Local Behavior of Airy Processes

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The Airy Process

It describes spatial fluctuations in a wide range of growth models, where each particular Airy process arising in each case depends on the geometry of the initial profile.

Last-Passage Percolation (Exponential)

It is an example of a growth model where the limit fluctuations are described by Airy processes.

Last-Passage Percolation

Point to point last-passage percolation time:

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- $\Pi^z(\mathbf{x}) = \{$ up-right paths from $(z, -z)$ to $\mathbf{x}$ $\}$;
- $z \in C^{\mathbf{x}} \mapsto L^z(\mathbf{x}) := \max_{\pi \in \Pi^z(\mathbf{x})} \sum_{\mathbf{y} \in \pi} \omega_{\mathbf{y}}$.

Last-Passage Percolation

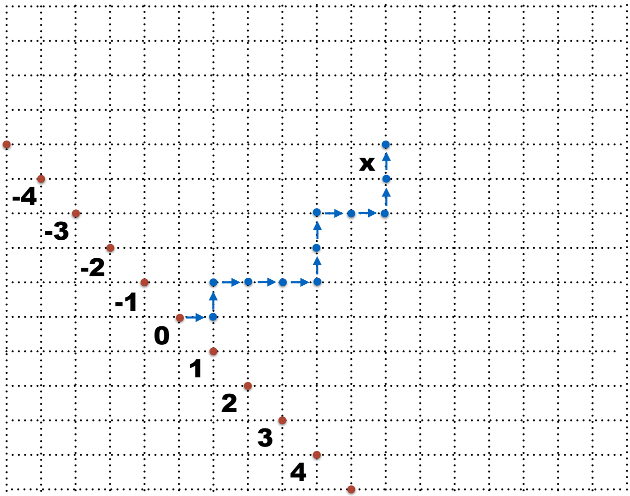

Figure: An up-right path from **0** to **x**.

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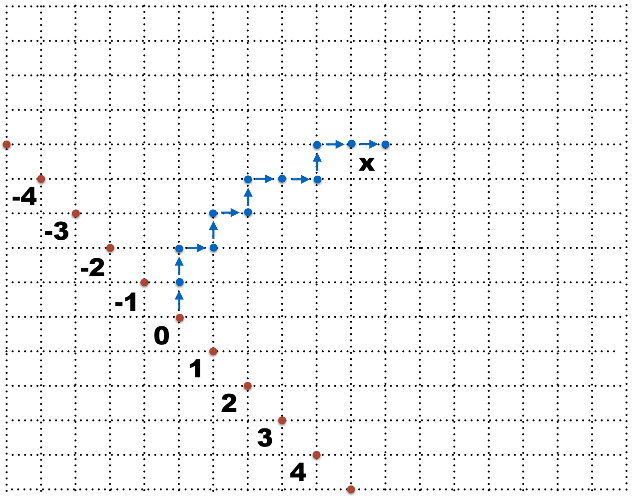

Figure: An up-right path from **0** to **x**

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Last-passage percolation time with an initial profile:

- $\mathrm{b} : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$, with $\mathrm{b}(0) = 0$;

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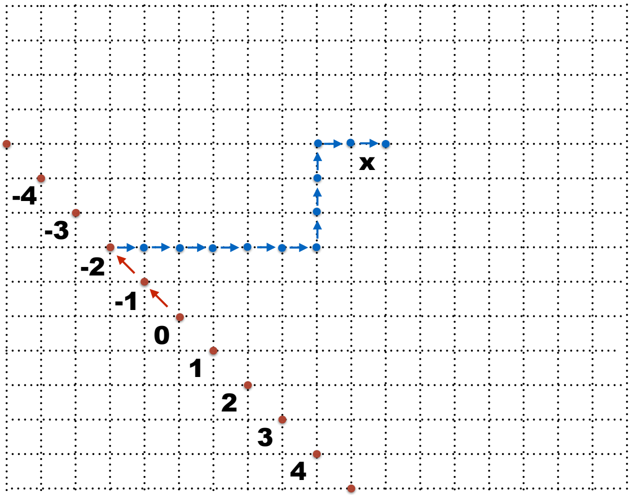

