n^{1/3}} \longrightarrow GUE \quad \text{(in distribution)}$$

($\sigma(s)$ is known explicitly in terms of polygamma functions ψ_1 and ψ_2).

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But how could it? We know the CLT holds, both quenched and averaged!

What is going on?!

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New process is another (classical) random walk. Nothing new.

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Averaged: average out the environment in the above.
What is the resulting process? (Again, not a classical random walk)

Busemann function

Theorem *(Balázs, R-A, Seppäläinen '16). For almost every choice of the environment* $\omega = \{p_x : x \in \mathbb{Z}_+\}$*, limit*

$$B^\zeta(x, y) = \lim_{n\to\infty} \Big[\log P^\omega(X_n \approx n\zeta \,|\, X_0 = x) - \log P^\omega(X_n \approx n\zeta \,|\, X_0 = y) \Big]$$

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exists and $H_q(s) = -s\mathbb{E}[B^\zeta(0,e_1)] - (1-s)\mathbb{E}[B^\zeta(0,e_2)]$
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This comes from the Markov property

$$P^\omega(X_n \approx n\zeta \,|\, X_0 = x) = p_x P^\omega(X_n \approx n\zeta \,|\, X_0 = x+e_1) + (1-p_x) P^\omega(X_n \approx n\zeta \,|\, X_0 = x+e_2)$$

(then divide by $P^\omega(X_n \approx n\zeta \,|\, X_0 = 0)$ *and take* $n \to \infty$*).*

Quenched conditioned RWRE

Define π^ζ *as a Doob transform of* p *by the harmonic function* $e^{-B^\zeta(0,x)}$*:*

$$\pi^\zeta_{x,x+e_1} = p_x \frac{e^{-B^\zeta(0,x+e_1)}}{e^{-B^\zeta(0,x)}} \quad \textit{and} \quad \pi^\zeta_{x,x+e_2} = (1-p_x)\frac{e^{-B^\zeta(0,x+e_2)}}{e^{-B^\zeta(0,x)}}.$$

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Note: $\zeta = \xi$ *gives* $B^\xi \equiv 0$ *and* $\pi^\xi \equiv p$*.*

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Note: *$\zeta = \xi$ gives $B^\xi \equiv 0$ and $\pi^\xi \equiv p$.*

So, if $\zeta \neq \xi$, the new process is another random walk in a stationary but very correlated random environment.

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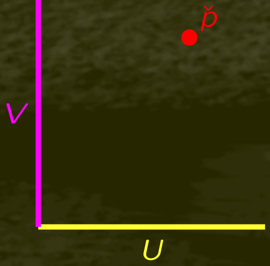
Let $\{U_{ke_1} : k \geq 0\}$ be i.i.d. Beta$(\alpha + \lambda, \beta)$.

Let $\{V_{ke_2}^{-1} : k \geq 0\}$ be i.i.d. Beta(λ, α).

Let $\{\check{p}_x : x \in \mathbb{N}^2\}$ be i.i.d. Beta(α, β).

All three families are mutually independent.

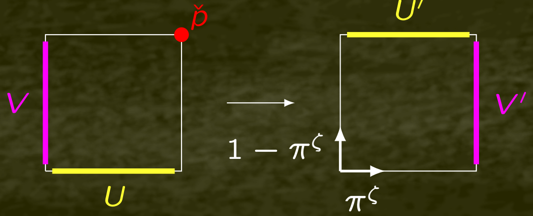


Distribution of π^ζ

For the rest of the edges of $\mathbb{Z}_+^2$ *define* U*s and* V*s via induction*

$$U' = \frac{\check{p}V + (1-\check{p})U}{V}, \qquad V' = \frac{\check{p}V + (1-\check{p})U}{U}.$$

And define $\pi^\zeta_{x,x+e_1} = \dfrac{V_x - 1}{V_x - U_x} \in (0,1)$ *and* $\pi^\zeta_{x,x+e_2} = 1 - \pi^\zeta_{x,x+e_1}$.



