A New Spectral Theory for Schur Polynomials and Applications

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A New Spectral Theory for Schur Polynomials and Applications

I.	λ objects, passive $\varepsilon > 0$	Special Enumeration
II.	λ random, active $\varepsilon > 0$	Special Probability
III.	$\Psi_\lambda(\cdot \mid \varepsilon)$, index λ	Special Functions
IV.	$\widehat{\mathcal{T}}(u \mid \varepsilon)\Psi_\lambda = T_\lambda(u)\Psi_\lambda$	Simultaneous Eigenfunctions
V.	$\widehat{\mathcal{T}}(u \mid \varepsilon)$, "$\hbar$" $= \varepsilon^2$	Quantum Integrability

Rewrite from Vantage of Hidden Classical Integrable System

On Randomness and Spectral Theory in Quantization

[von Neumann 1932] For $\widehat{\mathcal{T}}$ self-adjoint on Hilbert space $\mathcal{H}$, exist **spectral measures** μ **of** $\widehat{\mathcal{T}}$ **at** $\Psi \in \mathcal{H}$ defined by

$$\frac{\langle \Psi | \phi(\widehat{\mathcal{T}}) | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \int_{-\infty}^{+\infty} \phi(E) d\mu(E).$$

for bounded continuous $\phi : \mathbb{R} \to \mathbb{C}$.

[Born 1926] The random variable with law μ is outcome of measuring a quantum system in state Ψ with observable $\widehat{\mathcal{T}}$.

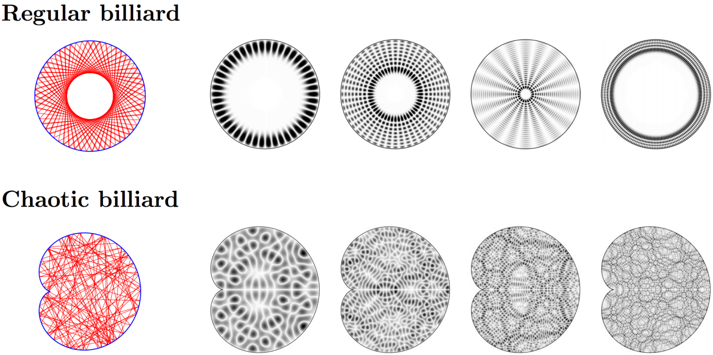
- If $\mathcal{H} = L^2(\Omega)$, spectral measure of $\widehat{q} = M(q)$ at Ψ is

$$d\mu(q) = \frac{|\Psi(q)|^2}{||\Psi||^2} dq$$

  random position distribution of Ψ.

$\star$ ***Born's Rule: Randomness from Non-Random Matrices*** $\star$

Bäcker *Quantum Chaos in Billiards* (2007)



- If $\mathcal{H} = L^2(\Omega)$, spectral measure of $\widehat{q} = M(q)$ at Ψ is

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⋆ ***Born's Rule: Randomness from Non-Random Matrices*** ⋆

Integrable Probability $\overset{?!}{\Longleftrightarrow}$ Quantum Chaos

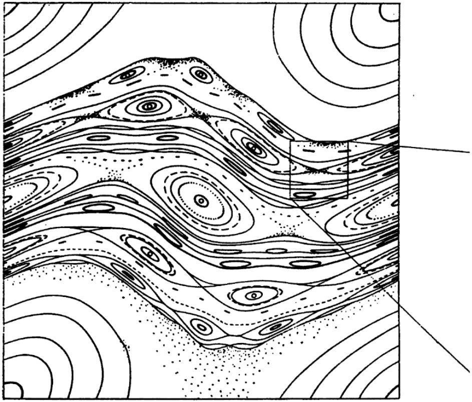

Fig. 26.
Regular and chaotic orbits

V. I. Arnold *Huygens and Barrow, Newton and Hooke* (1990)

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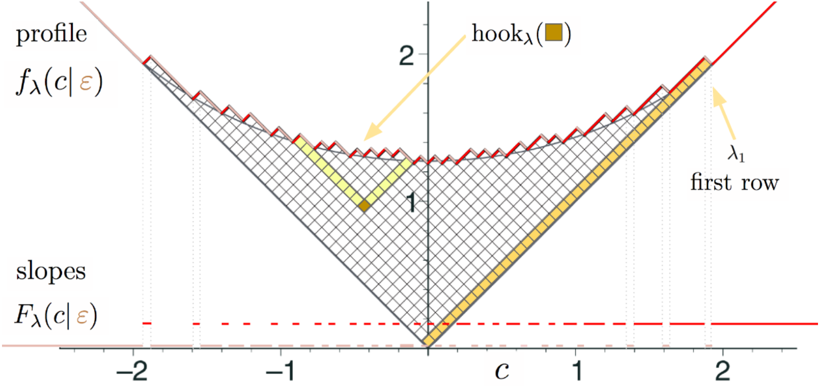
Rewrite from Vantage of Hidden Classical Integrable System