Figure: $L^{\mathrm{b}}(\mathbf{x})$

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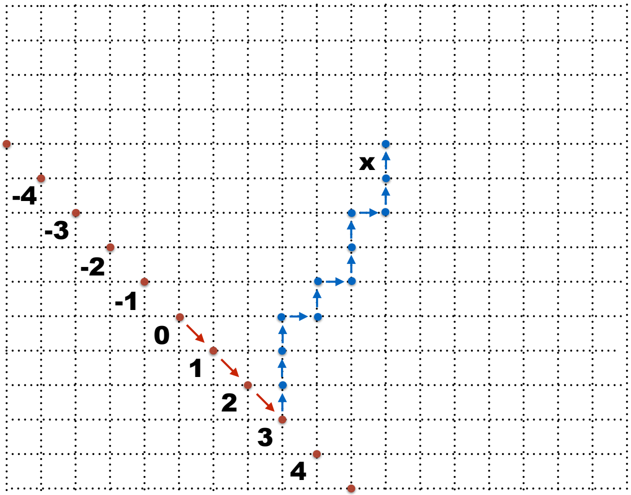

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Last-Passage Percolation

Narrow Wedge

Let

$$\mathrm{w}(k) = \begin{cases} 0 & \text{for } k = 0 \\ -\infty & \text{for } k \neq 0\,, \end{cases}$$

Then $L^{\mathrm{w}}(\mathbf{x}) = L^0(\mathbf{x})$.

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Flat

Let

$$\mathrm{f}(k) = 0\,,\ \forall\, k \in \mathbb{Z}\,.$$

Then $L^{\mathrm{f}}(\mathbf{x}) = \max_{z \in C^{\mathbf{x}}} L^z(\mathbf{x})$.

Last-Passage Percolation

Stationary

Let

$$s_\rho(k) = \begin{cases} \sum_{i=k+1}^{0} -\zeta_i & \text{for } k < 0\,, \\ 0 & \text{for } k = 0\,, \\ \sum_{i=1}^{k} \zeta_i & \text{for } k > 0\,, \end{cases}$$

where $\zeta_i = \zeta_i(\rho) \stackrel{dist.}{=} \mathrm{Exp}_1(1-\rho) - \mathrm{Exp}_2(\rho)$, and Exp_1 is independent of Exp_2.

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Mixed

Mixed profiles can be obtained by placing one condition on each half of $\mathbb{Z}$.

LPP Limit Fluctuations

For $u \in \mathbb{R}$

$$H_n^{\mathrm{b}}(u) = \frac{L^{\mathrm{b}}[2^{2/3}un^{2/3}]_n - 4n}{2^{4/3}n^{1/3}},$$

where $[x]_n \equiv (n + \lfloor x \rfloor, n - \lfloor x \rfloor)$.

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- $\lim_{n\to\infty} H_n^{\mathrm{w}}(u) \stackrel{dist.}{=} \mathcal{A}_2(u) - u^2$ (narrow wedge).
- $\lim_{n\to\infty} H_n^{\mathrm{f}}(u) \stackrel{dist.}{=} \mathcal{A}_1(u)$ (flat).
- $\lim_{n\to\infty} H_n^{\mathrm{s}_{1/2}}(u) \stackrel{dist.}{=} \mathcal{A}_0(u)$ (stationary).
- $\lim_{n\to\infty} H_n^{ij}(u) \stackrel{dist.}{=} \mathcal{A}_{ij}(u)$ (mixed).
- $\lim_{n\to\infty} H_n^{\mathrm{b}}(u) \stackrel{dist.}{=} \mathcal{A}_{\mathrm{b}}(u)$ (KPZ fixed point).

LPP Limit Fluctuations

Integrable Systems

The distribution of exponential LPP times can be written as Fredholm determinant of a certain operator (solvable model), and convergence to exact formulas can be obtained through asymptotic analysis.

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References

- Prähofer and Spohn '02
- Johansson '03
- Sasamoto '05
- Borodin, Ferrari, Prähofer and Sasamoto '07
- Baik, Ferrari and Péché '10
- Matetski, Quastel and Remenik '16

LPP Limit Fluctuations

Coupling Method

Construct joint realizations of the process starting from different profiles but with the same environment ω. Prove fluctuation results by comparing the evolution of the system with its equilibrium regime. It does not provide exact formulas for limiting distributions but it does provide critical exponents, functional tightness and asymptotic local fluctuations.

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References

- Fontes and Ferrari '94 (second-class particles)
- Cator and Groeneboom '06 (critical exponents)
- Balázs, Cator and Seppäläinen '06 (critical exponents)
- Cator and P. '15 (functional tightness and local convergence)

LPP Limit Fluctuations

Local Convergence

For a broad class of initial profiles b

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1/2} \left(A_{\mathrm{b}}(\epsilon x) - A_{\mathrm{b}}(0) \right) \overset{dist.}{=} \sqrt{2} B(x) \,,$$

where B is a standard Brownian Motion.

