Universality for fluctuation of the dimer model

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Joint work with Nathanael Berestycki and Benoit Laslier

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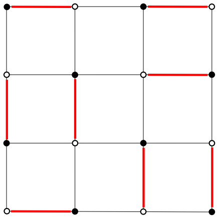
Dimer model

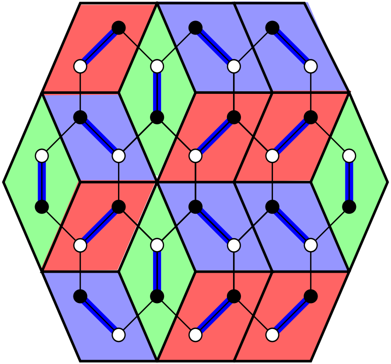
G : Planar bipartite graph (vertices can be colored black or white such that always black is adjacent to white)

Definition

A dimer configuration: perfect matching of a bipartite graph (every black vertex is connected to exactly one white vertex via an edge or dimer).

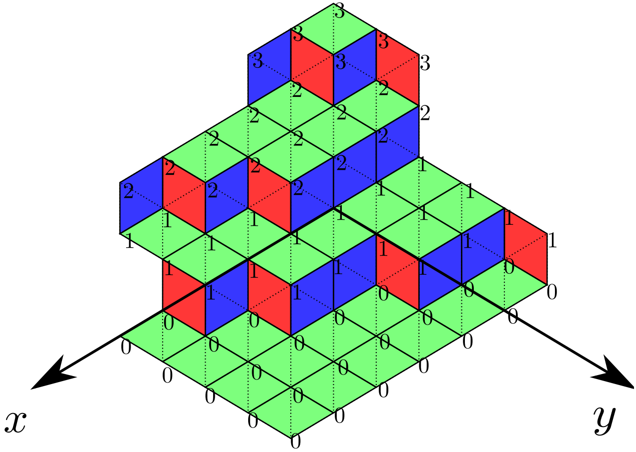


Dimer on honeycomb/ hexagonal lattice



- Describes a surface in $\mathbb{R}^3$.
- Boundary describes a curve in $\mathbb{R}^3$.

Height function



- Can be described for general planar bipartite graphs.
- Describes a surface in $\mathbb{R}^3$.

Dimer

Definition

Given a domain, pick a dimer configuration uniformly. This describes a random surface.

Question

- *"typical (mean) surface"?*
- *Fluctuations around typical surface (universality)?*

Random surfaces from dimer

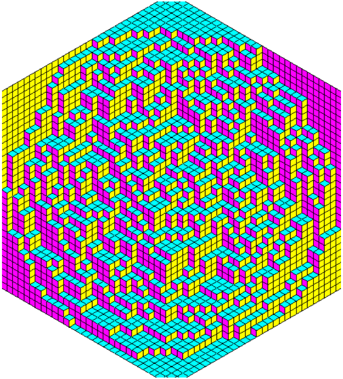


Figure: Arctic circle phenomenon ©Rick Kenyon

Random surfaces from dimer

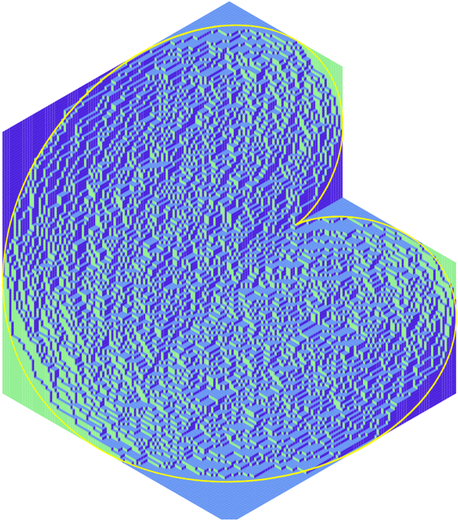


Figure: A cardioid shape $r(\theta) = 2(1 + \cos(\theta))$. (Fig: Okounkov.)

