Monotone Hurwitz Numbers

and

The HCIZ Integral

- Given: an $N \times N$ random Hermitian matrix, A_N.

- Question: what do the eigenvalues look like?

N random particles

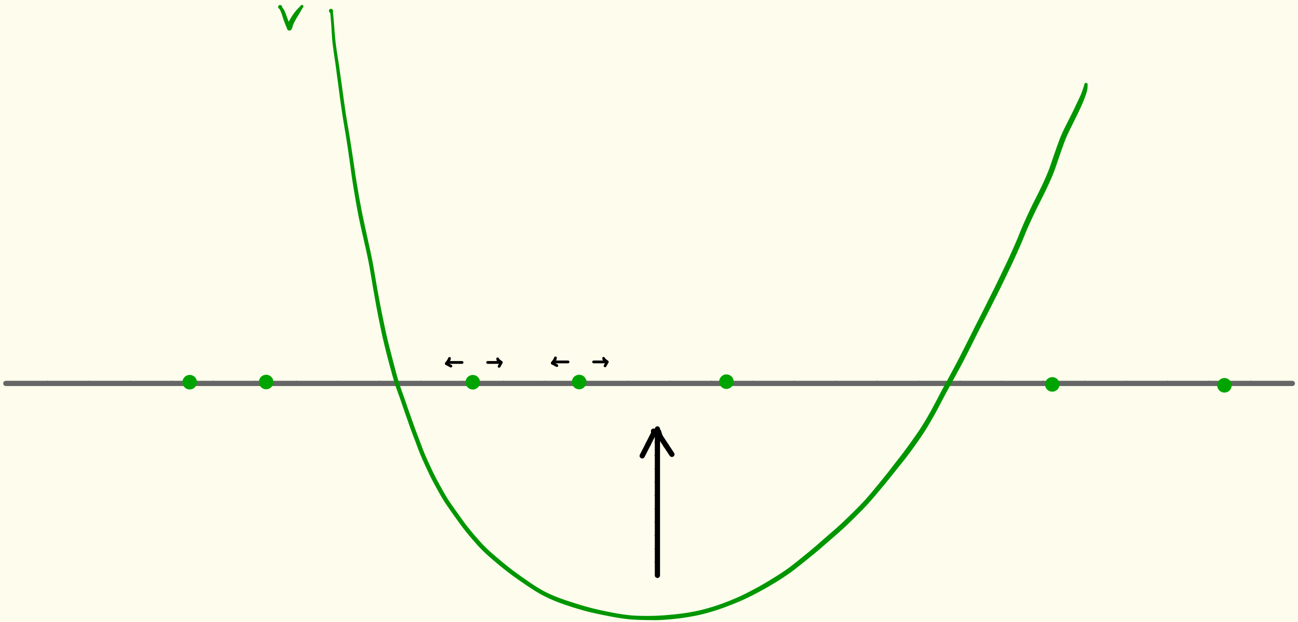
- Hermitian one-matrix model: probability measure on $N \times N$ Hermitian matrices with density proportional to

$$e^{-N \operatorname{Tr} V(A)}, \qquad V = \text{"potential"}.$$

- For A_N a random sample from this distribution, the eigenvalues form a Coulomb gas with confining potential V.

- This means: the probability to observe $a_1^{(N)} \geq \ldots \geq a_N^{(N)}$ at specified locations $a_1 \geq \ldots \geq a_N$ is proportional to

$$e^{-N^2 \mathcal{E}(a_1,\ldots,a_N)}, \qquad \mathcal{E}_N(a_1,\ldots,a_N) = \frac{1}{N}\sum_{i=1}^{N} V(a_i) - \frac{2}{N^2}\sum_{i<j} \log(a_i - a_j).$$



$$\mathcal{E}_N(a_1, \dots, a_N) = \frac{1}{N} \sum_{i=1}^{N} V(a_i) - \frac{2}{N^2} \sum_{i<j} \log(a_i - a_j)$$

- The Coulomb gas has a unique ground state configuration – "Fekete points."

- Example: if $V(x) = \frac{1}{2}x^2$ is the Gaussian potential, then Fekete points are the zeros of the N^{th} Hermite polynomial.

- Law of Large Numbers: as $N \to \infty$, eigenvalues "freeze" around ground state configuration.

- A_N a random sample from $e^{-N \operatorname{Tr} V(A)}$, eigenvalues $a_1^{(N)} \geq \ldots \geq a_N^{(N)}$ random sample from $e^{-N^2 \mathcal{E}(a_1,\ldots,a_N)}$.

- Empirical spectral measure: random probability measure σ_N which places equal mass at each particle.

Law of Large Numbers: As $N \to \infty$, σ_N converges weakly, in probability, to the unique minimizer σ_∞ of the action

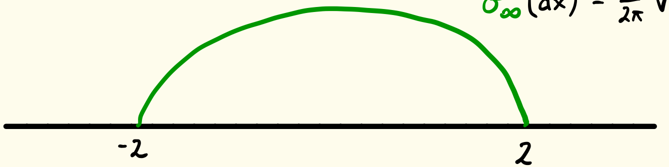
$$S(\sigma) = \int V(x)\,\sigma(dx) - \iint \log(x-y)\,\sigma(dx)\,\sigma(dy).$$

Minimizer σ_∞ is the "equilibrium measure."

- Density of the equilibrium measure σ_∞ is the "limit shape" of the ensemble $a_1^{(N)} \geq \ldots \geq a_N^{(N)}$.

- Example: if $V(x) = \frac{1}{2}x^2$, limit shape is the Wigner semicircle,

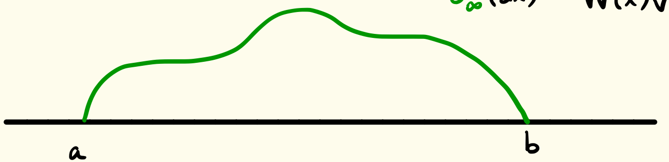
$$\sigma_\infty(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$



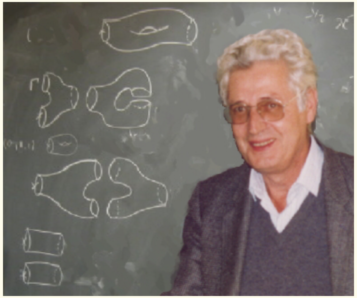
- If V is a perturbation of the Gaussian potential, σ_∞ is a perturbation of the semicircle,

$$\sigma_\infty(dx) = W(x)\sqrt{(x+a)(x-b)}\,dx$$

$$\deg W = \deg V - 2$$



- Claude Itzykson, Jean-Bernard Zuber (1978): What can be said about the asymptotics of the joint spectrum of a coupled pair A_N, B_N of random Hermitian matrices?



Itzykson, 1938–1995

Zuber, 1947–

- Itzykson-Zuber two-matrix model: probability measure on pairs (A,B) of $N\times N$ Hermitian matrices with density proportional to

$$e^{-N\mathrm{Tr}\left(V(A)+W(B)-zAB\right)}.$$

- Parameter z is the "coupling constant" – controls strength of interaction between A_N and B_N, where (A_N,B_N) a random sample.

- What is the joint distribution of the eigenvalues $a_1^{(N)}\geq\ldots\geq a_N^{(N)}$ and $b_1^{(N)}\geq\ldots\geq b_N^{(N)}$?

- The eigenvalues $a_1^{(N)} \geq \ldots \geq a_N^{(N)}$ and $b_1^{(N)} \geq \ldots \geq b_N^{(N)}$ behave like a two-component Coulomb gas with Boltzmann factor

$$e^{-N^2 \mathcal{E}(a_1,\ldots,a_N,b_1,\ldots,b_N)}$$

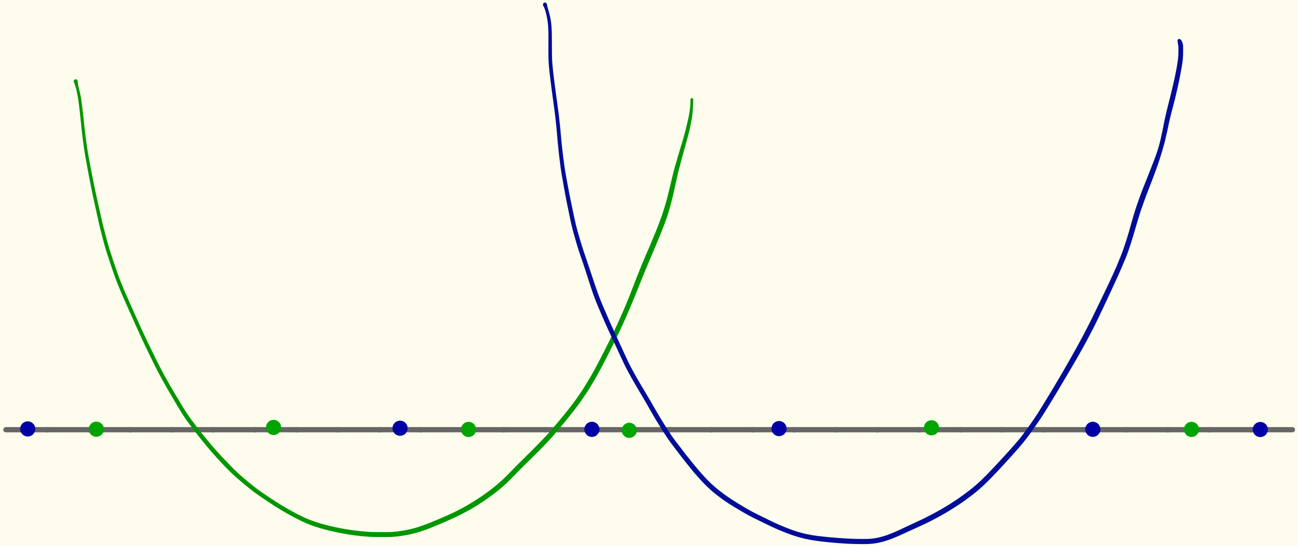
where

$$\mathcal{E}(a_1,\ldots,a_N,b_1,\ldots,b_N) = \mathcal{E}^V(a_1,\ldots,a_N) + \mathcal{E}^W(b_1,\ldots,b_N) - \mathcal{I}_N(z,a_1,\ldots,a_N,b_1,\ldots,b_N)$$

and

$$\boxed{\mathcal{I}_N(z,a_1,\ldots,a_N,b_1,\ldots,b_N) = \frac{1}{N^2}\log\int_{U(N)} e^{zN\mathrm{Tr}\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N\end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N\end{bmatrix} U^{-1}}\, dU}$$

is the Itzykson-Zuber "free energy."



$$\mathcal{E}_N(a_1,\dots,a_N;b_1,\dots,b_N) = \mathcal{E}_N^V(a_1,\dots,a_N) + \mathcal{E}_N^W(b_1,\dots,b_N) - \mathcal{J}_N^{z}(a_1,\dots,a_N;b_1,\dots,b_N)$$

?