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References

- Häag '08
- Quastel and Remenik '13
- Corwin and Hammond '14
- Cator and P. '15
- Matetski, Quastel and Remenik '16

Local Convergence

Exit point

Define

$$Z^{\mathrm{b}}(\mathbf{x}) := \max \arg\max_{z \in C_{\mathbf{x}}} \left\{ \mathrm{b}(z) + L^{z}(\mathbf{x}) \right\} \ ,$$

so that $L^{\mathrm{b}}(\mathbf{x}) = \mathrm{b}\left(Z^{\mathrm{b}}(\mathbf{x})\right) + L^{Z^{\mathrm{b}}(\mathbf{x})}(\mathbf{x})$.

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Spatial fluctuations

Define

$$\Delta_n^{\mathrm{b}}(u) := \frac{L^{\mathrm{b}}[un^{2/3}]_n - L^{\mathrm{b}}[0]_n}{2^{3/2} n^{1/3}} \, , u \in [0, C] \, ,$$

so that

$$H_n^{\mathrm{b}}(u) = H_n^{\mathrm{b}}(0) + 2^{1/6} \Delta_n^{\mathrm{b}}(2^{2/3} u) \, .$$

Local Convergence

Theorem (arXiv:1704.01903)

Assume that

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|Z^{\mathrm{b}}[Cn^{2/3}]_n| \geq rn^{2/3}\right) = 0\,. \qquad (1)$$

Then $\left\{\Delta_n^{\mathrm{b}}\,:\, n \geq 1\right\}$ *is tight (cadlag Skorohod), and any weak limit* Δ^{b} *is continuous almost surely. Furthermore,*

$$\lim_{\epsilon\downarrow 0}\epsilon^{-1/2}\Delta^{\mathrm{b}}(\epsilon x) \stackrel{dist.}{=} B(x) \qquad (2)$$

(functional convergence), where $(B(x)\,,\, x\in\mathbb{R})$ *is a standard two-sided Brownian Motion.*

Coupling Method

Checking (1)

Notice that

$$\left\{ Z^{\mathrm{b}}[0]_n \geq u \right\} = \left\{ \exists\, z \geq u \,:\, \mathrm{b}(z) + L^{z}[0]_n = L^{\mathrm{b}}[0]_n \right\}$$

Coupling Method

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Since

$$L^{z}[0]_n \leq L^{\mathrm{s}_\rho}[0]_n - s_\rho(z) \;\text{ and }\; L[0]_n = L^{0}[0]_n \leq L^{\mathrm{b}}[0]_n\,,$$

we have that

$$\left\{ Z^{\mathrm{b}}[0]_n \geq u \right\} \subseteq \left\{ \exists\, z \geq u \;:\; \mathrm{s}_\rho(z) - \mathrm{b}(z) \leq L^{\mathrm{s}_\rho}[0]_n - L^{0}[0]_n \right\},$$

Coupling Method

Checking (1)

Thus,

$$\mathbb{P}\left(Z^{\mathrm{b}}[0]_n \geq u\right) \leq \mathbb{P}\big(\mathrm{a}^{\mathrm{b}}_{\rho}(u) \leq L^{\mathrm{s}_\rho}[0]_n - L^0[0]_n\big)\,,$$

where

$$\mathrm{a}^{\mathrm{b}}_{\rho}(u) := \mathrm{m}^{\mathrm{b}}_{\rho}(u) + (\mathrm{s}_\rho, \mathrm{b})(u)\,,$$

$$(\mathrm{s}_\rho, \mathrm{b})(u) := \mathrm{s}_\rho(u) - \mathrm{b}(u)\,,$$

and

$$\mathrm{m}^{\mathrm{b}}_{\rho}(u) := \min\left\{(\mathrm{s}_\rho, \mathrm{b})(z) - (\mathrm{s}_\rho, \mathrm{b})(u)\ : z \geq u\right\}.$$

Coupling Method

Checking (1)

$$\begin{aligned}
\mathbb{P}\left(Z^{\mathrm{b}}[0]_n \geq u\right) &\leq \mathbb{P}\big(\mathrm{a}^{\mathrm{b}}_{\rho}(u) \leq L^{\mathrm{s}_\rho}[0]_n - L^0[0]_n\big) \\
&\leq \mathbb{P}\left(L^0[0]_n - 4n \leq -\frac{r^2 n^{1/3}}{c_1}\right) \\
&+ \mathbb{P}\left(\mathrm{a}^{\mathrm{b}}_{\rho}(u) \leq L^{\mathrm{s}_\rho}[0]_n - 4n + \frac{r^2 n^{1/3}}{c_1}\right)
\end{aligned}$$