Partitions λ: **microscopic configuration** of squares, mesh

$$\frac{1}{2}\mathrm{vol}(\square) = \varepsilon^2$$

Profiles $f_\lambda(c|\,\varepsilon)$: emergent **macroscopic interface**



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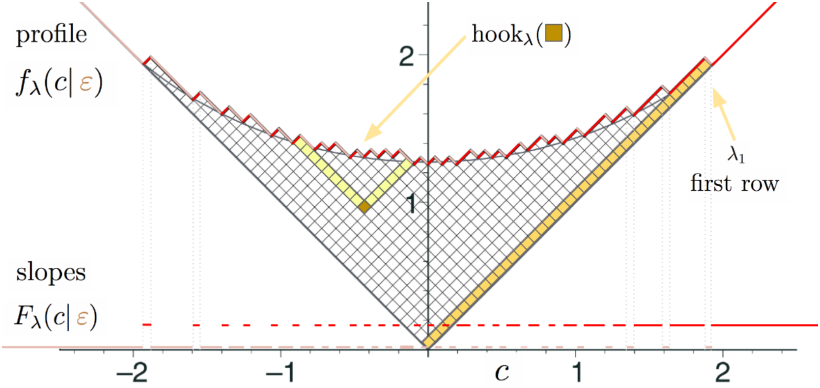
Rewrite from Vantage of Hidden Classical Integrable System

Poissonized Plancherel Measure $\lambda = \{0 \le \cdots \le \lambda_2 \le \lambda_1\}$

$$\mathrm{Prob}_{\varepsilon}(\lambda) \propto \prod_{\square \in \lambda} \left(\frac{1}{\varepsilon \, \mathrm{hook}_{\lambda}(\square)} \right)^2$$

Random Partitions $\lambda \Rightarrow$ Random Profiles $f_{\lambda}(\cdot | \varepsilon) : \mathbb{R} \to \mathbb{R}_{\ge 0}$



Poissonized Plancherel Measure $\lambda = \{0 \le \cdots \le \lambda_2 \le \lambda_1\}$

$$\mathrm{Prob}_{\varepsilon}(\lambda) \propto \prod_{\square \in \lambda} \left(\frac{1}{\varepsilon\, \mathrm{hook}_{\lambda}(\square)} \right)^2$$

Random Partitions $\lambda \Rightarrow$ Random Profiles $f_\lambda(\cdot|\,\varepsilon) : \mathbb{R} \to \mathbb{R}_{\ge 0}$

Let $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$ and $v(w) = w + \frac{1}{w}$.

1. [Kerov-Vershik 1977] As $\varepsilon \to 0$, limit shape

$$dF_\lambda(c|\,\varepsilon) \Rightarrow dF_{\star|v} = d(v_* \boldsymbol{\rho}_{\star|\mathbb{T}})(c)$$

push-forward of **uniform measure** $\boldsymbol{\rho}_{\star|\mathbb{T}}$ along $v(w)$

2. [Kerov 1993] As $\varepsilon \to 0$, Gaussian fluctuations

$$\tfrac{1}{\varepsilon}(F_\lambda(c|\,\varepsilon) - F_{\star|v}(\varepsilon)) \Rightarrow \Phi_{\star|v} := v_* \boldsymbol{\Phi}_{\mathbb{T}}$$

push-forward of Gaussian ***H***$^{1/2}$**-noise** $\boldsymbol{\Phi}_{\mathbb{T}}$ along $v(w)$

Poissonized Plancherel Measure $\lambda = \{0 \leq \cdots \leq \lambda_2 \leq \lambda_1\}$

$$\text{Prob}_{\varepsilon}(\lambda) \propto \prod_{\square \in \lambda} \left(\frac{1}{\varepsilon \, \text{hook}_{\lambda}(\square)} \right)^2$$

Definitions of Gaussian $H^{1/2}$-noise $\mathbf{\Phi}_{\mathbb{T}}$ on $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$