Continuum GFF (2 D): the universal fluctuation field

A "random function" $(h_x)_{x\in D}$

- Marginals $h_x \sim$ Gaussian.
- $Cov(h_x, h_y) \approx c \log|x-y|$.

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- $Cov(h_x, h_y) \approx c \log |x - y|$.

Definition (Gaussian free field)

$D \subset \mathbb{C}$.

- $h \equiv 0$ on ∂D.
- $(h, f) \sim N(0, \int G(x, y) f(x) f(y))$ for all test functions f.
  $G(x, y) \approx \log |x - y|$ as $x \to y$.

Can make sense as a **random distribution**. GFF is **conformaly invariant** in law.

GFF

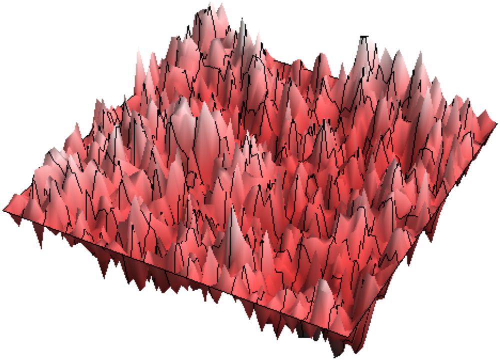


Figure: ©Scott Sheffield

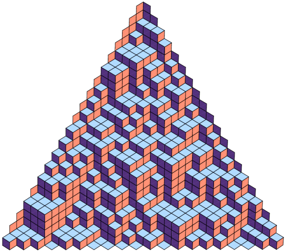
History

- Cohn, Larsen and Propp(NYJM '98) and Cohn, Kenyon and Propp (JAMS '01) the shape of a typical surface for general boundary condition.
- Kenyon (Ann. Probab. '00) Fluctuations of height function of domino tilings.
- Kenyon, Okounkov, Sheffield (Ann. Math '06) Surface tension and local Gibb's properties for periodic graphs.
- Petrov (Ann. Probab. '15)Fluctuations of lozenge tilings of Polygons.
- Li (AIHP '13) Fluctuations for Temperleyan version of isoradial graphs.

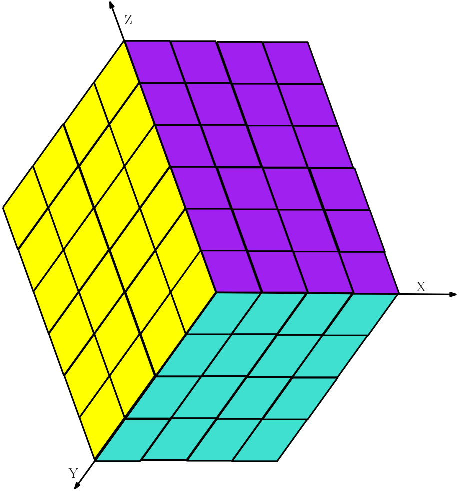
Our result setup: planar boundary

Definition

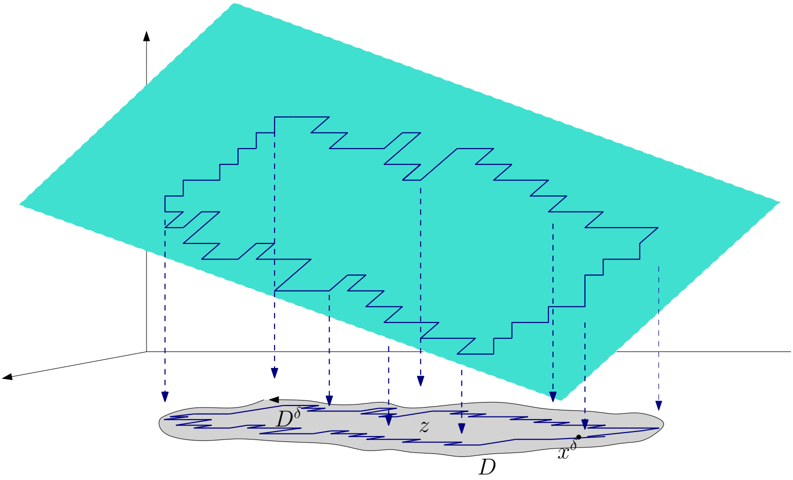
Planar boundary condition: Boundary curve in $\mathbb{R}^3$ lies within bounded distance from a plane.