Coupling Method

Checking (1)

$$\begin{aligned} \mathbb{P}(\cdots) &\leq \mathbb{P}\left(L^{s_\rho}[0]_n - 4n \geq \mathbb{E}a_\rho^b(u) - \frac{2r^2 n^{1/3}}{c_1}\right) \\ &+ \mathbb{P}\left(a_\rho^b(u) \leq \mathbb{E}a_\rho^b(u) - \frac{r^2 n^{1/3}}{c_1}\right), \end{aligned}$$

Coupling Method

Checking (1)

$$\mathbb{P}(\cdots) \leq \mathbb{P}\left(L^{s_\rho}[0]_n - \mathbb{E}L^{s_\rho}[0]_n \geq \Lambda - \frac{2r^2 n^{1/3}}{c_1}\right) + \mathbb{P}\left(\mathrm{a}^{\mathrm{b}}_\rho(u) \leq \mathbb{E}\mathrm{a}^{\mathrm{b}}_\rho(u) - \frac{r^2 n^{1/3}}{c_1}\right),$$

where

$$\Lambda(\rho, u, n) := 4n - \mathbb{E}L^{s_\rho}[0]_n + \mathbb{E}\mathrm{a}^{\mathrm{b}}_\rho(u).$$

Coupling Method

Checking (1)

General strategy: for n and $u = rn^{2/3}$ fixed, choose $\rho(u, n)$ to produce

$$\Lambda - \frac{2r^2 n^{1/3}}{c_1} \geq c_2 r^2 n^{1/3} .$$

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Apply Chebyshev's inequality

$$\mathbb{P}(\cdots) \leq \frac{\mathrm{Var}\, L^{s_\rho}[0]_n}{c_2^2 r^4 n^{2/3}} + c_1^2 \frac{\mathrm{Var}\, \mathrm{a}_\rho^{\mathrm{b}}(u)}{r^4 n^{2/3}} .$$

Coupling Method

Flat Profile (1)

For $\mathrm{b} = \mathrm{f} \equiv 0$,

$$(\mathrm{s}_\rho, \mathrm{b})(u) := \mathrm{s}_\rho(u)\,,$$

and

$$\mathrm{m}^0_\rho(u) := \min\left\{\mathrm{s}_\rho(z) - \mathrm{s}_\rho(u)\ :\ z \geq u\right\}\ .$$

$$\begin{aligned}
\Lambda &= 4n - \mathbb{E}L^{\mathrm{s}_\rho}[0]_n + \mathbb{E}\mathrm{a}^0_\rho(u) \\
&= \left(4 - \frac{1}{\rho(1-\rho)}\right) n + \frac{2\rho - 1}{\rho(1-\rho)} u + \mathbb{E}\mathrm{m}^0_\rho(u) \\
&= \frac{(4\rho(1-\rho) - 1)n + (2\rho - 1)u}{\rho(1-\rho)} + \mathbb{E}\mathrm{m}^0_\rho(u) \\
&\geq \frac{(4\rho(1-\rho) - 1)n + (2\rho - 1)u}{4} + \mathbb{E}\mathrm{m}^0_\rho(u)\,.
\end{aligned}$$

Coupling Method

Flat Profile (1)

Take

$$\rho(u,n) := \frac{1}{2} + \frac{u}{4n},$$

to get

$$\begin{aligned}
\Lambda &\geq \frac{(4\rho(1-\rho)-1)n + (2\rho-1)u}{4} + \mathbb{E}\mathrm{m}_{\rho}^{0}(u) \\
&= \frac{r^2}{16} n^{1/3} - \frac{1-\rho}{\rho(2\rho-1)} \\
&\geq \frac{r^2}{16} n^{1/3} - \frac{4}{r} n^{1/3}.
\end{aligned}$$

Coupling Method

Flat Profile (1)

Use that,

$$\limsup_{n\to\infty} \frac{\mathrm{Var} L^{\mathrm{s}_\rho}[0]_n}{n^{2/3}} \leq \limsup_{n\to\infty} \frac{\mathrm{Var} L^{\mathrm{s}_{1/2}}[0]_n}{n^{2/3}} + 4r \leq c_3 + 4r\,,$$

and that

$$\limsup_{n\to\infty} \frac{\mathrm{Var}\, \mathrm{a}^0_\rho(u)}{n^{2/3}} \leq \limsup_{n\to\infty} \frac{c_4}{n^{2/3}(2\rho-1)^2} \leq \frac{16c_4}{r^2}\,.$$