1. For $\mathbb{G}_k(\sigma)$ independent $\mathbb{C}$-Gaussians $\mathbb{E}[\overline{\mathbb{G}}_k(\sigma)\mathbb{G}_k(\sigma)] = \sigma^2 k$,

$$\mathbf{\Phi}_{\mathbb{T}}(w|\sigma^2) = \sum_{k=1}^{\infty} \frac{1}{k} \left(\mathbb{G}_k(\sigma) w^{-k} + \overline{\mathbb{G}}_k(\sigma) w^{+k} \right)$$

2. For $\phi \in H^{1/2}(\mathbb{T})$, pink noise

$$\oint \phi(z)(-\Delta)^{1/2} \mathbf{\Phi}_{\mathbb{T}}(z|\sigma) dz \quad \in \quad \mathcal{N}\left(0, \sigma^2 \sum_{k=-\infty}^{\infty} |k|^1 |\phi_k|^2 \right)$$

3. Periodic Fractional Brownian Motion Hurst index 0
4. Restriction of 2D Gaussian Free Field in any Ω to tiny loop $\mathbb{T}$

Poissonized Plancherel Measure $\lambda = \{0 \leq \cdots \leq \lambda_2 \leq \lambda_1\}$

$$\text{Prob}_{\varepsilon}(\lambda) \propto \prod_{\square \in \lambda} \left(\frac{1}{\varepsilon \, \text{hook}_{\lambda}(\square)} \right)^2$$

1. [Olshanski 2011*] Random Permutations $L^2(S(d), \mathit{Haar})$
2. [Gross-Witten 1980] Random Unitary Matrices with $\varepsilon = \frac{t}{N}$

$$\oint_{\mathbb{T}^N} \prod_{i<j} \left| w_i - w_j \right|^2 \prod_{i=1}^{N} e^{-\frac{N}{t}\left(w_i + \frac{1}{w_i} + \frac{1}{t}\right)} \frac{dw_i}{2\pi \mathbf{i} w_i} = \text{Prob}_{\varepsilon}\left(\lambda_1 \leq N\right)$$

3. [Okounkov 1999] 2D Conformal Field Theory: Free Fermions
4. [Okounkov 2000] Hurwitz Numbers: simply ramified $\Sigma \to \mathbb{P}^1$
5. [Nakajima 1997]$(-\varepsilon, \varepsilon)$-Equivariant Cohomology $\text{Hilb}_d(\mathbb{C}^2)$
6. [Nekrasov 2002] pure $\mathcal{N} = 2$ $U(1)$ Yang-Mills in $(\mathbb{R}^4, g_{-\varepsilon,\varepsilon})$
7. [Prähofer-Spohn 2000] PNG model: droplet initial data
8. [M.] Quantum-Classical Correspondence Principle for

$$\begin{cases} v_t + v v_x = 0 \\ v(x,0) = e^{\mathbf{i}x} + e^{-\mathbf{i}x} \end{cases}$$

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Rewrite from Vantage of Hidden Classical Integrable System

Poissonized Plancherel Measures $\lambda = \{0 \leq \cdots \leq \lambda_2 \leq \lambda_1\}$

$$\mathrm{Prob}_{\varepsilon}(\lambda) \propto \prod_{\square \in \lambda} \left(\frac{1}{\varepsilon \, \mathrm{hook}_{\lambda}(\square)} \right)^2 = \left| \frac{s_{\lambda}(1, 0, 0, \ldots | \, \varepsilon)}{||s_{\lambda}(\cdot | \, \varepsilon)||_{\varepsilon}} \right|^2$$

Schur Polynomials via Jacobi-Trudi Formula $\qquad V_k = \varepsilon \, p_k$

$$s_{\lambda}(V_1, V_2, \ldots \, | \, \varepsilon) = \det \left[\oint_{\mathbb{T}} w^{\lambda_i - i + j} \exp\left(\frac{1}{\varepsilon} \sum_{k=1}^{\infty} \frac{V_k \, w^{-k}}{k} \right) \frac{dw}{2\pi \mathbf{i} w} \right]_{i,j=1}^{\mathrm{length}(\lambda)}$$

are orthogonal for $\langle \cdot, \cdot \rangle_{\varepsilon}$ on $\mathbb{C}[V_1, V_2 \ldots]$ defined by declaring

$$\widehat{\mathcal{V}}_{-k} = \varepsilon^2 \, k \frac{\partial}{\partial V_k}$$

adjoint of operator $\widehat{\mathcal{V}}_{+k}$ of multiplication by V_k. $\widehat{\mathcal{V}}^{\dagger}_{\pm k} = \widehat{\mathcal{V}}_{\mp k}$