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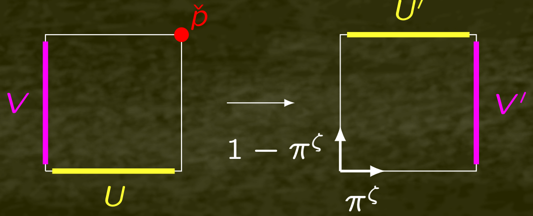

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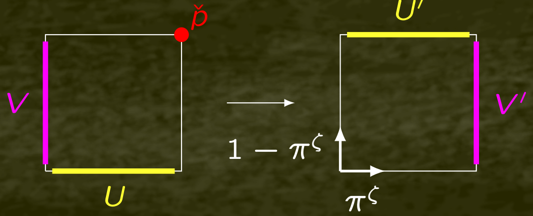

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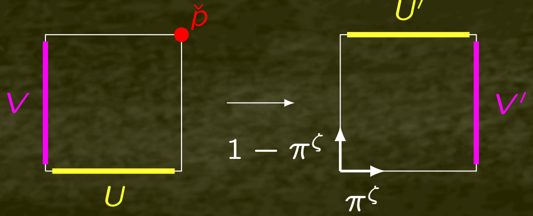

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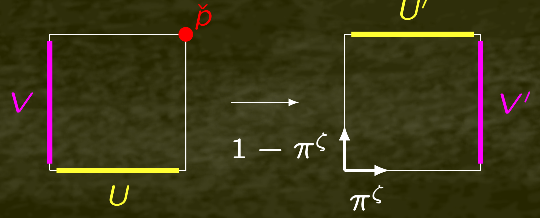

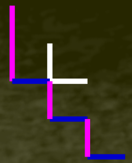

Distribution of π^ζ

For the rest of the edges of $\mathbb{Z}_+^2$ define Us and Vs via induction

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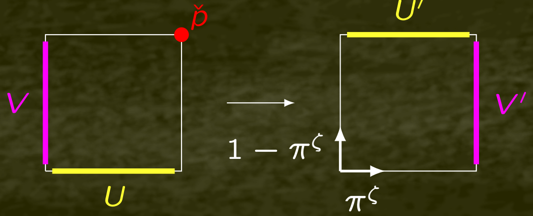

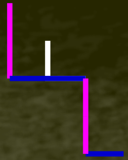

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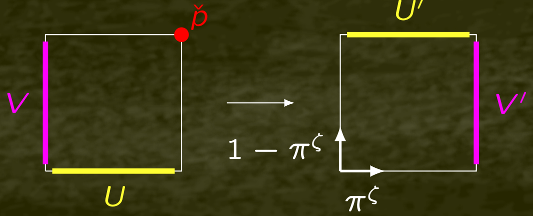

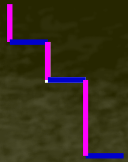

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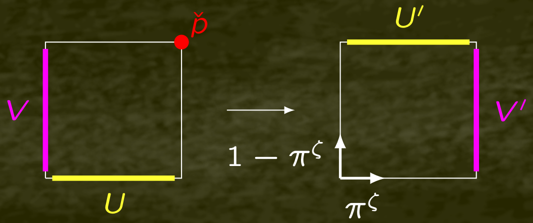

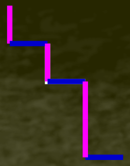

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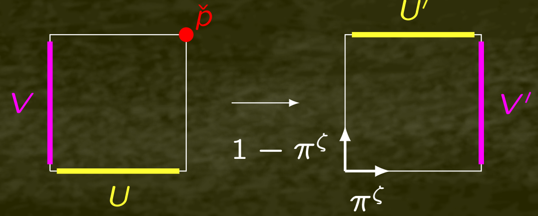

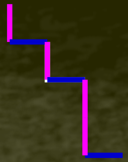

Lemma: (U', V', π^ζ) *has the same distribution as* $(U, V, \check{p})$.

Corollary: $\{\pi^\zeta_{x+y,x+y+e_1} : y \in \mathbb{Z}_+^2\}$ *has the same distribution as for* $x = 0$.

Bijection between velocity ζ and boundary parameter λ

$\lambda \in [0, \infty]$ *is in one-to-one correspondence with* ζ *via*

$$\zeta_1 = \frac{\psi_1(\lambda) - \psi_1(\alpha + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)} \in \Big[\frac{\alpha}{\alpha + \beta}, 1\Big], \quad \zeta_2 = 1 - \zeta_1$$

with $\lambda = 0 \iff \zeta = e_1$ *and* $\lambda = \infty \iff \zeta = \xi = (\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$.

ψ_1 *is the trigamma function:* $\psi_1 = (\log \Gamma)''$.