A non planar boundary

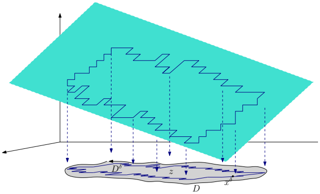


Setup



- $h^\delta(z)$ height with cube size δ.

Main theorem



- Honeycomb lattice.
- $h^\delta(z)$ height with cube size δ.

Theorem (Berestycki, Laslier, R.'2015)

For all slopes, for all $D \subset \mathbb{C}$, $\exists\ D^\delta$ (approximating D) such that

$$\frac{1}{\delta}\Big(h^\delta \circ \ell(\cdot) - \mathbb{E}(h^\delta \circ \ell(\cdot))\Big) \xrightarrow[\delta \to 0]{} \sqrt{2}GFF$$

ℓ : an explicit linear map depending only on slope.

Our work...

- Purely probabilistic viewpoint (without Kasteleyn matrices).

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- Robustness: Recovers and extends the work of Kenyon,Li.

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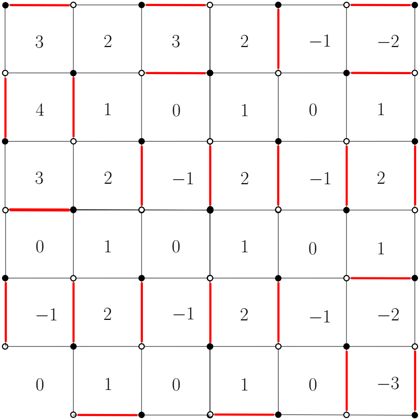
- Purely probabilistic viewpoint (without Kasteleyn matrices).
- Universality: use only CLT of random walk on certain graphs.
- Robustness: Recovers and extends the work of Kenyon,Li.
- Possible future applications: general topology, non-planar boundary, interacting dimers...

Dimers $\leftrightarrow$ UST: Temperley's bijection



A tool

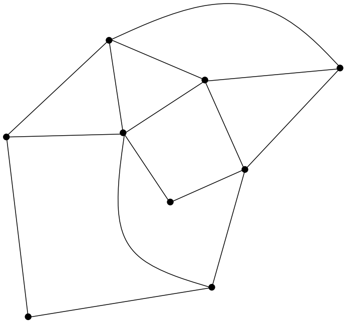
Uniform spanning tree in a certain graph $\xleftrightarrow{\text{winding}}$ Dimer, height function

Uniform spanning trees

Definition

A spanning tree of a graph G is its subgraph with no cycles and spanning all the vertices of G.

- A graph G.

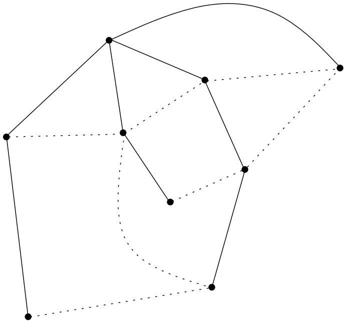


Uniform spanning trees

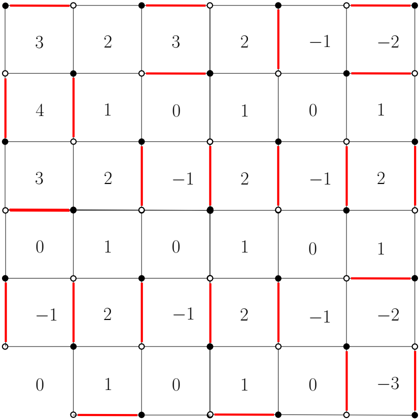
Definition

A spanning tree of a graph G is its subgraph with no cycles and spanning all the vertices of G.