Itzykson-Zuber Conjecture (1978)

Consider two sequences

$$\begin{array}{lll} a_1^{(1)} & & \\ a_1^{(2)} & a_2^{(2)} & \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ \vdots & \vdots & \vdots \end{array} \qquad \text{and} \qquad \begin{array}{lll} b_1^{(1)} & & \\ b_1^{(2)} & b_2^{(2)} & \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ \vdots & \vdots & \vdots \end{array}$$

of deterministic particle configurations

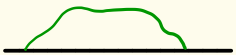
$$a_1^{(N)} \geq \dots \geq a_N^{(N)} \qquad \text{and} \qquad b_1^{(N)} \geq \dots \geq b_N^{(N)}$$

accumulating in a compact interval $[-M, M]$.

Suppose that the empirical measures of these configurations converge in moments: the limits

$$x_k = \lim_{N\to\infty} \frac{1}{N} p_k\left(a_1^{(N)}, \dots, a_N^{(N)}\right) \qquad \text{and} \qquad y_k = \lim_{N\to\infty} \frac{1}{N} p_k\left(b_1^{(N)}, \dots, b_N^{(N)}\right)$$

exist for each fixed $k \in \mathbb{N}$.



Associated to this data are two sequences of functions:

$$\underbrace{I_N(z) = \int_{U(N)} e^{zN\,\mathrm{Tr}\begin{bmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)} \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU}_{\text{entire}} \qquad \text{and} \qquad F_N(z) = \underbrace{\frac{1}{N^2}\log \int_{U(N)} e^{zN\,\mathrm{Tr}\begin{bmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)} \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU}_{\text{analytic near } z=0}$$

IZ Analytical: Limiting free energy exists.

There exists $\varepsilon > 0$ such that $F_N(z)$ converges uniformly on compact subsets of $\{|z| < \varepsilon\}$; put

$$F_\infty(z) := \lim_{N\to\infty} F_N(z).$$

IZ Combinatorial: Limiting free energy is a generating function.

Each derivative $F_\infty^{(d)}(0)$ is a polynomial in $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$, bihomogeneous of degree d with respect to the grading $\deg(x_k) = \deg(y_k) = k$, with integer coefficients:

$$F_\infty^{(d)}(0) = \sum_{\alpha, \beta \vdash d} IZ(\alpha,\beta)\, x_\alpha y_\beta, \qquad IZ(\alpha,\beta) \in \mathbb{Z}.$$

Example: $\mathcal{F}_{\infty}^{(4)}(0)$ is the following polynomial in x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4:

$$
\begin{aligned}
&6x_4y_4 - 24x_4y_3y_1 - 12x_4y_2^2 + 60x_4y_2y_1^2 - 30x_4y_1^4 \\
&-24x_3x_1y_4 + 96x_3x_1y_3y_1 + 48x_3x_1y_2^2 - 240x_3x_4y_2y_1^2 + 120x_3x_1y_1^4 \\
&-12x_2^2y_4 + 48x_2^2y_3y_1 + 18x_2^2y_2^2 - 108x_2^2y_2y_1^2 + 54x_2^2y_1^4 \\
&+60x_2x_1^2y_4 - 240x_2x_1^2y_3y_1 - 108x_2x_1^2y_2^2 + 576x_2x_1^2y_2y_1^2 - 288x_2x_1^2y_1^4 \\
&-30x_1^4y_4 + 120x_1^4y_3y_1 + 54x_1^4y_2^2 - 288x_1^4y_2y_1^2 + 144x_1^4y_1^4
\end{aligned}
$$

- Itzykson and Zuber were able to compute these limiting derivatives up to order 8!

Exclamation, not factorial

- Previous approaches (Matytsin, Guionnet-Zeitouni): IZ analytical first.

- New approach (Goulden, Guay-Paquet, Novak): IZ combinatorial first.

- Conceptual breakthrough: The Itzykson-Zuber numbers $IZ(\alpha, \beta)$ are Hurwitz numbers. Thus, they belong to the world of enumerative algebraic geometry (Gromov-Witten invariants etc.).

- This realization gives a whole new perspective on what $\mathcal{F}_N(z)$ is, and a new set of tools with which to address the Itzykson-Zuber conjecture.

(1) Probably invented the Exponential Formula:

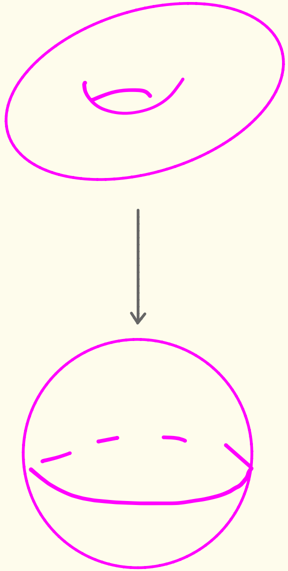
$$W = e^H$$

all structures

connected structures

ADOLF HURWITZ

(2) First to "count surfaces":



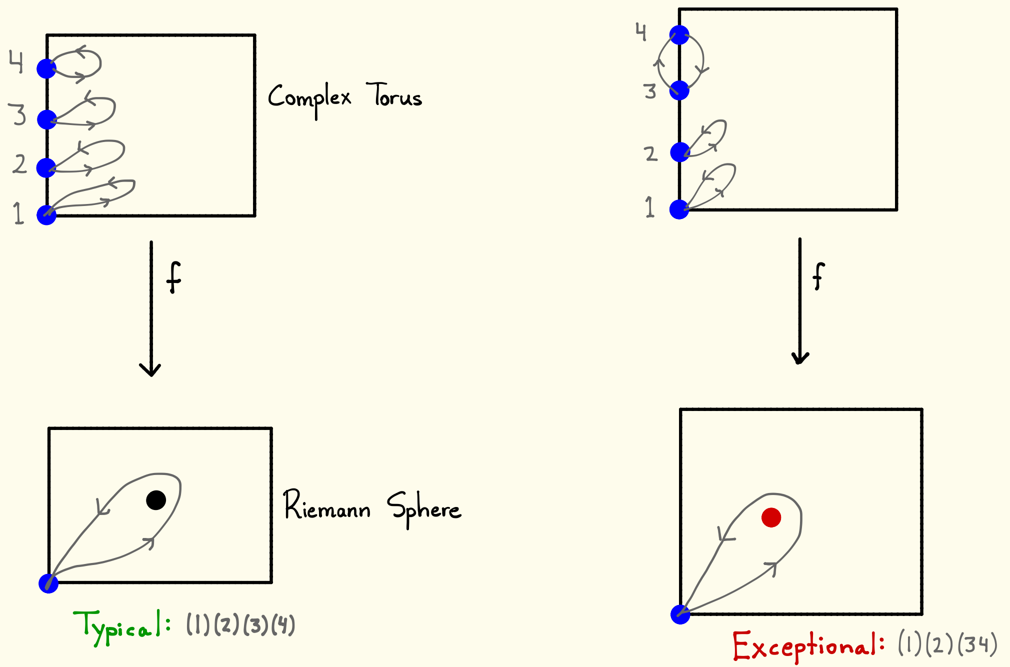
(3) Invented the braid groups:

$$B(n) = \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j|>1 \rangle$$

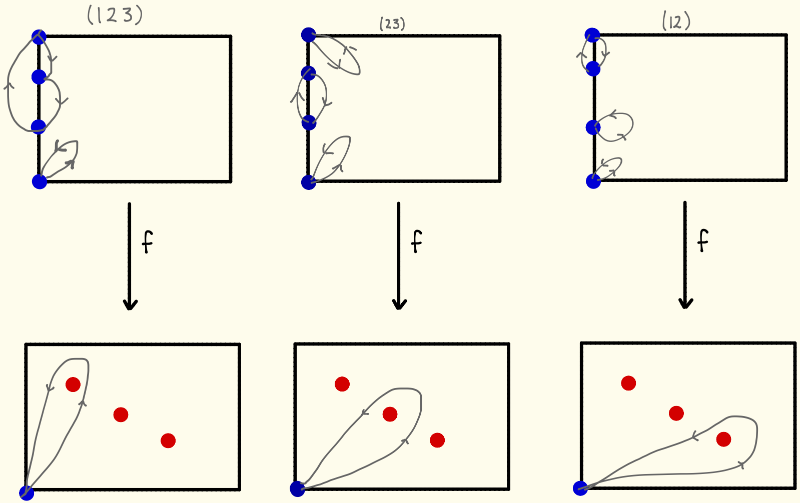
(4) Invented random matrices:

Invariant measures on compact matrix groups

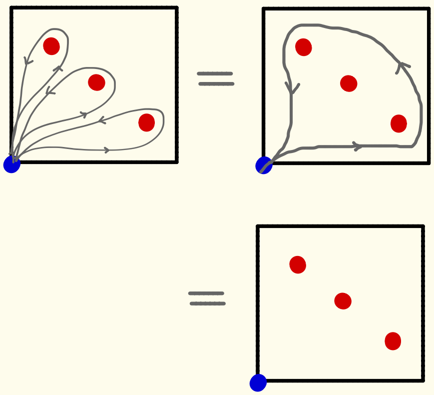
FACT: Every holomorphic function from a compact, connected Riemann surface to the Riemann sphere is a branched covering.



Conversely, given a topological branched covering $f: S \to \mathbb{P}^1$, there is a unique complex structure on S which makes f holomorphic.



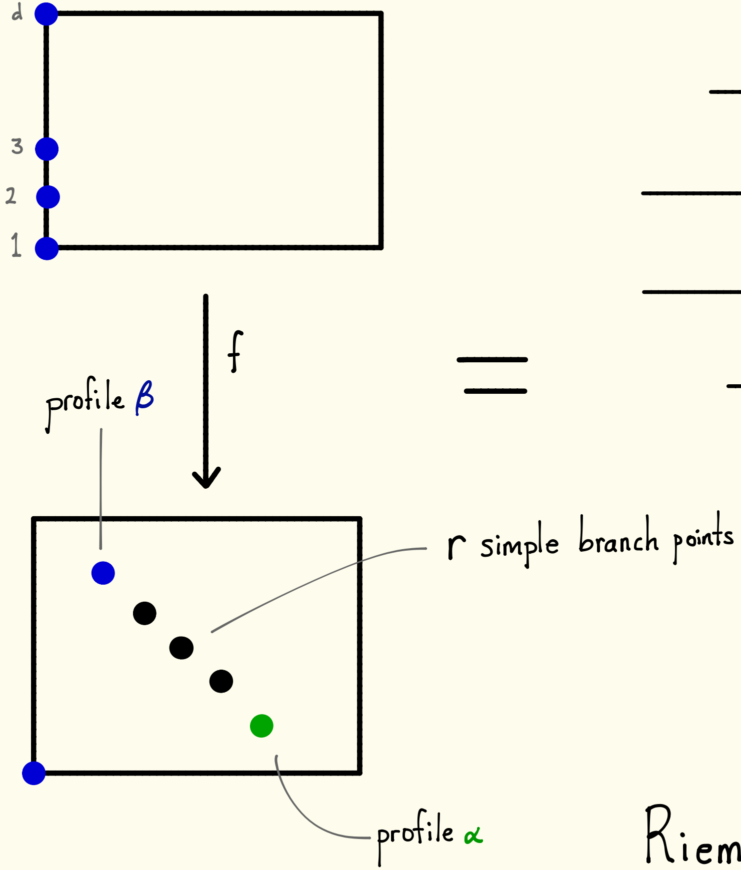
$$(123)(23)(12) = (1)(2)(3)(4)$$



GEOMETRY $\longleftrightarrow$ COMBINATORICS

$$\left\{\begin{array}{c}\text{degree } d \text{ branched covers}\\ \text{of } \mathbb{P}^1 \text{ with given ramification}\\ \text{data}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{transitive factorizations of}\\ (1)\ldots(d) \text{ in } S(d) \text{ with factors}\\ \text{in prescribed conjugacy classes}\end{array}\right\}$$