Proof of the Theorem

Lemma (Local Comparison)

Take a joint realization $(L_\omega^{\mathrm{b}_1}, L_\omega^{\mathrm{b}_2})$*. Let* $k \leq l$ *and* $n \geq 1$*. If*

$$Z^{\mathrm{b}_1}[l]_n \leq Z^{\mathrm{b}_2}[k]_n$$

then

$$L^{\mathrm{b}_1}[l]_n - L^{\mathrm{b}_1}[k]_n \leq L^{\mathrm{b}_2}[l]_n - L^{\mathrm{b}_2}[k]_n \,.$$

Proof of the Theorem

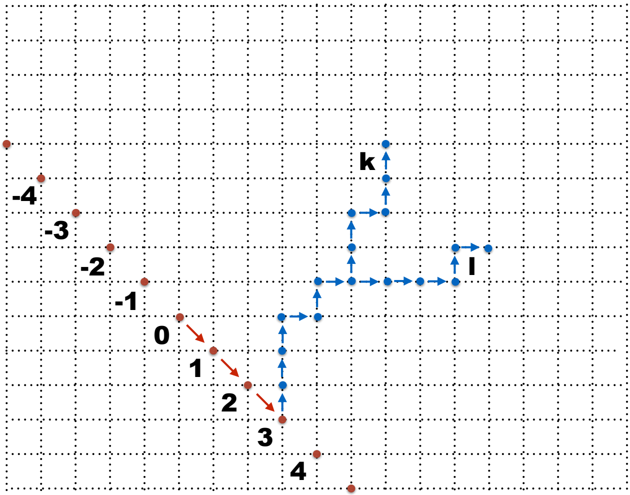

Figure: $Z^{b_2}[k]_n = 3$

Proof of the Theorem

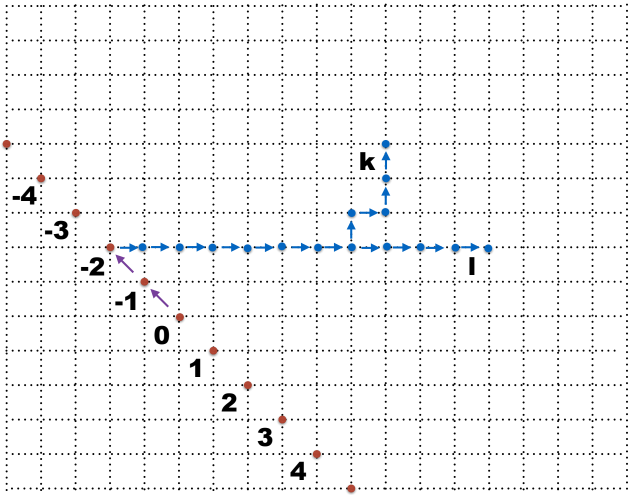

Figure: $Z^{b_1}[l]_n$=-2

Proof of the Theorem

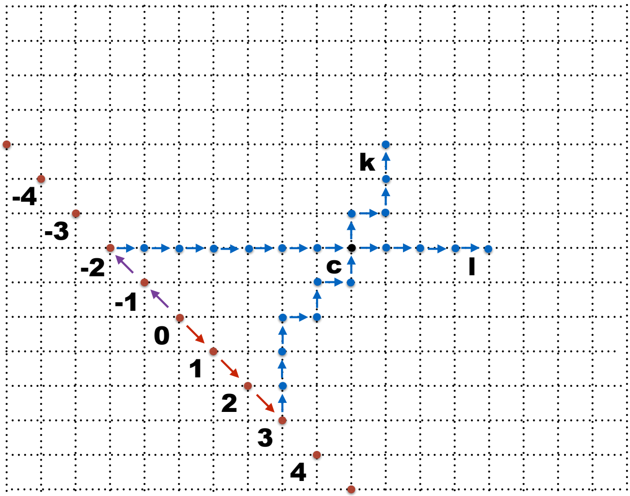

Figure: Geodesic Crossing **c**

Proof of the Theorem

Proof of Local Comparison

Set $z_2 = Z^{\mathrm{b}_2}[k]_n$. By superadditivity,

$$L^{\mathrm{b}_2}[l]_n \geq \mathrm{b}_2(z_2) + L^{z_2}[l]_n \geq \mathrm{b}_2(z_2) + L^{z_2}(\mathbf{c}) + L(\mathbf{c}, [l]_n)\,.$$