- A graph G.
- A spanning tree of G.



Dimers $\leftrightarrow$ UST: Temperley's bijection



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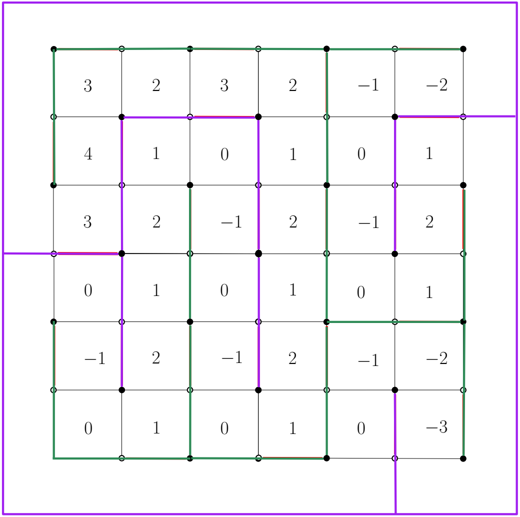

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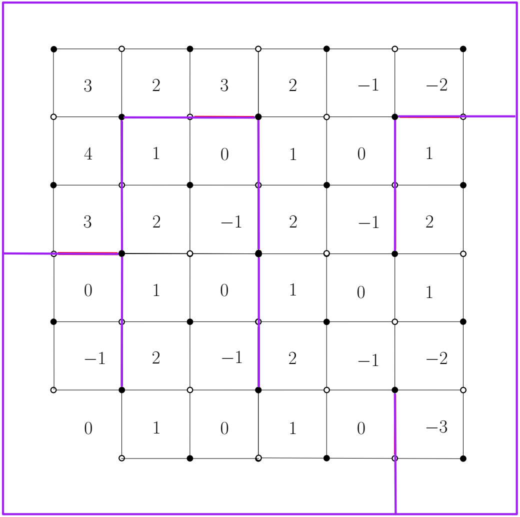
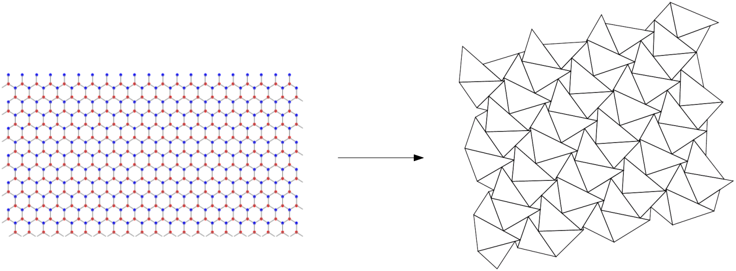


Such a connection also holds for hexagonal lattice and an associated graph called a T-graph. (Kenyon and Sheffield)



Height function of hexagonal lattice = winding of uniform spanning tree on T-graph.

Main idea

- Winding of uniform spanning tree (UST) branches $=$ height function.

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- Winding of uniform spanning tree (UST) branches = height function.
- Need to study winding field of UST and their scaling limits.

> Since dimers on $\mathcal{G}$ correspond to Uniform Spanning Trees (USTs) on Γ, and USTs can be generated from LERWs ([33]), this suggests universality of dimers when the underlying RW converges to BM. Remark also that the dimer height can be expressed in terms of windings of LERWs ([22]). However at this stage it is unclear how to implement rigorously this heuristic.

–Dubedat,Gheissari '14.

Our theorem...