Double Hurwitz Numbers: $H_g(\alpha, \beta)$



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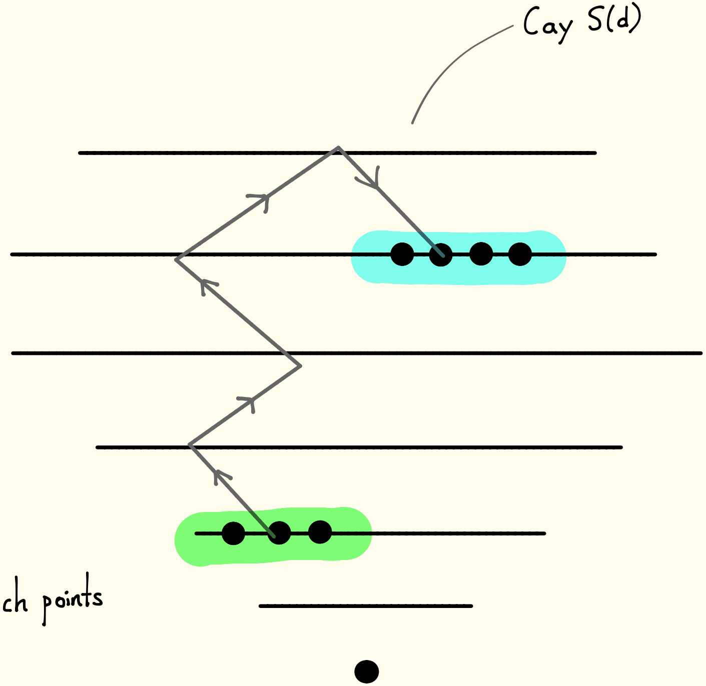
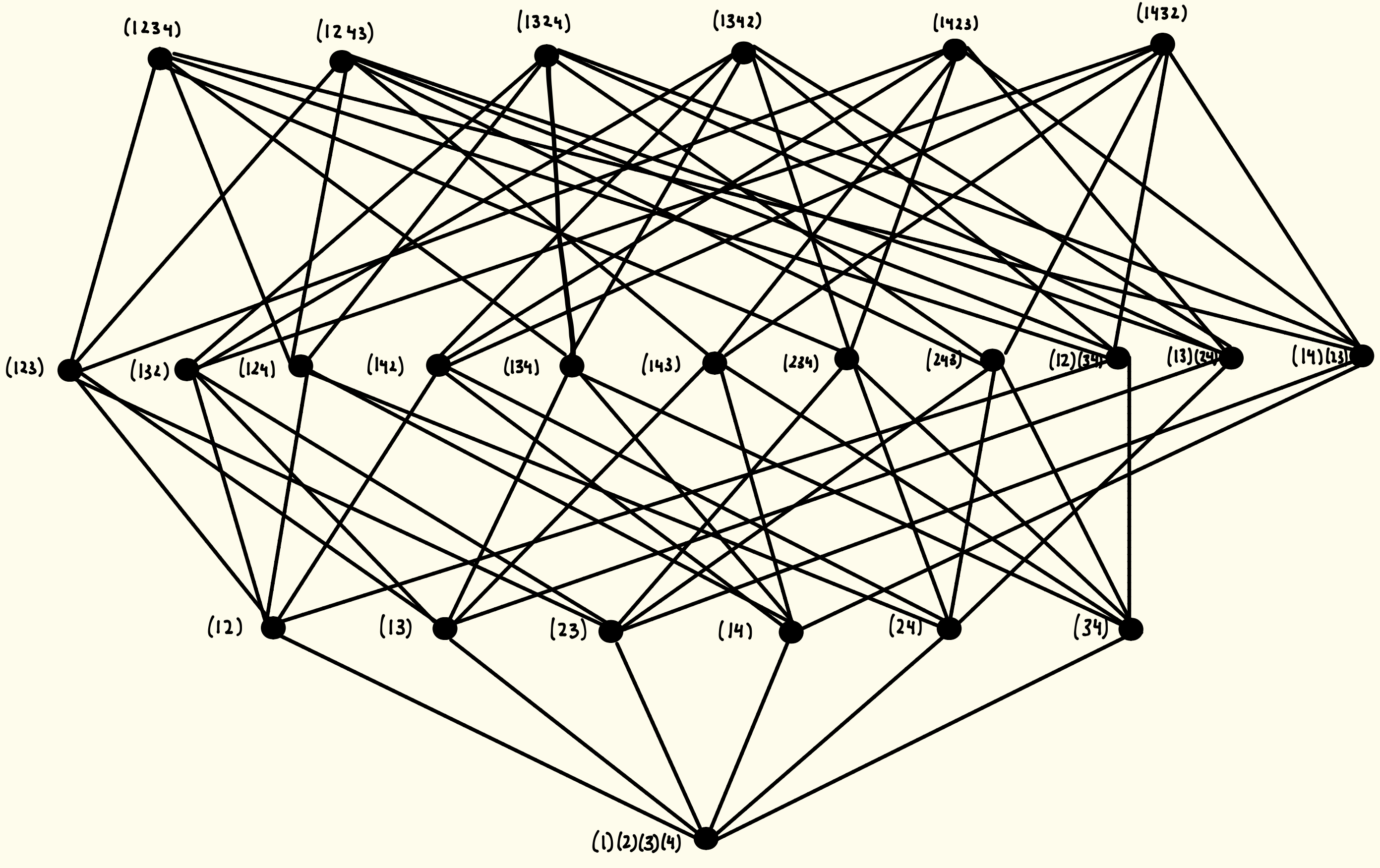


$$\sigma = \rho \tau_1 \dots \tau_r$$

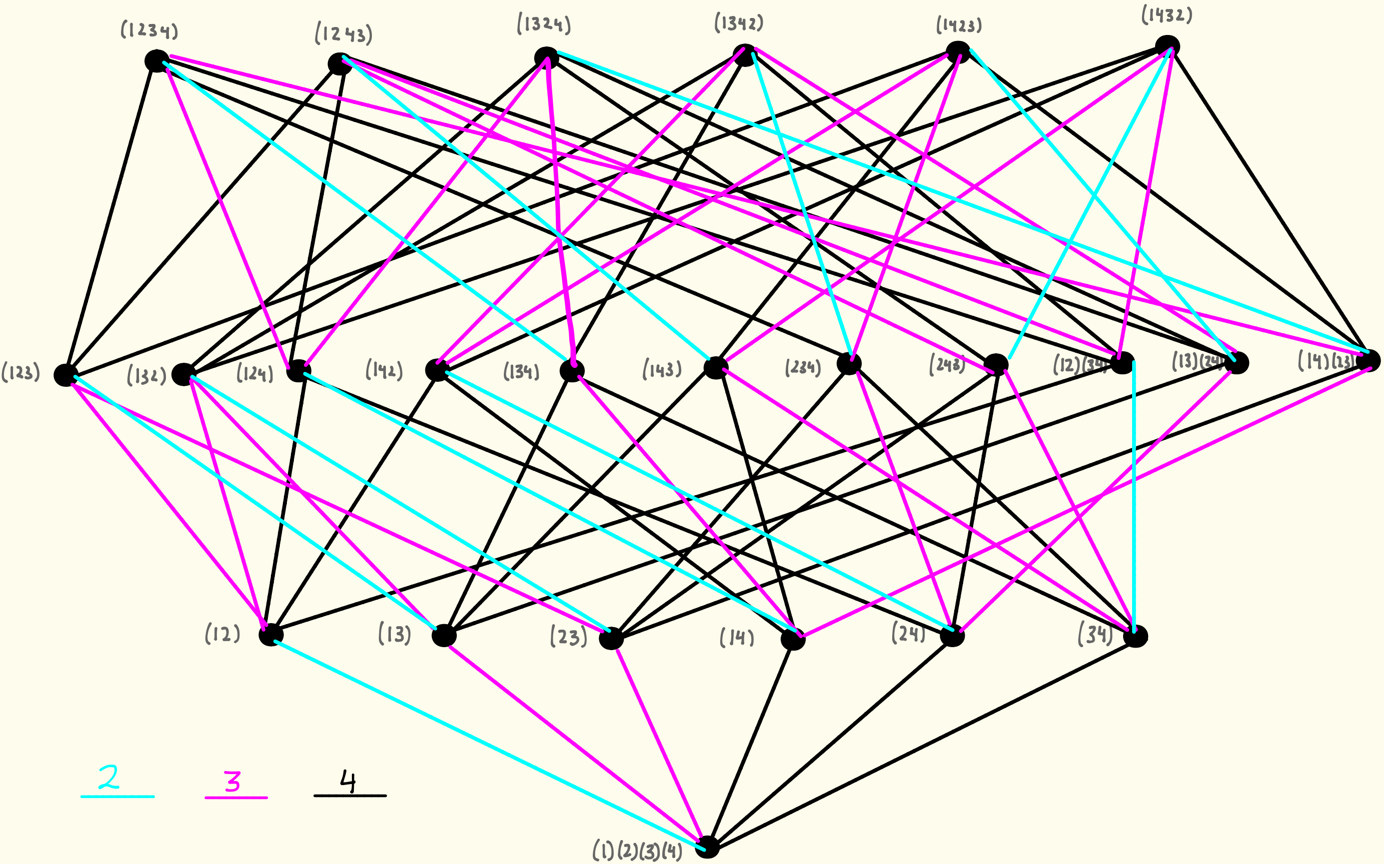
$\langle \rho, \tau_1, \dots, \tau_r, \sigma \rangle$ transitive

Riemann-Hurwitz: $r = 2g-2+\ell(\alpha)+\ell(\beta)$.