Since

$$\mathrm{b}_2(z_2) + L^{z_2}(\mathbf{c}) - L^{\mathrm{b}_2}[k]_n = -L(\mathbf{c}, [k]_n)\,,$$

thus

$$\begin{aligned} L^{\mathrm{b}_2}[l]_n - L^{\mathrm{b}_2}[k]_n &\geq \mathrm{b}_2(z_2) + L^{z_2}(\mathbf{c}) + L(\mathbf{c}, [l]_n) - L^{\mathrm{b}_2}[k]_n \\ &= L(\mathbf{c}, [l]_n) - L(\mathbf{c}, [k]_n)\,. \end{aligned}$$

Proof of the Theorem

Proof of Local Comparison

By superaddivity,

$$-L(\mathbf{c}\,,\,[k]_n) \geq L^{\mathrm{b}_1}(\mathbf{c}) - L^{\mathrm{b}_1}[k]_n\,,$$

and hence

$$\begin{aligned} L^{\mathrm{b}_2}[l]_n - L^{\mathrm{b}_2}[k]_n &\geq L(\mathbf{c},[l]_n) - L(\mathbf{c},[k]_n) \\ &\geq L(\mathbf{c},[l]_n) + L^{\mathrm{b}_1}(\mathbf{c}) - L^{\mathrm{b}_1}[k]_n \\ &= L^{\mathrm{b}_1}[l]_n - L^{\mathrm{b}_1}[k]_n\,. \end{aligned}$$

Proof of the Theorem

Local Equilibrium Sandwich

Choose

$$\rho_n^{\pm} = \frac{1}{2} \pm \frac{r}{n^{1/3}},$$

to have that

$$Z^{\mathrm{s}_{\rho_n^-}}[Cn^{2/3}]_n \leq Z^{\mathrm{b}}[0]_n \text{ and } Z^{\mathrm{b}}[Cn^{2/3}]_n \leq Z^{\mathrm{s}_{\rho_n^+}}[0]_n, \qquad (3)$$

with high probability:

$$R_1(r) := \limsup_{n\to\infty} \mathbb{P}\left(\{(3)\}^c\right) \to 0, \text{ as } r \to \infty.$$

Proof of the Theorem

Local Equilibrium Sandwich

Let

$$B_n^{\pm}(u) := \frac{\left(L^{\rho_n^{\pm}}[un^{2/3}]_n - L^{\rho_n^{\pm}}[0]_n\right) - m_{\rho_n^{\pm}} un^{2/3}}{2^{3/2} n^{1/3}} .$$

On the event (3),

$$B_n^{-}(v) - B_n^{-}(u) - 3\sqrt{2}(v-u)r \leq \Delta_n^{\mathrm{b}}(v) - \Delta_n^{\mathrm{b}}(u) ,$$

and

$$\Delta_n^{\mathrm{b}}(v) - \Delta_n^{\mathrm{b}}(u) \leq B_n^{+}(v) - B_n^{+}(u) + 3\sqrt{2}(v-u)r ,$$

for all $u, v \in [0, C]$.

Proof of the Theorem

Local Equilibrium Sandwich

On the event (3),

$$W_{\Delta_n^b}(\delta) \leq \max\left\{ W_{B_n^-}(\delta)\,,\ W_{B_n^+}(\delta) \right\} + 3\sqrt{2}\delta r\,,$$

where

$$W_X(\delta) := \sup_{u,v\in[0,C],\,|u-v|\leq\delta} |X(u) - X(v)|\,.$$

Proof of the Theorem

Local Equilibrium Sandwich

Fix $\beta \in (0,1)$ and set

$$r = r_\delta := \delta^{-\beta} \to \infty\,, \text{ as } \delta \downarrow 0\,,$$

then

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \mathbb{P}\left(W_{\Delta_n^b}(\delta) > \eta \right) = 0\,,$$

which implies tightness, and continuity of weak limits.

Proof of the Theorem

Local Limit (2)

Set $X^\epsilon(x) := \epsilon^{-1/2}X(\epsilon x)$. On the event (3),

$$B_n^{-,\epsilon}(v) - B_n^{-,\epsilon}(u) - 3\sqrt{2}(v-u)\epsilon^{1/2}r \leq \Delta_n^{\mathrm{b},\epsilon}(v) - \Delta_n^{\mathrm{b},\epsilon}(u)\,,$$

and

$$\Delta_n^{\mathrm{b},\epsilon}(v) - \Delta_n^{\mathrm{b},\epsilon}(u) \leq B_n^{+,\epsilon}(v) - B_n^{+,\epsilon}(u) + 3\sqrt{2}(v-u)\epsilon^{1/2}r\,.$$

for all $u, v \in [0, C]$.