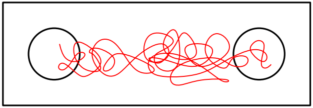
Theorem (Berestycki, Laslier, R.' 2016)

$D \subset \mathbb{C}$ a domain with locally connected boundary. $D^\delta \subset G^\delta$ approximating D. h^δ : winding of UST from a marked point $x^\delta \in \partial D^\delta$.

$$h^\delta(\cdot) - \mathbb{E}(h^\delta(\cdot)) \to \sqrt{2} GFF$$

with Dirichlet boundary condition **if G satisfies the following conditions.**

- Simple random walk satisfies a CLT. (CLT holds for T graphs (Laslier' 14)).
- Uniform crossing condition:



This holds for T graphs (not obvious!).

Scaling limits

Theorem (Lawler, Schramm, Werner '03, Schramm '00)

$D \subset \mathbb{C}$

- *Uniform spanning tree on* $D \cap \delta\mathbb{Z}^2 \rightarrow$ *"A continuum tree" (continuum uniform spanning tree).*

Continuum UST

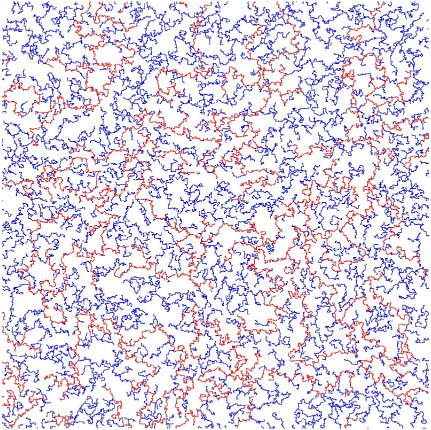


Figure: UST in a square with 1000 branches

Scaling limits

Theorem (Lawler, Schramm, Werner '03, Schramm '00)

$D \subset \mathbb{C}$

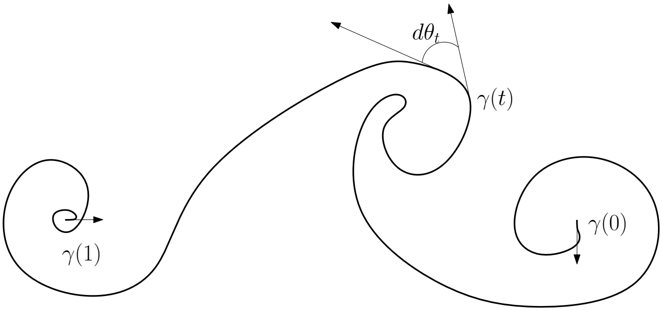
- *Uniform spanning tree on* $D \cap \delta\mathbb{Z}^2 \to$ *"A continuum tree" (continuum uniform spanning tree).*
- *Branches of the continuum uniform spanning tree are* SLE_2 *curves.*

SLE

Theorem (Schramm)

SLE_κ is the only one parameter family of random curves satisfying Domain Markov property + Conformal invariance.

Winding on rough curves?



- $\theta_t = \arg(\gamma'(t))$ taken continuously.
- Intrinsic winding: $\int_0^1 d\theta_t = -13\pi/2$.
- Defined only on smooth curves.

Winding on rough curves? Topological winding

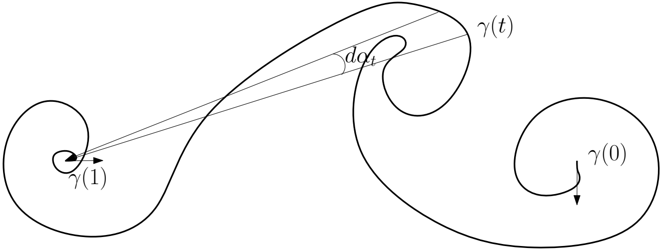
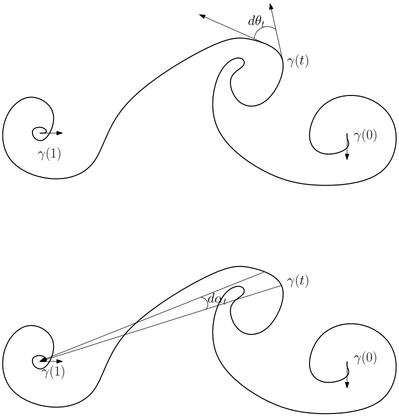