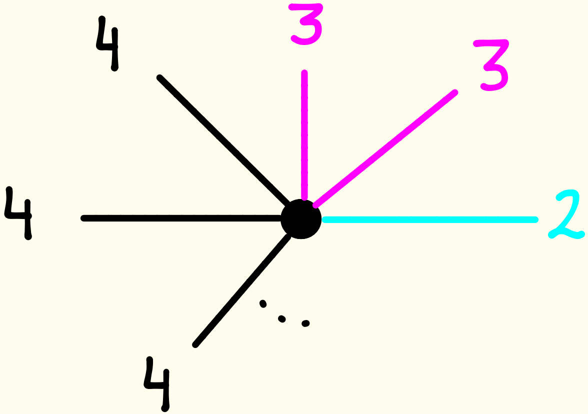
The Group S(4)


(1234)
(1243)
(1324)
(1342)
(1423)
(1432)
(123)
(132)
(124)
(142)
(134)
(143)
(234)
(243)
(12)(34)
(13)(24)
(14)(23)
(12)
(13)
(23)
(14)
(24)
(34)
(1)(2)(3)(4)

Biane-Stanley edge-labelling of S(4)

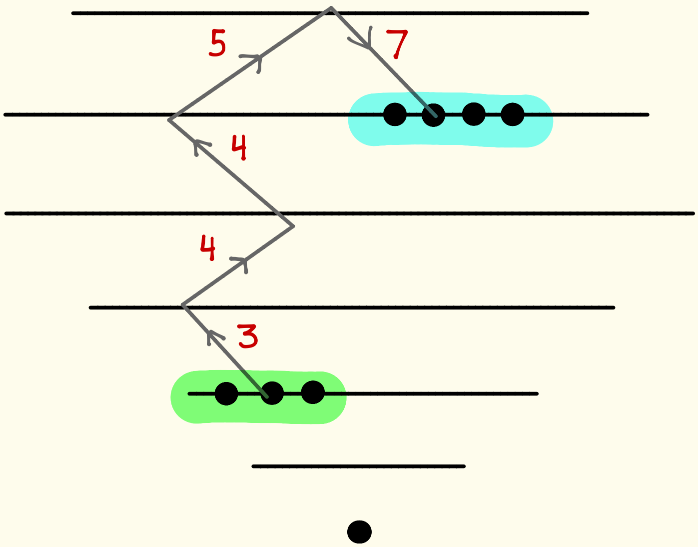


A fragment of S(d)



$$T = \begin{bmatrix} (12) & (13) & (14) \\ & (23) & (24) \\ & & (34) \end{bmatrix}$$

Monotone Double Hurwitz Numbers: $\vec{H}_g(\alpha, \beta)$



Leading Derivatives Theorem (Goulden, Guay-Paquet, Novak)

For any $1 \leq d \leq N$, the d^{th} derivative of the Itzykson-Zuber free energy expands as a generating function for monotone double Hurwitz numbers of degree d:

$$\mathcal{F}_N^{(d)}(0) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{\alpha,\beta \vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_g(\alpha,\beta) \frac{p_\alpha(a_1^{(N)},\dots,a_N^{(N)})}{N^{\ell(\alpha)}} \frac{p_\beta(b_1^{(N)},\dots,b_N^{(N)})}{N^{\ell(\beta)}}.$$

The series is absolutely convergent.

- Leading order is $g=0$:

$$\mathcal{F}_N^{(d)}(0) \sim \sum_{\alpha,\beta \vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_0(\alpha,\beta)\, x_\alpha y_\beta.$$

Corollary of LDT

For each fixed $d \in \mathbb{N}$, the limit $\mathcal{J}^{(d)}_\infty(0) := \lim_{N\to\infty} \mathcal{J}^{(d)}_N(0)$ exists:

$$\mathcal{J}^{(d)}_\infty(0) = \sum_{\alpha,\beta\vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_0(\alpha,\beta)x_\alpha y_\beta.$$

- Actually, the LDT yields more: if

$$\frac{1}{N}p_k\left(a_1^{(N)},\dots,a_N^{(N)}\right) = x_k + o(N^{-2h}) \quad \text{and} \quad \frac{1}{N}p_k\left(b_1^{(N)},\dots,b_N^{(N)}\right) = y_k + o(N^{-2h})$$

as $N\to\infty$, then

$$\mathcal{J}^{(d)}_N(0) = \sum_{g=0}^{h} \frac{1}{N^{2g}} \sum_{\alpha,\beta\vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_g(\alpha,\beta)x_\alpha y_\beta + o\left(\frac{1}{N^{2h}}\right).$$

- Now want the analytical part of the Itzykson - Zuber conjecture: the free energy $\mathcal{F}_N(z)$ converges uniformly on compact subsets of a complex domain.

- At present, only know convergence of each derivative, $\lim_{N\to\infty} \mathcal{F}_N^{(d)}(0) = \mathcal{F}_\infty^{(d)}(0)$, together with a beautiful but subtle combinatorial description of the limit.

- Strategy: establish absolute summability of the formal power series

$$\mathcal{F}_\infty(z) = \sum_{d=1}^{\infty} \mathcal{F}_\infty^{(d)}(0)\frac{z^d}{d!},$$

then compare analytic functions...

- Question: What is known about monotone Hurwitz numbers?

- Answer: Exactly what is known about classical Hurwitz numbers.

- Illustrate this via explicit formulas.

Theorem (Hurwitz, 1891):

$$H_0(1^d, (d)) = (d-1)!\, d^{d-2}$$

Theorem (Goulden, Guay-Paquet, N., 2012):

$$\overrightarrow{H}_0(1^d, (d)) = (d-1)!\, \frac{1}{d}\binom{2d-2}{d-1}.$$

Theorem (Hurwitz, 1891):

$$H_0(1^d, \beta) = \frac{d!}{|\mathrm{Aut}(\beta)|} (d+\ell(\beta)-2)! \, d^{\ell(\beta)-3} \prod_{j=1}^{\ell(\beta)} \frac{\beta_j^{\beta_j}}{\beta_j!}.$$

Theorem (Goulden, Guay-Paquet, N., 2012):

$$H_0(1^d, \beta) = \frac{d!}{|\mathrm{Aut}(\beta)|} (2d+1)^{\overline{\ell(\beta)-3}} \prod_{j=1}^{\ell(\beta)} \binom{2\beta_j}{\beta_j}.$$

Theorem (Vakil, 2001):

$$H_1(1^d, \beta) = \frac{d!}{|\mathrm{Aut}(\beta)|}\,(d+\ell(\beta))!\prod_{j=1}^{\ell(\beta)} \frac{\beta_j^{\beta_j}}{\beta_j!} \cdot \frac{1}{24}\left(d^{\ell(\beta)} - d^{\ell(\beta)-1} - \sum_{k=2}^{\ell(\beta)} (k-2)!\, d^{\ell(\beta)-k}\, e_k(\beta)\right).$$

Theorem (Goulden, Guay-Paquet, N., 2012):

$$\vec{H}_1(1^d, \beta) = \frac{d!}{|\mathrm{Aut}(\beta)|}\prod_{j=1}^{\ell(\beta)} \binom{2\beta_j}{\beta_j} \cdot \frac{1}{24}\left((2d+1)^{\overline{\ell(\beta)}} - 3(2d+1)^{\overline{\ell(\beta)-1}} - \sum_{k=2}^{\ell(\beta)} (k-2)!\, (2d+1)^{\ell(\beta)-k}\, e_k(2\beta-1)\right).$$

Theorem (ELSV, 2001): For each pair $(g,\ell) \notin \{(0,1),(0,2)\}$, there exists a polynomial P_g in ℓ variables such that

$$H_g(1^d, \beta) = \frac{d!}{|\mathrm{Aut}(\beta)|}(d+\ell+2g-2)! \prod_{j=1}^{\ell} \frac{\beta_j^{\beta_j}}{\beta_j!} \cdot P_g(\beta_1, \dots, \beta_\ell)$$

for all partitions β with ℓ parts, where $d = |\beta|$.

Theorem (Goulden, Guay-Paquet, N.): For each pair $(g,\ell) \notin \{(0,1),(0,2)\}$, there exists a polynomial $\vec{P}_g$ in ℓ variables such that

$$\vec{H}_g(1^d, \beta) = \frac{d!}{|\mathrm{Aut}(\beta)|} \prod_{j=1}^{\ell} \binom{2\beta_j}{\beta_j} \cdot \vec{P}_g(\beta_1, \dots, \beta_\ell)$$

for all partitions β with ℓ parts, where $d = |\beta|$.

Theorem (Goulden, Guay-Paquet, Novak)

For any $g \geq 0$, the series

$$\vec{H}_g(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta \vdash d} \vec{H}_g(\alpha,\beta)$$

has radius of convergence at least $\frac{1}{54}$ and at most $\frac{2}{27}$.

Remark: Conjecturally, each of these series has radius $\frac{2}{27}$. Thus, by Pringsheim, they have a common dominant singularity at $z_c = \frac{2}{27}$. This common singular behaviour is a hallmark of generating functions related to 2D quantum gravity.

Remark (Di Francesco): Let Γ_N denote # of isomorphism classes of groups of order p^N. Then

$$\lim_{N\to\infty} \frac{1}{N^3} \log \Gamma_N = \frac{2}{27} \log p.$$

Theorem (Goulden, Guay-Paquet, Novak)

Let

$$\begin{array}{ccc} a_1^{(1)} & & \\ a_1^{(2)} & a_2^{(2)} & \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ \vdots & \vdots & \vdots \end{array} \quad \text{and} \quad \begin{array}{ccc} b_1^{(1)} & & \\ b_1^{(2)} & b_2^{(2)} & \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ \vdots & \vdots & \vdots \end{array}$$

be two sequences of particle configurations accumulating in the origin-centred disc of radius M. Suppose the limits

$$x_k = \lim_{N\to\infty} \frac{1}{N} p_k\left(a_1^{(N)}, \dots, a_N^{(N)}\right) \quad \text{and} \quad y_k = \lim_{N\to\infty} \frac{1}{N} p_k\left(b_1^{(N)}, \dots, b_N^{(N)}\right)$$

exist for each fixed $k \in \mathbb{N}$. Associate to this data the function sequences

$$\underbrace{\mathcal{I}_N(z) = \int_{U(N)} e^{zN\operatorname{Tr}\begin{bmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)}\end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)}\end{bmatrix} U^{-1}} dU}_{\text{entire}} \quad \text{and} \quad \underbrace{\mathcal{F}_N(z) = \frac{1}{N^2} \log \int_{U(N)} e^{zN\operatorname{Tr}\begin{bmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)}\end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)}\end{bmatrix} U^{-1}} dU}_{\text{analytic near } z=0}$$

→ cont'd

There exists $\varepsilon > 0$ such that $\mathcal{F}_N(z)$ converges to the generating function

$$\mathcal{F}_\infty(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta \vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_0(\alpha,\beta)\, x_\alpha y_\beta$$

for genus zero monotone double Hurwitz numbers, uniformly on compact subsets of $\{|z| < \varepsilon\}$.

- This proves the conjecture of Itzykson and Zuber, and explains what the limiting free energy is.

- It is probably impossible to give an explicit formula for $\mathcal{F}_\infty(z)$.