- Monomial basis $V_{\mu} = {V_1}^{d_1} {V_2}^{d_2} \cdots$ also orthogonal $\langle V_{\mu}, V_{\mu} \rangle_{\varepsilon} = \prod_{k=1}^{\infty} \langle \widehat{\mathcal{V}}^{d_k}_{-k} \widehat{\mathcal{V}}^{d_k}_{k} \rangle_{\varepsilon} = \prod_{k=1}^{\infty} k! (\varepsilon^2 \, k)^{d_k}$

Poissonized Plancherel Measures $\Leftarrow$ Schur Measures

$$\mathrm{Prob}_{v,\varepsilon}(\lambda) \propto \prod_{\square \in \lambda} \left(\frac{1}{\varepsilon \, \mathrm{hook}_\lambda(\square)} \right)^2 \Leftarrow \left| \frac{s_\lambda(V_1, V_2, \ldots | \varepsilon)}{||s_\lambda(\cdot | \varepsilon)||_\varepsilon} \right|^2$$

Random Partitions $\lambda \Rightarrow$ Random Profiles $f_\lambda(\cdot | \varepsilon) : \mathbb{R} \to \mathbb{R}_{\geq 0}$

Let $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$ and $v : \mathbb{T} \to \mathbb{R}$ Fourier modes V_k

1. [Okounkov 2003] As $\varepsilon \to 0$, limit shape

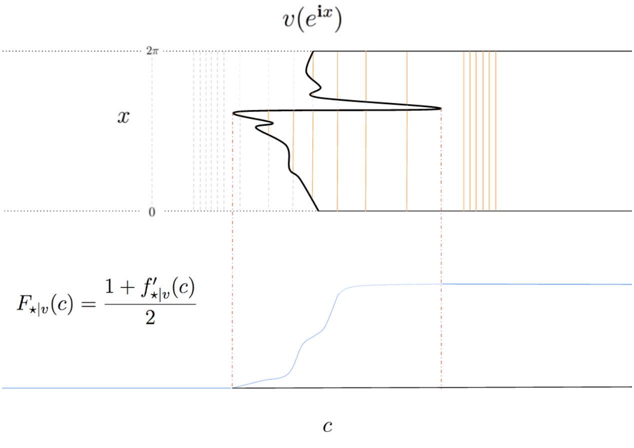
$$dF_\lambda(c | \varepsilon) \Rightarrow dF_{\star|v} = d(v_* \boldsymbol{\rho}_{\star|\mathbb{T}})(c)$$

push-forward of **uniform measure** $\rho_{\star|\mathbb{T}}$ along $v(w)$

2. [M.] As $\varepsilon \to 0$, Gaussian fluctuations

$$\frac{1}{\varepsilon}(F_\lambda(c | \varepsilon) - F_{\star|v}(\varepsilon)) \Rightarrow \Phi_{\star|v} := v_* \boldsymbol{\Phi}_\mathbb{T}$$

push-forward of Gaussian ***$H^{1/2}$*-noise** $\boldsymbol{\Phi}_\mathbb{T}$ along $v(w)$



$$F_{\star|v}(c) = (v_*\boldsymbol{\rho}_{\mathbb{T}})\big((-\infty, c]\big) = \oint_{\mathbb{T}} \mathbb{1}_{v(w)\leq c} \frac{dw}{2\pi\mathbf{i}w}$$

- In one-cut regime due to ***regularity of*** $v : \mathbb{T} \to \mathbb{R}$
- Unlike all $f_\lambda(c|\,\varepsilon)$, limit shape $f_{\star|v}(c)$ is ***convex***
- Match [Breuer-Duits 2013] for biorthogonal ensembles

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Rewrite from Vantage of Hidden Classical Integrable System

$\mathbf{P} = \{\tau \text{ probability measures on } \mathbb{R}\}$
$\mathbf{P}^\vee = \{f : \mathbb{R} \to \mathbb{R} : \text{1-Lipshitz}, \ \int_0^{\pm\infty} \frac{1 \mp f'(c)}{1+|c|} dc < \infty\}$

- Rayleigh function $\quad F_f = \frac{1+f'}{2}$
- Rayleigh measure $\quad dF_f(c) = dF^+(c) - dF^-(c)$
- Interlacing measures $\quad \{dF^-(c), dF^+(c)\}$
- Shifted Rayleigh function $\quad \xi_f = F_f - \mathbb{1}_{[0,\infty)}$

[Kerov 1996] the T-**observable** of a profile $f \in \mathbf{P}^\vee$

$$\int_{-\infty}^{+\infty} \frac{d\tau_f(c)}{u-c} = T(u)\Big|_f = \frac{1}{u} \cdot \exp\left(- \int_{-\infty}^{+\infty} \frac{\xi_f(c)dc}{u-c} \right)$$

Stieltjes transform of $\tau_f \in \mathbf{P}$, the transition measure of f.
This *Kerov-Markov-Krein Transform* $\mathbf{P}^\vee \to \mathbf{P}$ is bijective!