Figure: Topological winding around $\gamma(1)$

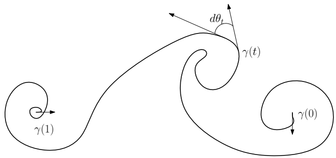
- $\alpha_t := \arg(\gamma(t) - \gamma(1)) - \arg(\gamma(0) - \gamma(1))$ taken continuously.
- Topological winding around $\gamma(1)$: $\int_0^1 d\alpha_t = -4\pi$.
- Defined on curves smooth near target point (in this case $\gamma(1)$).

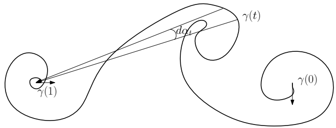
Winding on rough curves?



- Topological winding around $\gamma(1)$: $= \int_0^1 d\alpha_t = -4\pi$.
- Topological winding around $\gamma(0) = -5\pi/2$
- Intrinsic winding: $\int_0^1 d\theta_t = -13\pi/2$.

Winding on rough curves?





For smooth curves:

Topological winding around$(\gamma(1)+\gamma(0))$ **= Intrinsic winding.**

Lemma

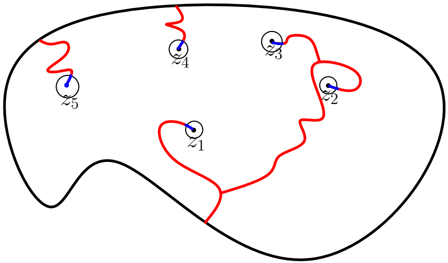
Let γ be a rough curve which is smooth near endpoints obtained as a limit of a discrete curve. Then intrinsic winding of the discrete curve converges and is equal to sum of topological windings of the continuous curve around the end points.

Proof ideas

- Parametrize the tree branches according to capacity. This means

$$\text{Conformal radius } (D \setminus \gamma[0,t]) = e^{-t} \tag{1}$$

- h_t^δ : topological winding up to capacity t around end points.
- $h^\delta = h_t^\delta + e^\delta$.



Second moment convergence

$$\mathbb{E}((\int_D h^\delta(z)f(z)dz)^2) = \int_{D^2} h^\delta(z_1)h^\delta(z_2)f(z_1)f(z_2)dz_1dz_2$$

Second moment convergence

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$$\mathbb{E}((\int_D h^\delta(z)f(z)dz)^2)$$
$$= \int_{D^2}(h_t^\delta(z_1) + e^\delta(z_1))(h_t^\delta(z_2) + e^\delta(z_2))f(z_1)f(z_2)dz_1dz_2$$

- h_t^δ is continuous in δ. Need to somehow show that the truncated continuous windings converge to GFF.
- Done if e^δ are independent since we subtract the mean.

Proof ideas

- h_t^δ : topological winding up to capacity t around end points.
- h_t : topological winding of continuum UST branch up to capacity t.

Steps in proof:

- Step 1: $h_t^\delta \xrightarrow[\delta\to 0]{} h_t$ (discrete winding up to capacity $t \to$ continuum winding up to capacity t).

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- Step 2: $h_t - \mathbb{E}h_t \xrightarrow[t \to \infty]{} \sqrt{2}GFF$.
- Step 3: Discrete winding from t to ∞ are independent from point to point.

Step 1

- h_t^δ : topological winding up to capacity t around end points.

Theorem (Berestycki, Laslier, R. 2016)

h_t^δ (winding in the discrete) $\to h_t$ in the sense of convergence of moments.

Proof.