CONJECTURE (Goulden, Guay-Paquet, Novak)

Let

$$\begin{array}{ccc} a_1^{(1)} & & \\ a_1^{(2)} & a_2^{(2)} & \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ \vdots & \vdots & \vdots \end{array} \quad \text{and} \quad \begin{array}{ccc} b_1^{(1)} & & \\ b_1^{(2)} & b_2^{(2)} & \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ \vdots & \vdots & \vdots \end{array}$$

be two sequences of particle configurations accumulating in the origin-centred disc of radius M. Suppose there exists $h \in \mathbb{N}$ such that

$$\frac{1}{N} p_k\left(a_1^{(N)}, \ldots, a_N^{(N)}\right) = x_k + o\left(N^{-2h}\right) \quad \text{and} \quad \frac{1}{N} p_k\left(b_1^{(N)}, \ldots, b_N^{(N)}\right) = y_k + o\left(N^{-2h}\right)$$

as $N \to \infty$ for each $k \in \mathbb{N}$. Then, there exists $\varepsilon > 0$ such that

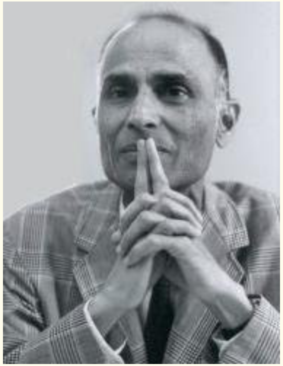
$$\mathcal{F}_N(z) = \sum_{g=0}^{h} \frac{1}{N^{2g}} \mathcal{F}_{\infty,g}(z) + o\left(\frac{1}{N^{2h}}\right)$$

uniformly on compact subsets of $\{|z| < \varepsilon\}$, where

$$\mathcal{F}_{\infty,g}(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta \vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_g(\alpha,\beta)\, x_\alpha y_\beta .$$

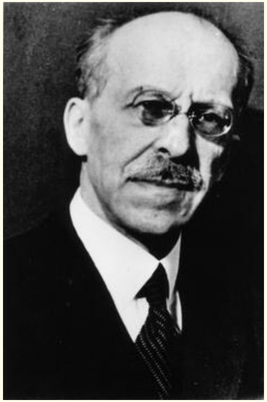
- The Itzykson-Zuber integral is closely related to the representation theory of $GL_N(\mathbb{C})$.

- This connection is due to Harish-Chandra.



1923 – 1983

- Rep theory of $GL_N(\mathbb{C})$ was developed by Issai Schur.



1875-1941

- Schur: irreps of $GL_N(\mathbb{C})$ are parameterized by configurations of N hard particles on $\mathbb{Z}$.

$\vdots$

- Schur's character formula: for any $A \in GL_N(\mathbb{C})$ with eigenvalues $z_1, \ldots, z_N \in \mathbb{C}^\times$

$$\chi^{(b_1,\ldots,b_N)}(z_1,\ldots,z_N) = \frac{\det\left[\cdots \; z_i^{b_j} \; \cdots\right]}{\det\left[\cdots \; z_i^{N-j} \; \cdots\right]} .$$

- Harish-Chandra's character formula: for any $B \in GL_N(\mathbb{C})$ with eigenvalues $e^{a_1}, \ldots, e^{a_N} \in \mathbb{C}^\times$

$$\frac{\chi^{(b_1,\ldots,b_N)}(e^{a_1},\ldots,e^{a_N})}{\chi^{(b_1,\ldots,b_N)}(1,\ldots,1)} = \prod_{i<j} \frac{a_i - a_j}{e^{a_i} - e^{a_j}} \int_{U(N)} e^{\mathrm{Tr}\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{bmatrix} U^{-1}} \, dU.$$

- Combining the Harish-Chandra formula with monotone Hurwitz theory opens up a new path into asymptotic representation theory — and large 2D random structures.

- A statistical mechanics model with two ingredients:

  (1) A tiling of the plane by equilateral triangles;

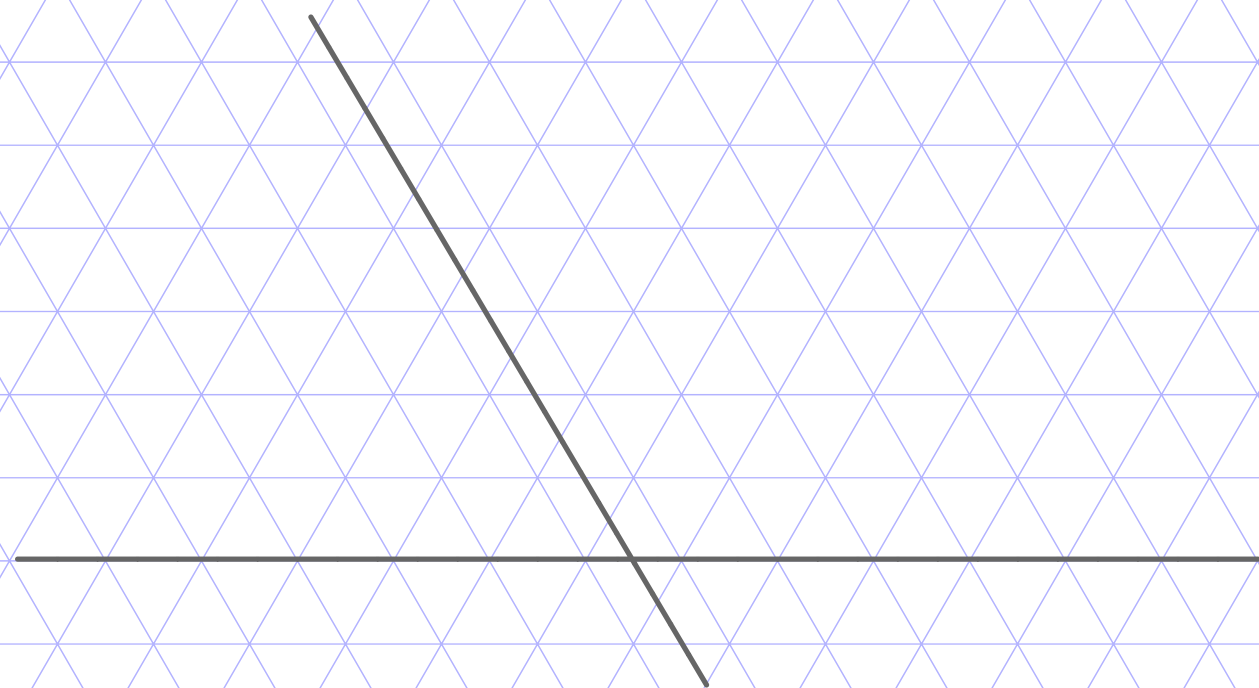
  (2) A triangular array of integers,

$$\begin{array}{ccccc} & & b_1^{(1)} & & \\ & b_1^{(2)} & & b_2^{(2)} & \\ b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\ \iddots & & & & \ddots \end{array}$$

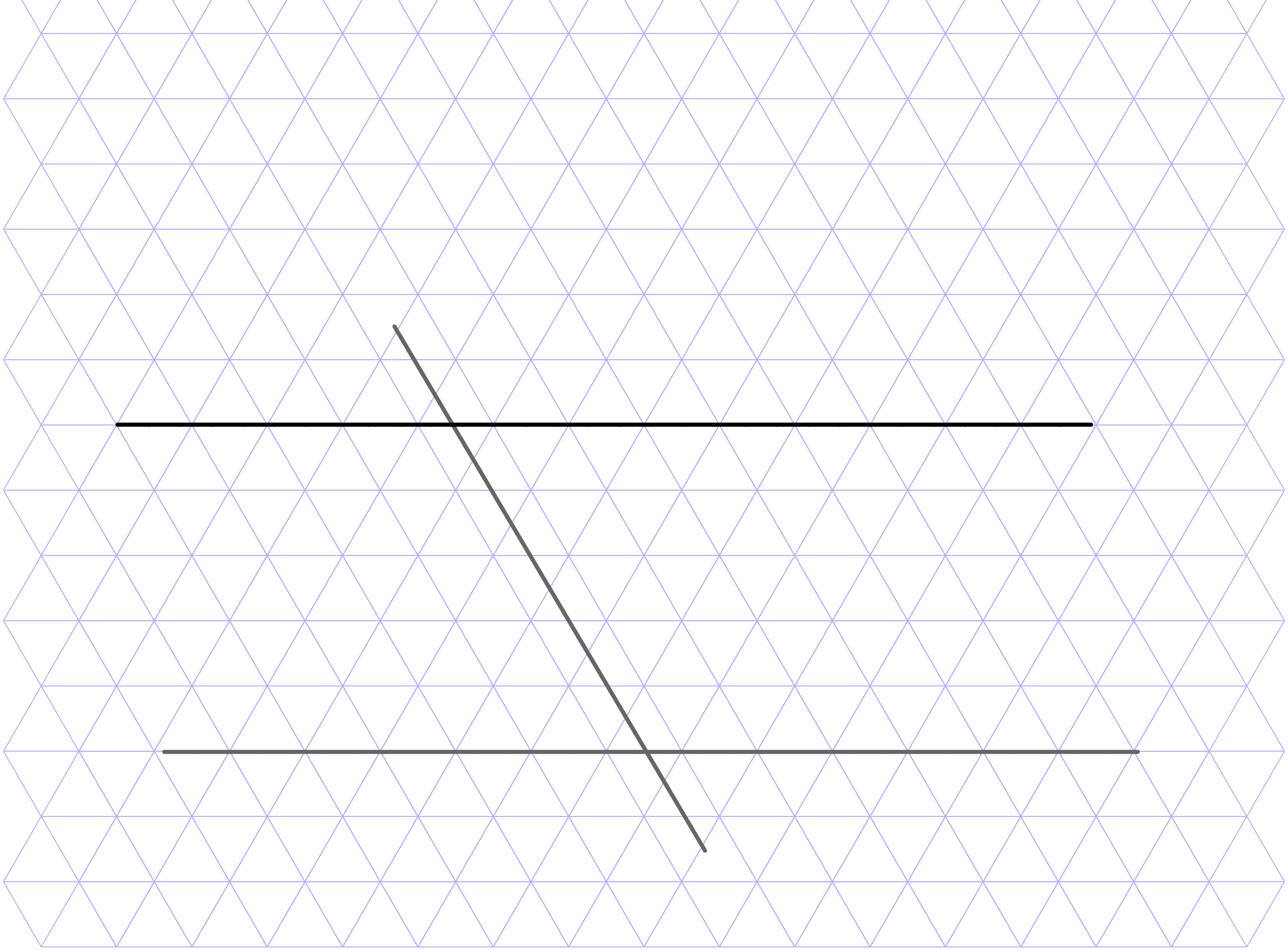
- Using this data, build a sequence of simply-connected planar domains,