For rest of velocities, $\zeta_1 \in [0, \frac{\alpha}{\alpha+\beta}]$, *switch role of Us and Vs.*

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$$= s\psi_0\big(\alpha+\beta+\lambda(\zeta)\big) + (1-s)\psi_0\big(\lambda(\zeta)\big) - \psi_0\big(\alpha+\lambda(\zeta)\big)$$

for $s \in [\frac{\alpha}{\alpha+\beta}, 1]$.

(For $s \in [0, \frac{\alpha}{\alpha+\beta})$ *switch the role of the axes.)*

(Barraquand and Corwin '15 got this formula first, by a more direct computation.)

KPZ behavior of averaged conditioned RWRE

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Theorem: *For* $\zeta \neq \xi$, $\exists C, c$: $\forall n \in \mathbb{N}$ *and* b *large,*

$$\mathbb{E}P^{\pi^\zeta}\{|X_n - n\zeta| \geq bn^{2/3}\} \leq Cb^{-3}$$

and

$$\mathbb{E}P^{\pi^\zeta}[|X_n - n\zeta|] \geq cn^{2/3}.$$

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Complements the aforementioned results saying KPZ fluctuations exponent (for $\log P^\omega(X_n \approx n\zeta)$*) is 1/3.*

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In both models, solvability comes from the Beta-Gamma algebra. Namely:

If A is Gamma$(a + b, c)$ and B is an independent Beta(a, b), then AB and $A(1 - B)$ are independent Gamma(a, c) and Gamma(b, c).

Existence of Busemann limit: coupling

For $\lambda > 0$ recall the system of edge variables U and V.

Denote them by U^λ and V^λ.

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So we can use these as transitions for the Beta RWRE.

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Given ζ*, let* $\lambda = \lambda(\zeta)$ *and define*

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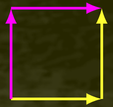

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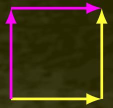

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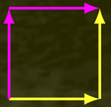

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Then define $B^{\zeta}(x, y) = B^{\zeta}(0, y) - B^{\zeta}(0, x)$ *and we have the cocycle property:* $B^{\zeta}(x, y) + B^{\zeta}(y, z) = B^{\zeta}(x, z)$*.*

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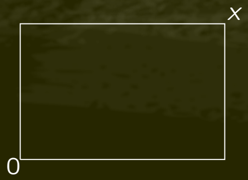
Take ζ, $\lambda = \lambda(\zeta)$, $x \in \mathbb{N}^2$ *and consider the rectangle with corners* 0 *and* x.



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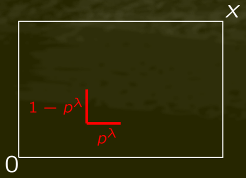

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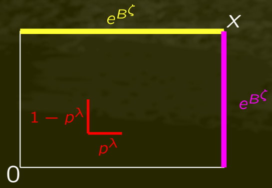

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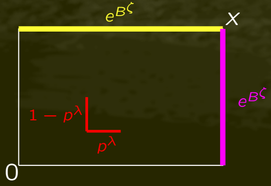

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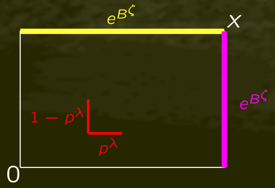

Note how path $x_{0,n}$ *accumulates a product of* p*'s and* $(1-p)$*'s,*
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I.e. $B^\zeta(0,x)$ *is almost the same as* $\log P^\omega(X_n = x \,|, X_0 = 0)$.

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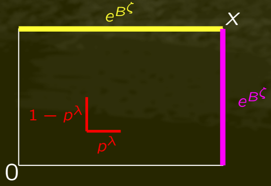

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Remark: *This connects the RWRE to a polymer with boundary conditions, which leads to the KPZ wandering exponent.*

Existence of Busemann limit: comparison lemma

By a monotonicity of $B^\zeta(0,x)$ *in the edge weights* σ *the above gives:*

Lemma: *With probability one, for* n *large and* $\eta' \cdot e_1 < \zeta \cdot e_1 < \eta \cdot e_1$

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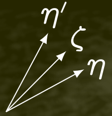

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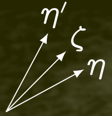

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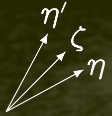

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Thank You!