Use Yadin, Yehudayoff along with crossing estimate to prove exponential tail of winding in annulus. □

Theorem (Yadin, Yehudayoff)

CLT for random walk $\Rightarrow$ branches of UST $\to SLE_2$.

Convergence of continuum winding

Step 2:

Theorem (Berestycki, Laslier, R. 2016)

$$h_t \xrightarrow[t\to\infty]{L^p} \sqrt{2} h_{GFF}$$

for any $p > 1$ in $H^{-1-\eta}$ for all $\eta > 0$.

Proof.

- Convergence of joint moment of winding at k points. (Need to understand how winding behaves under conformal maps).

$\square$

Convergence of continuum winding

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$$h_t \xrightarrow[t\to\infty]{L^p} \sqrt{2}h_{GFF}$$

for any $p > 1$ *in* $H^{-1-\eta}$ *for all* $\eta > 0$.

Proof.

- Convergence of joint moment of winding at k points. (Need to understand how winding behaves under conformal maps).
- Identification of the limit using SLE/GFF coupling. (winding field, UST) pair behaves similarly as coupled (GFF,UST) pair.

□

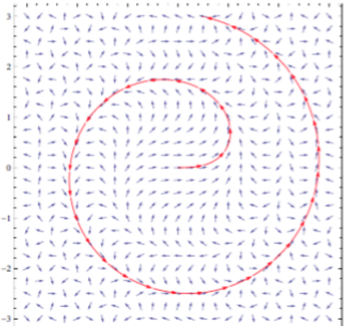
Imaginary geometry

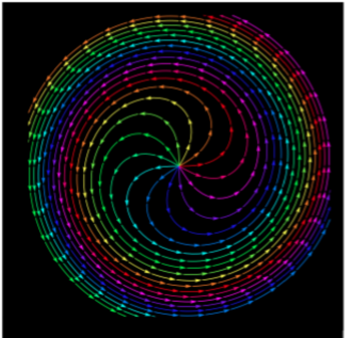
- Work of Dubédat and Miller, Sheffield (Imaginary geometry).

Given a GFF h in a domain, a **flow line** η at angle θ is the solution to the following equation

$$\frac{\partial \eta(t)}{\partial t} = e^{i\left(\frac{h(\eta(t))}{\chi}+\theta\right)}$$

$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$.





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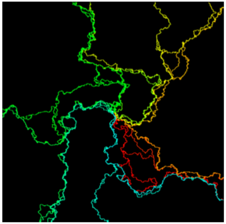
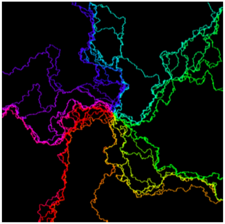
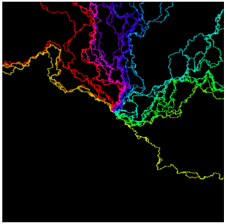
Flow lines



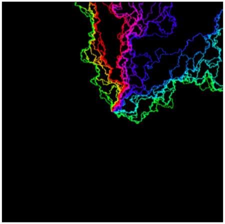
(a) $\alpha = -\frac{1}{2}\chi$; π range of angles



(b) $\alpha = 0$; 2π range of angles



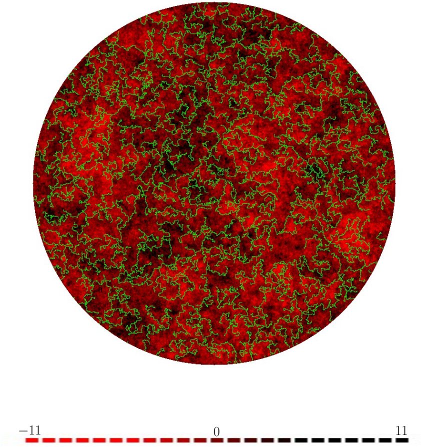
(c) $\alpha = \chi$; 4π range of angles

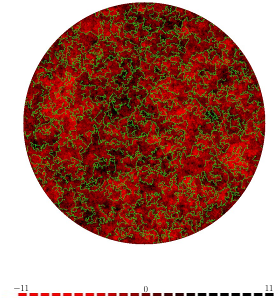


(d) $\alpha = 2\chi$; 6π range of angles

Figure: ©Jason Miller

UST/GFF ($\kappa = 2$)