  $\Omega^{(1)}, \Omega^{(2)}, \Omega^{(3)}, \ldots$

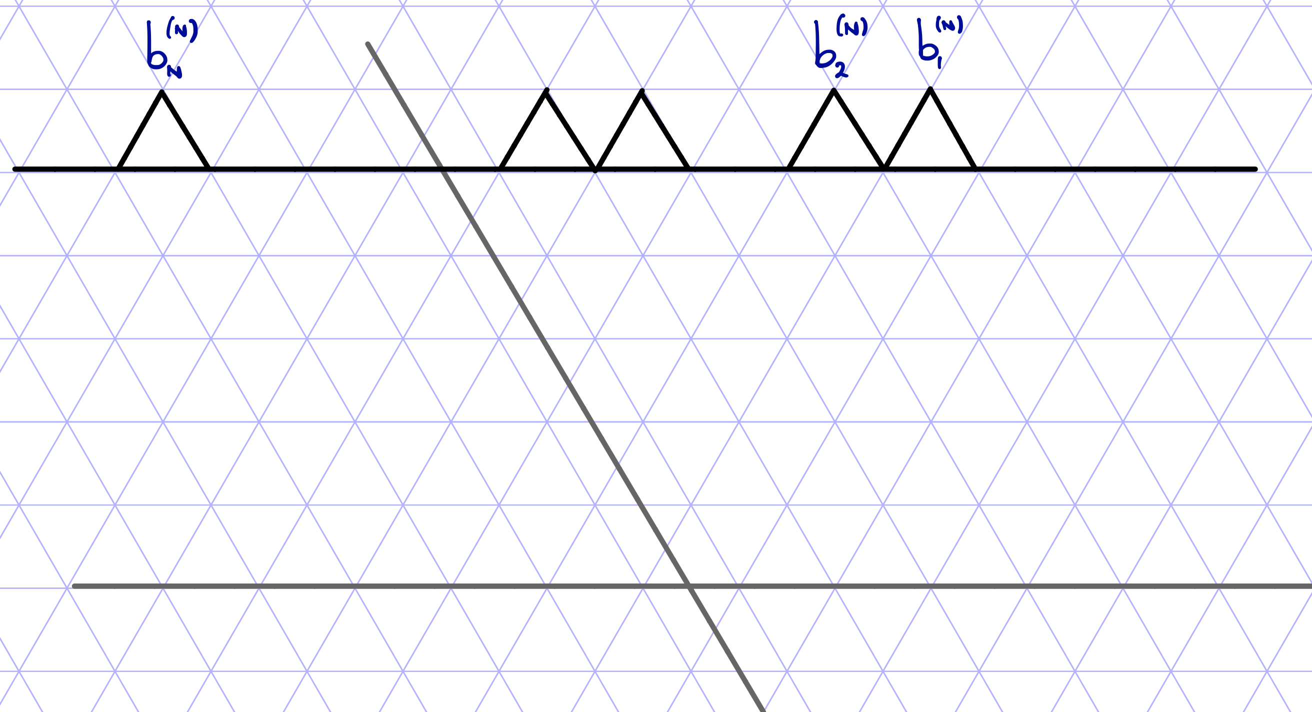
- STEP ZERO: Introduce a coordinate system.



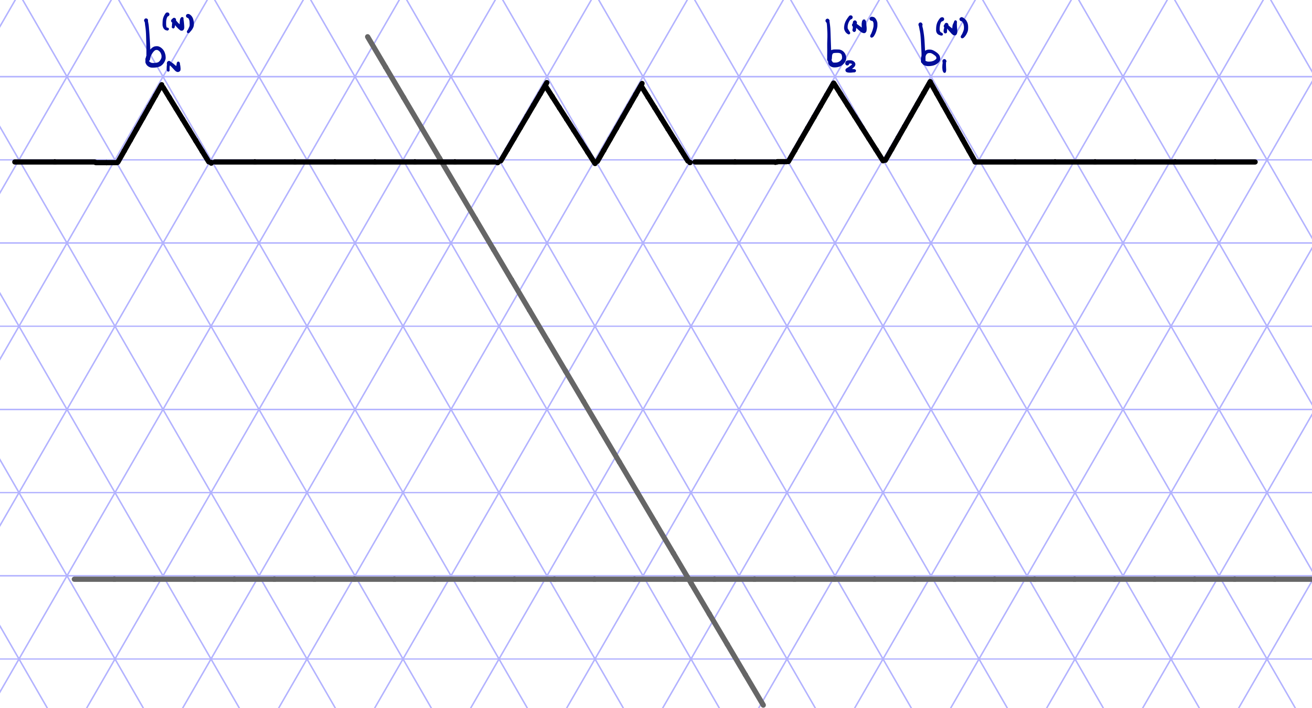
• STEP ONE: Construct the horizontal line through (O,N).



- STEP TWO: Construct N outward-facing unit triangles on the top line such that the midpoints of their bases have horizontal coordinates $b_1^{(N)} > \ldots > b_N^{(N)}$.

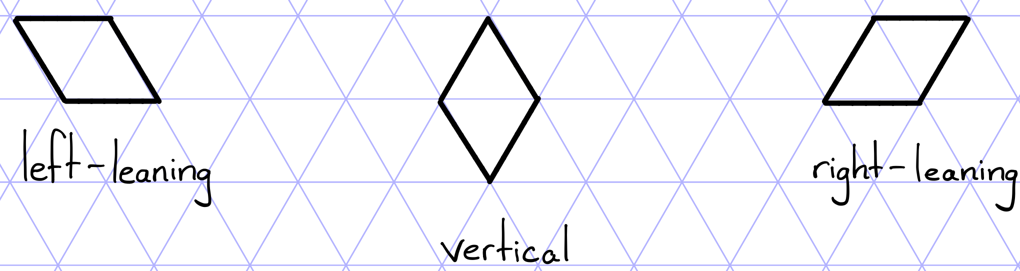


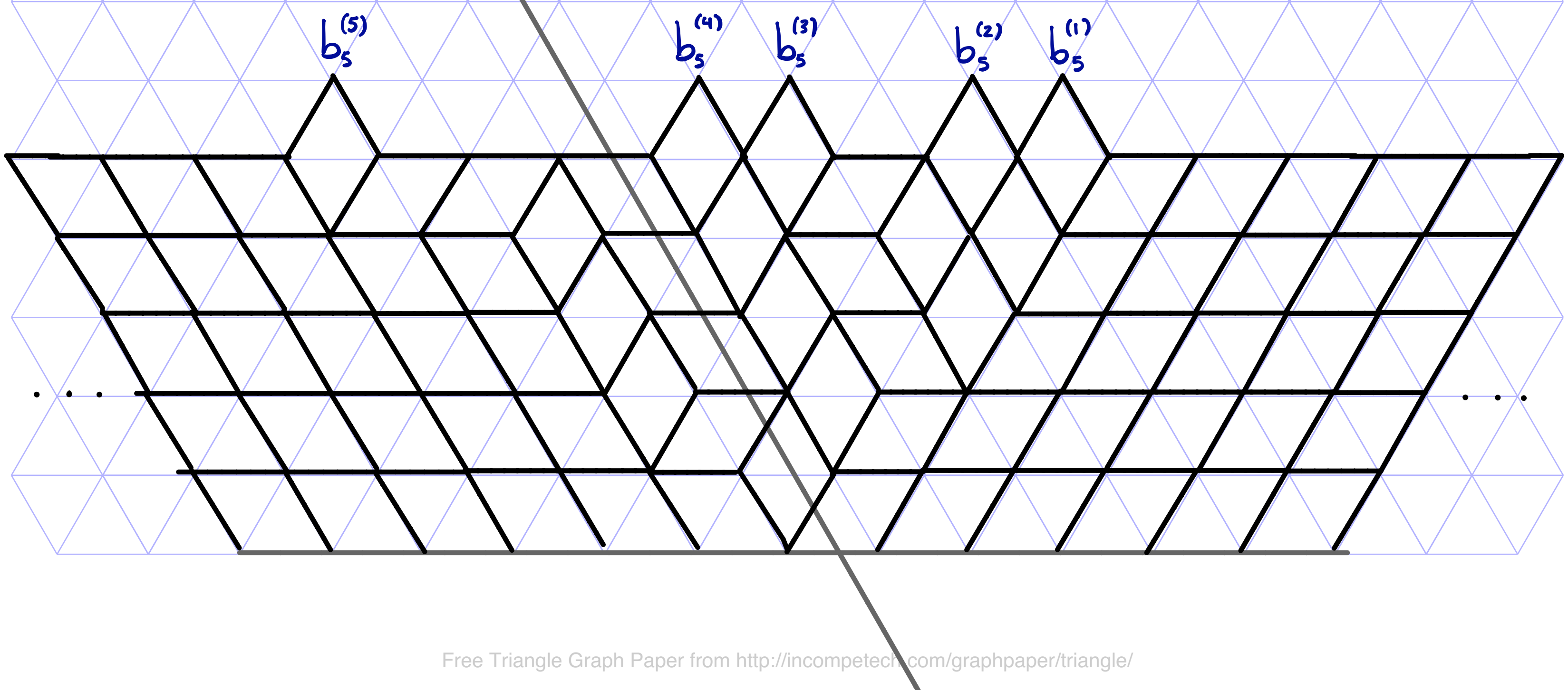
- STEP THREE: Erase the bases of the triangles.