- Nota Bene: If F is *bounded variation* then may write

$$T(u)\Big|_f = \exp\left(\int_{-\infty}^{+\infty} \log\left[\frac{1}{u-c}\right] dF_f(c) \right)$$

Theorem [OPRL] If $L_\bullet$ self-adjoint on $H_\bullet$, $\Psi_0 \in H_\bullet$ cyclic for $L_\bullet$, and $L_\bullet\Big|_{\mathrm{span}\{L_\bullet^\ell \Psi_0\}}$ is *essentially self-adjoint*, then

$$\langle \Psi_0 | R_\bullet(u) | \Psi_0 \rangle = T(u) = \frac{\det_{H_+}(u - L_+)}{\det_{H_\bullet}(u - L_\bullet)}.$$

1. Resolvent $R_\bullet(u) = \frac{1}{u - L_\bullet}$ with spectral parameter $u \in \mathbb{C} \setminus \mathbb{R}$
2. Perturbation determinant via Fredholm determinant

$$\frac{\det_{H_\bullet}(u - L_+)}{\det_{H_+}(u - L_\bullet)} = \det_{H_\bullet}\Big(\mathbb{1} - (L_\bullet - L_+) R_\bullet(u)\Big)$$

   can be defined if $L_\bullet - L_+$ is trace class
3. $\Psi_0 \in H_\bullet$ is cyclic for $L_\bullet$ if span of $\{L_\bullet^\ell\}_{\ell=0}^\infty$ dense in $H_\bullet$
4. $H_0 = \mathbb{C}\,|\Psi_0\rangle$, $H_\bullet = H_0 \oplus H_+$ with projections $\pi_\bullet = \pi_0 + \pi_+$
5. (Ψ_0, Ψ_0)-minor L_+ of $L_\bullet$ defined by $L_+ = \pi_+ L_\bullet \pi_+$

Theorem [OPRL] If $L_\bullet$ self-adjoint on $H_\bullet$, $\Psi_0 \in H_\bullet$ cyclic for $L_\bullet$, and $L_\bullet|_{\mathrm{span}\{L_\bullet^\ell \Psi_0\}}$ is *essentially self-adjoint*, then

$$\langle \Psi_0 | R_\bullet(u) | \Psi_0 \rangle = T(u) = \frac{\det_{H_+}(u - L_+)}{\det_{H_\bullet}(u - L_\bullet)}.$$

- For $||\Psi_0|| = 1$, spectral measure τ of $L_\bullet$ at $\Psi_0 \in H_\bullet$

$$\int_{-\infty}^{+\infty} \frac{d\tau(c)}{u - c} = \langle \Psi_0 | R_\bullet(u) | \Psi_0 \rangle = T(u) \qquad \text{[Nevanlinna 1922]}$$

- $L_\bullet, L_+$ self-adjoint, $L_\bullet - L_+$ trace class, spectral shift function

$$\text{[Krein 1953]} \qquad T(u) = \frac{\det_{H_\bullet}(u - L_+)}{\det_{H_\bullet}(u - L_\bullet)} = \exp\left(-\int_{-\infty}^{+\infty} \frac{\xi(c)dc}{u - c} \right)$$

Corollary [Kerov 1996*] $T(u)$ is T-**observable** of $f \in \mathbf{P}^\vee$

$$\int_{-\infty}^{+\infty} \frac{d\tau_f(c)}{u - c} = T(u)\Big|_f = \frac{1}{u} \cdot \exp\left(-\int_{-\infty}^{+\infty} \frac{\xi_f(c)dc}{u - c} \right)$$

- transition measure τ_f is spectral measure of $L_\bullet$ at Ψ_0 is
- shifted Rayleigh $\xi_f = F_f - \mathbb{1}_{[0,\infty)}$ spectral shift of $L_\bullet, L_+$