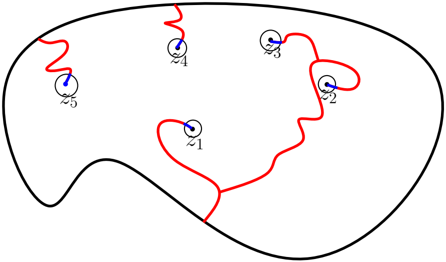


Lemma (Uniqueness)

Our limit of winding satisfies some uniqueness properties of the flowlines.

Coupling

Step 3



Theorem (Berestycki, Laslier, R. 2016)

The blue parts are roughly independent.

Coupling

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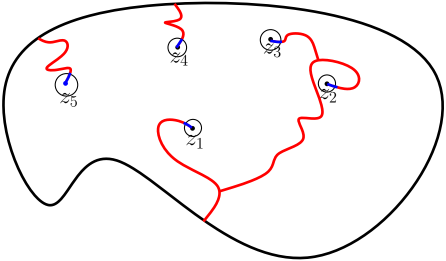

Theorem (Berestycki, Laslier, R. 2016)

The blue parts are roughly independent.

The proof is technical and uses multiscale arguments. Thus overall

$$h = h_t + e_t$$

$e_t - \mathbb{E}(e_t) \approx$ independent from point to point with mean zero.

Step 3.

Main idea: Discrete coupling with a full plane UST.
Main tool:

Lemma (Schramm's finiteness lemma.)

For all $\epsilon > 0$ *there exists* $j(\epsilon)$ *branches such that all other branches except these* $j(\epsilon)$ *branches have diameter* $\leq \epsilon$ *with probability* $\geq 1 - \epsilon$

Extension: Riemann surfaces (work in progress!)

- Temperleyan bijection extends to cycle-rooted spanning forests. All cycles non-contractible.

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- Temperleyan bijection extends to cycle-rooted spanning forests. All cycles non-contractible.
- Height function no longer a function. But height difference (or winding) along the forest make sense (i.e. makes sense as a 1-form, which is exact).
- Hodge theorem

$$\omega = h \bigoplus df$$

ω : **closed** 1-form, h : harmonic 1-form, f-function. So

$$\textbf{Height difference } \to d(GFF) \bigoplus h = \textbf{ compactified GFF}$$

h: Harmonic 1-form (Instanton component).

Extension: Riemann surfaces (work in progress!)

- On torus, for $\mathbb{Z}^2$, full convergence proved by Dubédat. (ref. Dimers and analytic torsion.)
- On torus, for general graphs satisfying CLT, convergence of instanton component proved by Dubedat and Gheissari.

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We prove (disclaimer: writing in progress!!)

- Convergence in a Riemann surface with finitely many handles and holes.
- Full convergence to compacified GFF for general graphs on torus.
- Identification in general? Imaginary geometry on general surfaces.

More science fiction!

Generalize to $\kappa \neq 2$. Let $\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$ (constant in imaginary geometry).

$$\begin{tikzcd}
\text{spanning tree via Temperleyan bijection} \arrow[r, leftrightarrow, "\text{winding}"] \arrow[d, "\text{scaling limit}"] & \text{Interacting Dimer, height function} \arrow[d, "\text{Fluctuations (Giuliani,Mastropietro, Toninelli)}"] \\
\text{Flow line tree} \arrow[r, leftrightarrow, "\text{winding??}"'] & 1/\chi \text{ Gaussian free field}
\end{tikzcd}$$

Thanks for listening!