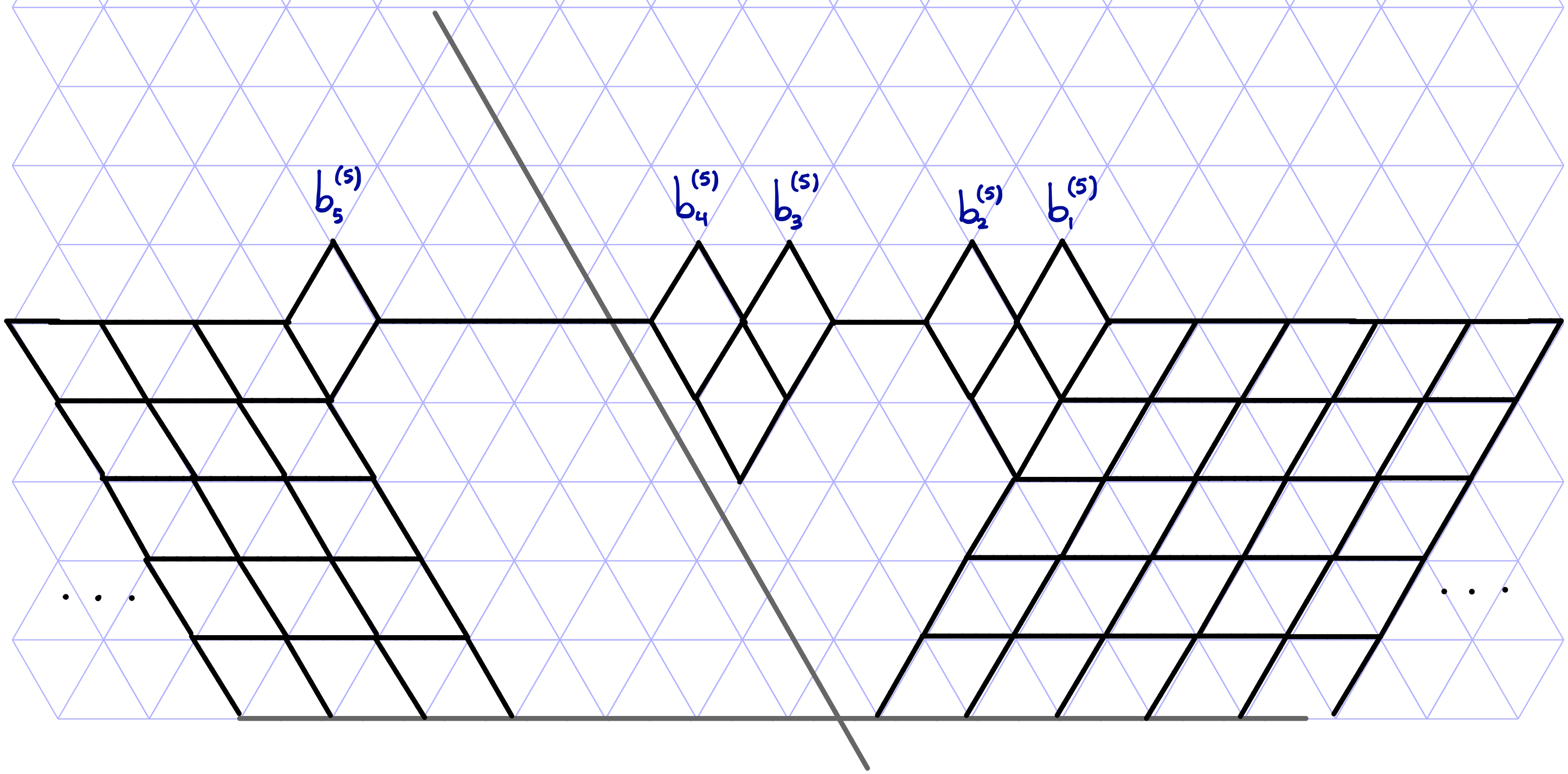
- You have now constructed the sawtooth domain $\Omega^{(N)}$ of rank N with boundary conditions $(b_1^{(N)}, \ldots, b_N^{(N)})$.

• FACT: $\Omega^{(N)}$ can be tessellated using tiles of three types, called lozenges.





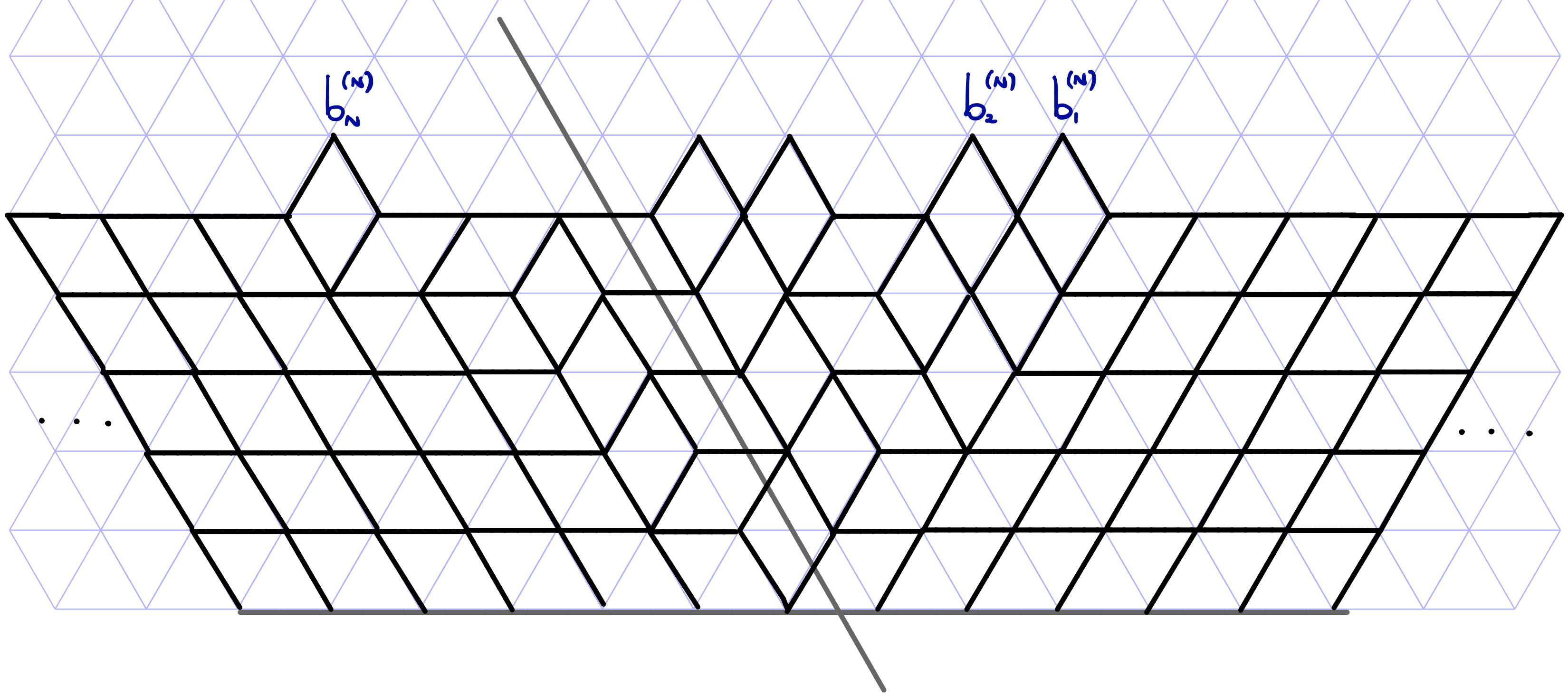
- FACT: $\Omega^{(N)}$ admits finitely many lozenge tilings.



- Ensemble: for each $N \in \mathbb{N}$, $T^{(N)}$ is a uniformly random tiling of $\Omega^{(N)}$.

- The ensemble $T^{(N)}$ is defined by the boundary condition data:

$$
\begin{array}{ccccc}
 & & b_1^{(1)} & & \\
 & b_1^{(2)} & & b_2^{(2)} & \\
b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\
\ddots\!\!\!\!\!\!\!\!\!\!\!\! & & & & \ddots
\end{array}
$$

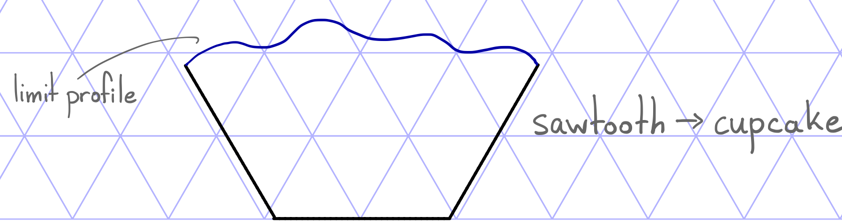


- Assumptions on the boundary data

$$
\begin{array}{ccccc}
 & & b_1^{(1)} & & \\
 & b_1^{(2)} & & b_2^{(2)} & \\
b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\
\iddots & & & & \ddots
\end{array}
$$

- Assumption 1: There exists a constant $M \geq 1$ such that $\{b_1^{(N)} > \ldots > b_N^{(N)}\} \subseteq \{MN > \ldots > -MN\}$ for all $N \in \mathbb{N}$.

- Assumption 2: For each $k \in \mathbb{N}$, the limit $y_k = \lim\limits_{N\to\infty} \frac{1}{N} p_k\left(\frac{b_1^{(N)}}{N}, \ldots, \frac{b_N^{(N)}}{N}\right)$ exists.