$L_\bullet$ *Toeplitz operators* $L_\bullet(v)$ scalar symbol $v : \mathbb{T} \to \mathbb{R}$

$$v(w) = \sum_{k=-\infty}^{+\infty} V_k w^{-k} \quad \Rightarrow \quad L_\bullet(v) = \begin{bmatrix} V_0 & V_1 & V_2 & \cdots \\ V_{-1} & V_0 & V_1 & \cdots \\ V_{-2} & V_{-1} & V_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- <u>Reality</u> $v : \mathbb{T} \to \mathbb{R}$ .... <u>Conjugacy</u> $V_{-k} = \overline{V_k}$
- <u>Regularity</u> of v .... <u>Decay</u> of V_k
- <u>Delocalized</u> $\chi^{(k)}(w) = w^k$ .. <u>Localized</u> $\chi_h^{(k)} = \delta(h-k) = |k\rangle$
- $L(v)$ multiplication on $H = L^2(\mathbb{T})$ $\quad L(v)\phi(w) = v(w)\phi(w)$
- $H_\bullet$ Hardy space for $\mathbb{C}[w]$ in H with Szegő projection $\pi_\bullet$
- $H_0 = \mathbb{C}\,|0\rangle$ with projection π_0
- H_+ from $H_\bullet = H_0 \oplus H_+$ with Szegő projection π_+
- **Toeplitz Operator** with symbol v $\quad L_\bullet(v) = \pi_\bullet L(v) \pi_\bullet$
- **Toeplitz** $(0,0)$-**Minor** $\quad L_+(v) = \pi_+ L_\bullet(v) \pi_+$

Theorem* [OPRL] If $L_\bullet$ self-adjoint on $H_\bullet$, $\Psi_0 \in H_\bullet$ cyclic for $L_\bullet$, and $L_\bullet|_{\mathrm{span}\{L_\bullet^\ell \Psi_0\}}$ is *essentially self-adjoint*, then

$$\langle \Psi_0 | R_\bullet(u) | \Psi_0 \rangle = T(u) = \frac{\det_{H_+}(u - L_+)}{\det_{H_\bullet}(u - L_\bullet)}.$$

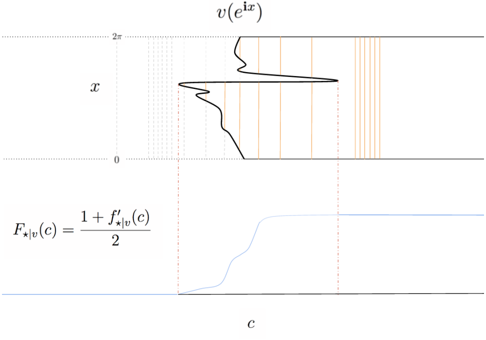
[Szegő 1915] perturbation det is geometric mean

$$\frac{\det_{H_+}(u - L_+(v))}{\det_{H_\bullet}(u - L_\bullet(v))} = \exp\left(\oint_{\mathbb{T}} \log\left[\frac{1}{u - v(w)}\right] \frac{dw}{2\pi \mathbf{i} w} \right)$$

Corollary* [M.] our ***limit shape*** $f_{\star|v} \in \mathbf{P}^\vee$ has T-observable

$$\int_{-\infty}^{+\infty} \frac{d\tau_{\star|v}(c)}{u - c} = T(u)\Big|_{f_{\star|v}} = \frac{1}{u} \cdot \exp\left(- \int_{-\infty}^{+\infty} \frac{\xi_{\star|v}(c)\,dc}{u - c} \right)$$

- transition measure $\tau_{\star|v}(c)$ is *spectral measure* of Toeplitz operator $L_\bullet(v)$ at vector $\Psi_0 = |0\rangle \in H_\bullet$
- shifted Rayleigh function $\xi_{\star|v}(c) = F_{\star|v}(c) - \mathbb{1}_{[0,\infty)}$ *spectral shift function* of self-adjoint $L_\bullet(v), L_+(v)$



$$F_{\star|v}(c) = (v_*\boldsymbol{\rho}_{\mathbb{T}})\big((-\infty, c]\big) = \oint_{\mathbb{T}} \mathbb{1}_{v(w)\leq c} \frac{dw}{2\pi \mathbf{i} w}$$

- [M.] Toeplitz spectral shifts as **conserved density** $dF_{\star|v}(c)$ of the classical Hopf equation

$$\begin{cases} v_t + v v_x = 0 \\ \quad v(x,0) = v(x) \text{ periodic \& analytic} \end{cases}$$