Proof of the Theorem

Local Limit (2)

Then

$$\mathbb{P}\left(W_{\Delta^{\mathrm{b},\epsilon}}(\delta) > \eta\right) \leq 2\mathbb{P}\left(W_B(\delta) > \eta - 3\sqrt{2}\delta\epsilon^{1/2}r\right) + R_1(r)\,.$$

If we now set $r_\epsilon := \epsilon^{-\beta}$, with $\beta \in (0, 1/2)$, then

$$\limsup_{\epsilon\downarrow 0} \mathbb{P}\left(W_{\Delta^{\mathrm{b},\epsilon}}(\delta) > \eta\right) \leq 2\mathbb{P}\left(W_B(\delta) > \eta\right) \ ,$$

and hence

$$\lim_{\delta\downarrow 0} \limsup_{\epsilon\downarrow 0} \mathbb{P}\left(W_{\Delta^{\mathrm{b},\epsilon}}(\delta) > \eta\right) = 0\,,$$

which implies tightness of $\Delta^{\mathrm{b},\epsilon}$.

Proof of the Theorem

Local Limit (2)

For finite dimensional convergence,

$$\mathbb{P}\left(\cap_{i=1}^{j}\left\{\Delta^{\mathrm{b},\epsilon}(u_i) \leq x_i\right\}\right) \leq \mathbb{P}\left(\cap_{i=1}^{j}\left\{B(u_i) \leq x_i + 3\sqrt{2}\epsilon^{1/2-\beta}\right\}\right) + R_1(r_\epsilon),$$

and

$$\mathbb{P}\left(\cap_{i=1}^{j}\left\{\Delta^{\mathrm{b},\epsilon}(u_i) \leq x_i\right\}\right) \geq \mathbb{P}\left(\cap_{i=1}^{j}\left\{B(u_i) \leq x_i - 3\sqrt{2}\epsilon^{1/2-\beta}\right\}\right) + R_1(r_\epsilon).$$

Other results (in preparation)

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- Time ergodicity of space increments of the KPZ fixed point.
- Decorrelation for large times of space increments;
- An Airy sheet is locally an additive Brownian motion;

Ergodicity of space increments of the KPZ fixed point

Let

$$\Delta_n^{\mathrm{b}}(u,t) := \frac{L^{\mathrm{b}}[un^{2/3}]_{nt} - L^{\mathrm{b}}[0]_{nt}}{2^{3/2}n^{1/3}}\,, u \in [0,C]\,,$$

so that

$$H_n^{\mathrm{b}}(u,t) = H_n^{\mathrm{b}}(0) + 2^{1/6}\Delta_n^{\mathrm{b}}(2^{2/3}u,t)\,.$$

Recall that

$$\Delta^{1/2}(\cdot,t) \overset{dist.}{=} \lim_{n\to\infty} \Delta_n^{1/2}(\cdot,t)\,,$$

is a Brownian Motion for all $t \geq 0$.

Ergodicity of space increments of the KPZ fixed point

Theorem

Denote

$$d_C(f,g) := \sup_{x\in[0,C]} |f(x)-g(x)|\,.$$

Consider a joint weak limit $(\Delta^{\mathrm{b}}(\cdot,t),\Delta^{1/2}(\cdot,t))$ of the last-passage percolation model. Then,

$$\lim_{t\to\infty} \mathbb{P}\left(d_C(\Delta^{1/2}(\cdot,t),\Delta^{\mathrm{b}}(\cdot,t)) > \eta\right) = 0\,,$$

for all $\eta > 0$.

Proof of Ergodicity

Lemma (Attractiveness)

Assume that $\mathrm{b}_1(l) - \mathrm{b}_1(k) \leq \mathrm{b}_2(l) - \mathrm{b}_2(k)$ for all $k < l$. Then

$$L^{\mathrm{b}_1}[l]_n - L^{\mathrm{b}_1}[k]_n \leq L^{\mathrm{b}_2}[l]_n - L^{\mathrm{b}_2}[k]_n \,,$$

for all $k < l$.

Proof of Ergodicity

Proof

Denote

$$z_1 := Z^{b_1}[l]_n \quad \text{and} \quad z_2 := Z^{b_2}[k]_n \,.$$

If $z_1 \leq z_2$ then it follows by local comparison.

Proof of Ergodicity

Proof

Denote

$$z_1 := Z^{\mathrm{b}_1}[l]_n \text{ and } z_2 := Z^{\mathrm{b}_2}[k]_n \,.$$

If $z_1 \leq z_2$ then it follows by local comparison.