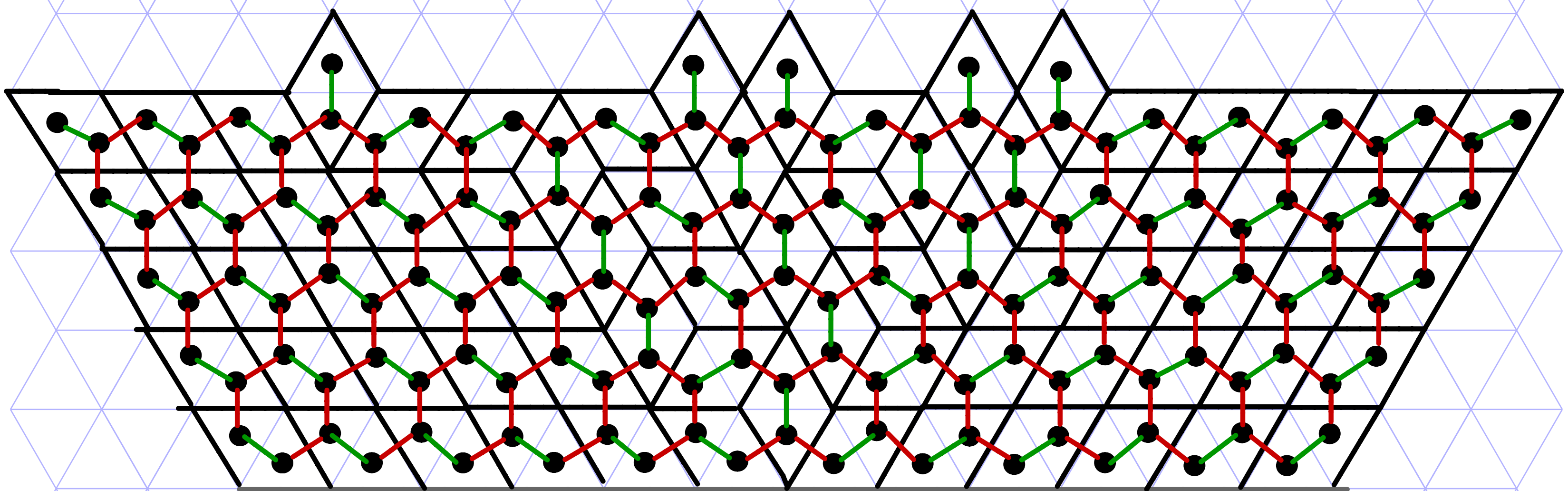


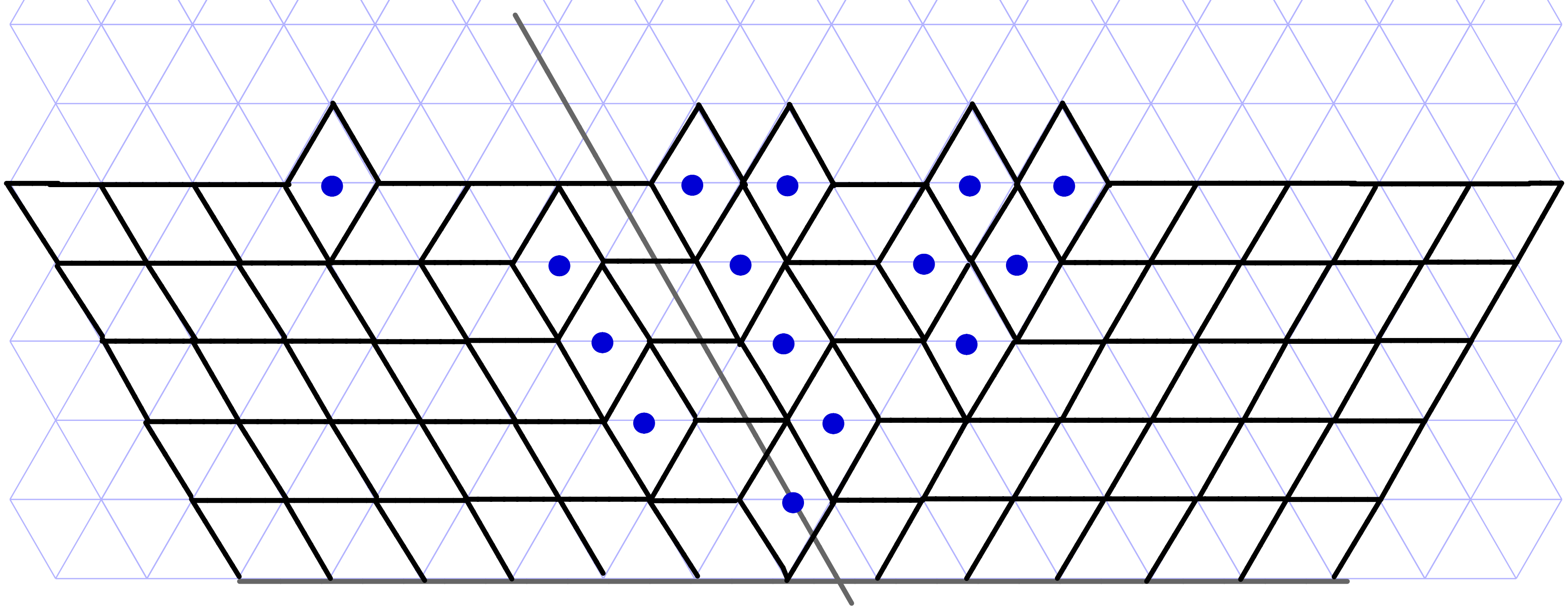
- GOAL: Describe $N \to \infty$ emergent features of $T^{(N)}$ in terms of the limit profile.

- Emergent features: limit shape, fluctuations, etc.

- How to get started?

- Bifurcation: dimers or particles?

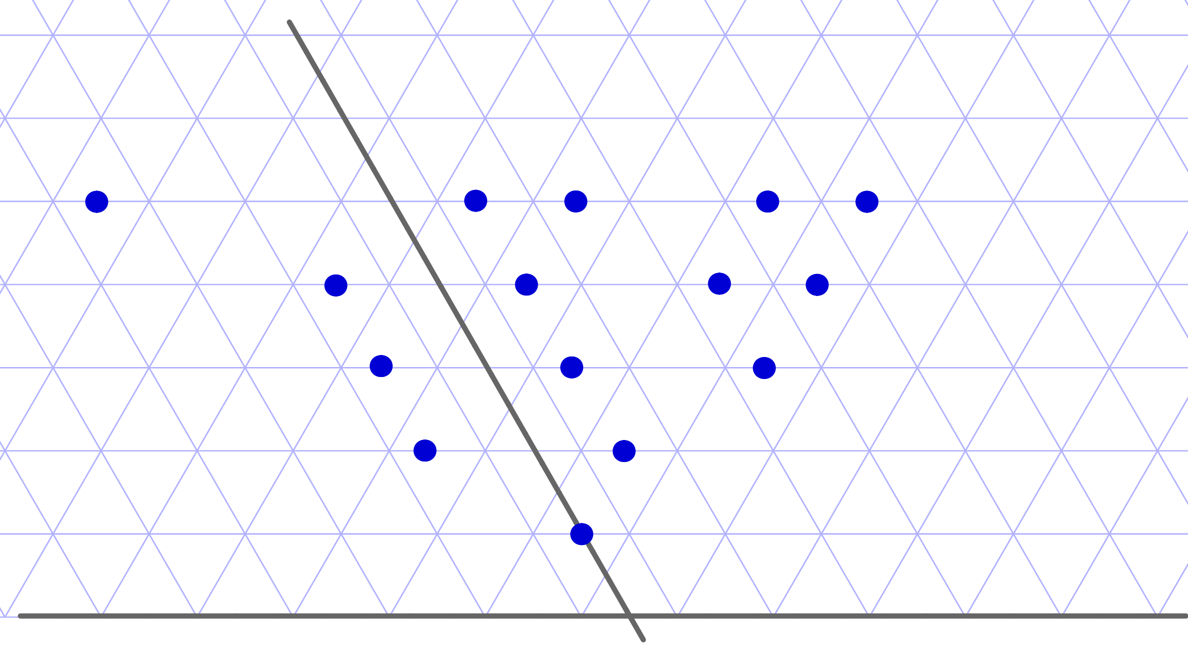




- From the particle perspective, $T^{(N)}$ is a sequence of N random vectors,

$$\left(b_{k1}^{(N)}, \dots, b_{kk}^{(N)}\right), \quad 1 \le k \le N,$$

where $b_{ki}^{(N)}$ is the horizontal coordinate of the i^{th} particle from right on the k^{th} wire from bottom.



- Reasonable starting point: understand the joint distribution of the particles $b_{k1}^{(N)} > \dots > b_{kk}^{(N)}$ on the k^{th} wire through $T^{(N)}$.

- Reasonable goal: compute the Laplace transform

$$\mathbb{E}\left[e^{(a_1,\dots,a_k)\cdot(b_{k1}^{(N)},\dots,b_{kk}^{(N)})}\right] = \sum_{\{b_1>\dots>b_k\}\subset\mathbb{Z}} \mathbb{P}\left(b_{k1}^{(N)}=b_1,\dots,b_{kk}^{(N)}=b_k\right)e^{a_1b_1+\dots+a_kb_k}$$

in terms of the boundary data $b_1^{(N)} > \dots > b_N^{(N)}$.

- No formula for this object.

- Flash of inspiration: trade the random vector $(b_{k1}^{(N)},\ldots,b_{kk}^{(N)})$ for the random matrix

$$B_k^{(N)} = U_k \begin{bmatrix} b_{k1}^{(N)} & & \\ & \ddots & \\ & & b_{kk}^{(N)} \end{bmatrix} U_k^{-1},$$

where U_k is a uniformly random $k\times k$ unitary matrix.

- Take the Laplace transform of $B_k^{(N)}$:

$$\mathbb{E}\left[e^{\mathrm{Tr}\, AB_k^{(N)}}\right] = \sum_{\{b_1>\cdots>b_k\}\subset\mathbb{Z}} \mathbb{P}\left(b_{k1}^{(N)}=b_1,\ldots,b_{kk}^{(N)}=b_k\right)\int_{U(k)} e^{\mathrm{Tr}\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_k \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_k \end{bmatrix} U^{-1}} dU,$$

where A is a $k\times k$ complex semisimple matrix with eigenvalues $a_1,\ldots,a_k$.

- There is a formula for this matrix Laplace transform:

$$\boxed{\mathbb{E}\left[e^{\mathrm{Tr}\, A B_k^{(N)}}\right] = \left(\prod_{i=1}^{k} \frac{a_i}{e^{a_i} - 1}\right)^{N-k} \int_{U(N)} e^{\mathrm{Tr}\begin{bmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_k & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU.}$$

- Claim: all emergent features of $T^{(N)}$ can be extracted from this formula.

Theorem: There exists $\varepsilon > 0$ such that, for each fixed $k \in \mathbb{N}$,

$$\frac{1}{N} \log \mathbb{E}\left[e^{\operatorname{Tr} A B_k^{(N)}}\right] \sim \sum_{i=1}^{k} \log\left(\frac{a_i}{e^{a_i} - 1}\right) + \sum_{g=0}^{\infty} \frac{Q_g(a_1, \dots, a_k)}{N^{2g}},$$

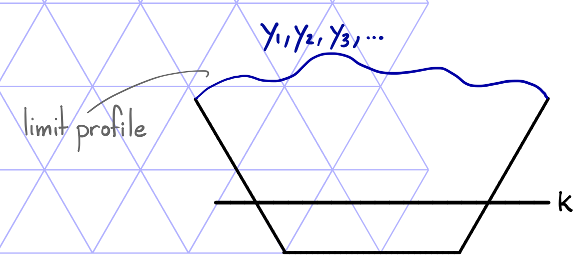
uniformly on compact subsets of $\{A \in \operatorname{End} \mathbb{C}^k : A^*A = AA^*, \|A\| < \varepsilon\}$.

Moreover,

$$Q_g(a_1, \dots, a_k) = Q_g(a_1) + \dots + Q_g(a_k),$$

where $Q_g(a)$ is, essentially, the generating function for one-part monotone double Hurwitz numbers in genus g:

$$Q_g(a) = \sum_{d=1}^{\infty} \frac{a^d}{d!} \sum_{\beta \vdash d} (-1)^{1+\ell(\beta)} \vec{H}_g((d), \beta)\, y_\beta.$$



- FACT: the generating function Q_0 is the antiderivative of the Voiculescu R-transform of the limit profile of $\Omega^{(N)}$:

$$Q_0(a) = \sum_{d=1}^{\infty} \frac{a^d}{d} \kappa_d,$$

where $\kappa_1 = y_1$, $\kappa_2 = y_2 - y_1^2, \ldots$ are the free cumulants of the limit profile.

- Soft conclusion: statistics of the random particles $b_{k1}^{(N)} > \ldots > b_{kk}^{(N)}$ are expressible in terms of the limiting boundary conditions via Free Probability.

– END –