$L_\bullet$ *Toeplitz operators* $L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))$ with $\widehat{\mathfrak{gl}_1}$ symbol "$\hbar$" $= \varepsilon^2$

$$\widehat{\mathbf{v}}(w|\,\varepsilon) = \sum_{k=-\infty}^{+\infty} \widehat{\mathcal{V}}_{+k}(\varepsilon) w^{-k} \;\Rightarrow\; L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon)) = \begin{bmatrix} 0 & \widehat{\mathcal{V}}_{+1} & \widehat{\mathcal{V}}_{+2} & \cdots \\ \widehat{\mathcal{V}}_{-1} & 0 & \widehat{\mathcal{V}}_{+1} & \cdots \\ \widehat{\mathcal{V}}_{-2} & \widehat{\mathcal{V}}_{-1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- $\widehat{\mathcal{V}}_{+k}(\varepsilon)$ multiplication by V_k on $\mathbb{C}[V_k]$ $\qquad k = 1, 2, \ldots$
- $\widehat{\mathcal{V}}_{-k}(\varepsilon) = \varepsilon^2\, k \frac{\partial}{\partial V_k}$ weighted differentiation $\qquad \hbar\omega_k = \varepsilon^2\, k$
- Fock Space $\mathcal{F} = \mathbb{C}[V_1, V_2, \ldots]$ $\qquad \langle \cdot, \cdot \rangle_\varepsilon$ so $\widehat{\mathcal{V}}^\dagger_{\pm k} = \widehat{\mathcal{V}}_{\mp k}$
- Fock-$L^2(\mathbb{T})$ Space $\qquad \mathcal{H} = \mathcal{F} \otimes \mathbb{C}[w, w^{-1}]$
- Fock-Hardy Space $\qquad \mathcal{H}_\bullet = \mathcal{F} \otimes \mathbb{C}[w]$
- Fock-Vacuum $\qquad \mathcal{H}_0 = \mathcal{F} \otimes H_0$
- Fock-Complement $\qquad \mathcal{H}_+ = \mathcal{F} \otimes H_+$
- **Fock-Block Toeplitz Operator** $\qquad L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))$
- **Fock-Block Toeplitz** $(\mathcal{H}_0, \mathcal{H}_0)$**-Minor** $\qquad L_+(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))$

Theorem* [OPRL] If $L_\bullet$ self-adjoint on $H_\bullet$, $\Psi_0 \in H_\bullet$ cyclic for $L_\bullet$, and $L_\bullet|_{\mathrm{span}\{L_\bullet^\ell \Psi_0\}}$ is *essentially self-adjoint*, then

$$\langle \Psi_0 | R_\bullet(u) | \Psi_0 \rangle = T(u) = \frac{\det_{H_+}(u - L_+)}{\det_{H_\bullet}(u - L_\bullet)}.$$

[Nazarov-Sklyanin 2013] $L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))$ resolvent VEV $\mathcal{H}_0 \to \mathcal{H}_0$

$$(\pi_0 \otimes \mathbb{1})\widehat{\mathcal{R}}(u|\,\varepsilon)(\pi_0 \otimes \mathbb{1}) =: \widehat{\mathcal{T}}(u|\,\varepsilon)$$

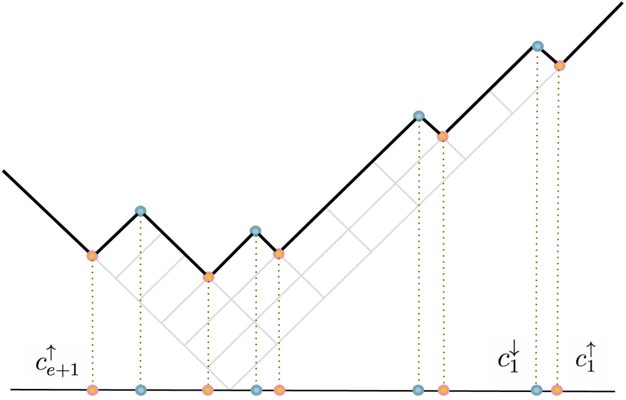
simultaneously diagonalized on Schur polynomials!!!!!!!!!!!!!!

$$\widehat{\mathcal{T}}(u|\,\varepsilon)\Big|_{s_\lambda(V_1, V_2, \ldots)} = T(u)\Big|_{f_\lambda(\cdot|\,\varepsilon)}.$$

Corollary* [M.] partition ***profiles*** $f_\lambda(\cdot|\,\varepsilon) \in \mathbf{P}^\vee$ T-observable

$$\int_{-\infty}^{+\infty} \frac{d\tau_\lambda(c|\,\varepsilon)}{u - c} = T(u)\Big|_{f_\lambda(\cdot|\,\varepsilon)} = \frac{1}{u} \cdot \exp\left(- \int_{-\infty}^{+\infty} \frac{\xi_\lambda(c|\,\varepsilon) dc}{u - c} \right)$$