Now, $L^{\mathrm{b}_2}[l]_n - L^{\mathrm{b}_2}[k]_n - \left(L^{\mathrm{b}_1}[l]_n - L^{\mathrm{b}_1}[k]_n\right)$ equals

$$L^{\mathrm{b}_2}[l]_n - \left(\mathrm{b}_2(z_1) + L_{z_1}[l]_n\right) \quad + \quad \Big(L^{\mathrm{b}_1}[k]_n - \left(\mathrm{b}_1(z_2) + L_{z_2}[k]_n\right)\Big)$$
$$+ \left(\mathrm{b}_2(z_1) - \mathrm{b}_2(z_2)\right) \quad - \quad \left(\mathrm{b}_1(z_1) - \mathrm{b}_1(z_2)\right) \geq 0 \,,$$

if $z_1 > z_2$. (Use super-additivity for the first two terms, and the assumption for hte third one.)

Proof of Ergodicity

Set $\delta_t := Ct^{-2/3}$ and

$$\rho^{\pm}_{n,t} := \frac{1}{2} \pm \frac{\delta_t^{-\alpha}}{n^{1/3}} \,,$$

where $\alpha \in (0, 1/2)$. Let $E(n,t)$ denote the event

$$\left\{ Z^{\rho^{+}_{n,t}}[0]_{\lfloor tn \rfloor} \geq Z^{\mathrm{b}}[Cn^{2/3}]_{\lfloor tn \rfloor} \text{ and } Z^{\rho^{-}_{n,t}}[Cn^{2/3}]_{\lfloor tn \rfloor} \leq Z^{\mathrm{b}}[0]_{\lfloor tn \rfloor} \right\} ,$$

and write $R(t) := \limsup_{n\to\infty} \mathbb{P}\left(E(n,t)^c\right)$. Then, under Assumption 1,

$$\lim_{t\to\infty} R(t) = 0 \,.$$

Proof of Ergodicity

We couple the stationary regimes in such way that initially they are ordered. One can do it by setting

$$\zeta_i(\rho) = \frac{1}{2(1-\rho)}\mathrm{Exp}_{1,i}(1/2) - \frac{1}{2\rho}\mathrm{Exp}_{2,i}(1/2)\,.$$

Thus, by attractiveness,

$$\Delta_n^{\rho_{n,t}^-}(u,t) \le \Delta_n^{1/2}(u,t) \le \Delta_n^{\rho_{n,t}^+}(u,t)\,.$$

for all $u \in [0, C]$.

Proof of Ergodicity

By local comparison, on the event $E(n, t)$,

$$\Delta_n^{\rho_{n,t}^-}(u, t) \leq \Delta_n^{\mathrm{b}}(u, t) \leq \Delta_n^{\rho_{n,t}^+}(u, t)\,,$$

for all $u \in [0, C]$, and hence, on the event $E(n, t)$,

$$|\Delta_n^{1/2}(u, t) - \Delta_n^{\mathrm{b}}(u, t)| \leq \Delta^{\rho_{n,t}^+}(u, t) - \Delta^{\rho_{n,t}^-}(u, t)\,,$$

for all $u \in [0, C]$.

Proof of Ergodicity

Attractiveness implies that

$$0 \leq \Delta^{\rho^+_{n,t}}(u,t) - \Delta^{\rho^-_{n,t}}(u,t) \leq \Delta^{\rho^+_{n,t}}(C,t) - \Delta^{\rho^-_{n,t}}(C,t)\,,$$

which yields to

$$\sup_{u \in [0,C]} |\Delta_n^{1/2}(u,t) - \Delta_n^{\mathrm{b}}(u,t)| \leq \Delta^{\rho^+_{n,t}}(C,t) - \Delta^{\rho^-_{n,t}}(C,t)\,.$$

Proof of Ergodicity

Since $Cn^{2/3} = \delta_t (tn)^{2/3}$

$$\mathbb{E}\left(\Delta^{\rho^+_{n,t}}(C,t) - \Delta^{\rho^-_{n,t}}(C,t)\right) \le c_1 C^{1/2} \delta_t^{1/2-\alpha} ,$$

we finally get that,

$$\limsup_{n\to\infty} \mathbb{P}\left(d_C(\Delta_n^{1/2}(\cdot,t), \Delta_n^{\mathrm{b}}(\cdot,t)) > \eta\right) \le R(t) + \frac{c_1 C^{1/2} \delta_t^{1/2-\alpha}}{\eta} .$$

Recall that

$$\lim_{t\to\infty} R(t) = 0 \text{ and } \delta_t = Ct^{-2/3} .$$