Introductory school at CIRM, Jan 9-13, 2017

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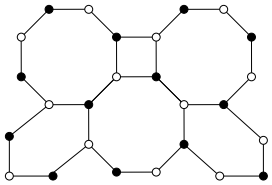
## Exercises

1. *Kasteleyn's theorem.*

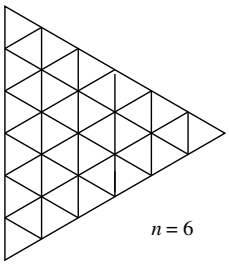
   (a) Find a Kasteleyn orientation of the following graph.



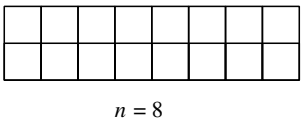
   (b) Prove the following result of Kasteleyn: *Let* $\mathsf{G}$ *be a finite, planar graph. Suppose that edges are oriented so that all inner faces are cw odd. Then, each cycle containing in its interior an even (resp. odd) number of vertices is cw odd (resp. even).*

2. *Counting tilings.*

   (a) Count the number of tilings of an equilateral triangle of side $n$.




   (b) Count the number of tilings of a rectangle of side $2 \times n$.

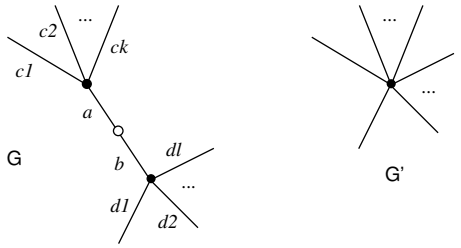



3. *Dimer surgery.*

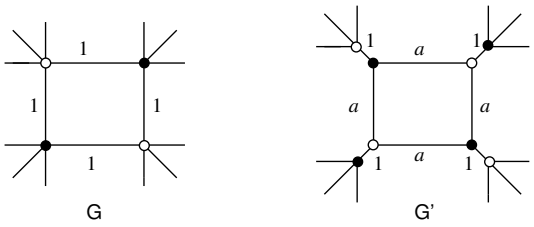
   If a graph has a degree 2 vertex, consider the graph $\mathsf{G}'$ obtained by merging this vertex with its two neighbors.

   (a) Exhibit a bijection between dimer configurations of $\mathsf{G}$ and $\mathsf{G}'$.

   (b) Find weights on $\mathsf{G}'$ such that this bijection is weight preserving.




Let $\mathsf{G}$ and $\mathsf{G}'$ be two graphs differing by an "urban renewal":

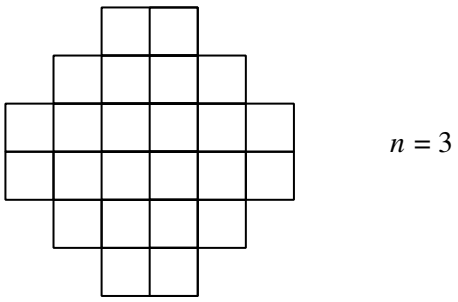



(a) Exhibit a local mapping between dimer configurations of $\mathsf{G}$ and $\mathsf{G}'$.

(b) Find $a$ and $b$ such that $Z(\mathsf{G}) = bZ(\mathsf{G}')$.

4. *Number of tilings of the Aztec diamond.*

Le $Z_n$ be the number of tilings of the Aztec diamond of size $n$.




Using "fusion" and "urban renewal", determine a relation between $Z_n$ and $Z_{n-1}$, and then compute $Z_n$ as a function of $n$.

5. Using the identity

$$\mathrm{cn}(u+v)\mathrm{cn}(u) = \mathrm{cn}(v) - \mathrm{sn}(u+v)\mathrm{sn}(u)\mathrm{dn}(v)$$

, prove that, for every $u \in \mathbb{T}(k)$, for every $\alpha, \beta \in \mathbb{R}$ such that $\beta - \alpha = \theta$:

$$\mathrm{cn}\left(\frac{u-\beta}{2}\right) - \mathrm{cn}\left(\frac{u-\alpha}{2}\right) = \frac{\mathrm{sn}\,\theta}{1+\mathrm{cn}\,\theta}\left[\mathrm{sn}\left(\frac{u-\alpha}{2}\right)\mathrm{dn}\left(\frac{u-\beta}{2}\right) - \mathrm{sn}\left(\frac{u-\beta}{2}\right)\mathrm{dn}\left(\frac{u-\alpha}{2}\right)\right].$$

*(The* cn *and* dn *functions are even, the* sn *function is odd).*