- transition measure $\tau_\lambda(c|\,\varepsilon)$ is *spectral measure* of Fock-Block Toeplitz operator $L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))$ at $\Psi_\lambda(\cdot|\,\varepsilon) = |0\rangle \otimes s_\lambda(\cdot|\,\varepsilon) \in \mathcal{H}_\bullet$
- shifted Rayleigh function $\xi_\lambda(c|\,\varepsilon) = F_\lambda(c|\,\varepsilon) - \mathbb{1}_{[0,\infty)}$ *spectral shift function* of self-adjoint $L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon)), L_+(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))$



- [M.] interlacing sequence of extrema of profile $f_\lambda(c|\,\varepsilon)$ as eigenvalues of $L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))|_{\mathcal{H}_\lambda(\varepsilon)}, L_+(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))|_{\mathcal{H}_\lambda(\varepsilon)}$ where $\mathcal{H}_{\bullet,\lambda}(\varepsilon)$ is the *finite-dimensional* **orbit** of $\Psi_\lambda(\cdot|\,\varepsilon) = s_\lambda(\cdot|\,\varepsilon) \otimes |0\rangle$ under $L_\bullet(\widehat{\mathbf{v}}(\cdot|\,\varepsilon))$
- Compare [Nekrasov 2016], [Kimura-Pestun 2016], [Maulik-Okounkov 2012]

Poissonized Plancherel Measures $\Leftarrow$ Schur Measures

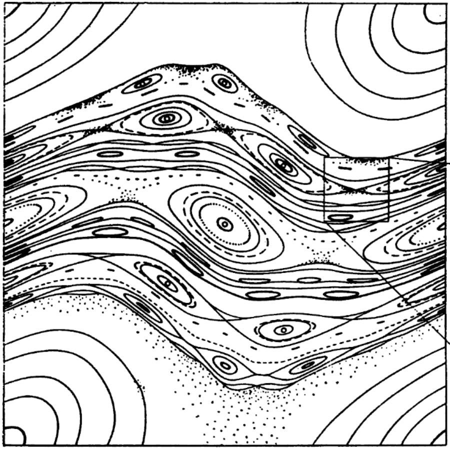
$$\mathrm{Prob}_{v,\varepsilon}(\lambda) \propto \prod_{\square\in\lambda}\left(\frac{1}{\varepsilon\,\mathrm{hook}_\lambda(\square)}\right)^2 \Leftarrow \left|\frac{s_\lambda(V_1, V_2, \ldots |\,\varepsilon)}{||s_\lambda(\cdot|\,\varepsilon)||_\varepsilon}\right|^2$$

1. [Olshanski 2011*] Uniform Permutations $L^2(S(d), Haar)$
2. [Gross-Witten 1980] Random Unitary Matrices with $\varepsilon = \frac{t}{N}$

$$\oint_{\mathbb{T}^N}\prod_{i<j}\left|w_i - w_j\right|^2\prod_{i=1}^N e^{-\frac{N}{t}\left(w_i+\frac{1}{w_i}+\frac{1}{t}\right)}\frac{dw_i}{2\pi\mathbf{i}w_i} = \mathrm{Prob}_\varepsilon\left(\lambda_1 \le N\right)$$

3. [Okounkov 1999] 2D Conformal Field Theory: Free Fermions
4. [Okounkov 2000] Hurwitz Numbers: simply ramified $\Sigma \to \mathbb{P}^1$
5. [Nakajima 1997]$(-\varepsilon,\varepsilon)$-Equivariant Cohomology $\mathrm{Hilb}_d(\mathbb{C}^2)$
6. [Nekrasov 2002] pure $\mathcal{N} = 2$ $U(1)$ Yang-Mills in $(\mathbb{R}^4, \Omega_{-\varepsilon,\varepsilon})$
7. [Prähofer-Spohn 2000] PNG model: droplet initial data
8. [M.] Quantum-Classical Correspondence Principle for

$$\begin{cases} v_t + vv_x = 0 \\ \quad v(x,0) = v(x) \text{ periodic \& analytic} \end{cases}$$



81

Fig. 26.
Regular and chaotic orbits

The Fortune of Integrability: a ***promise*** and a ***warning***

V. I. Arnold *Huygens and Barrow, Newton and Hooke* (1990)

Le Bateleur *Le Tarot de* ***Marseille*** (1715)