Regionally proximal relation of higher order: results and questions

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Sections

1. Background

In 1977 Furstenberg gave a beautiful ergodic proof of the well known Szemeredi's theorem.

One of the main steps in the proof is the so-called: multiple ergodic recurrent theorem. This leads the question: Does

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\dots f_d(T^{dn}x), \tag{1.1}$$

converge in L^2-norm or a.e.? where $(X,\mathcal{B},\mu,T)$ is a measure preserving transformation, $f_i \in L^\infty$, $1 \le i \le d$.

1. Background

The general form is:

Let Γ be a nilpotent group acting on a probability space $(X, \mathcal{X}, \mu)$ and $T_1, T_2, \ldots, T_d \in \Gamma$, do the averages

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{k}(T_1^{p_{1,j}(n)}T_2^{p_{2,j}(n)}\cdots T_d^{p_{d,j}(n)})f_j$$

always converge in L^2-norm or a.e. for every $f_1, \ldots, f_k \in L^\infty$ and every set of integer valued polynomials $p_{i,j}$?

Remark: it is not true for solvable groups (Bergelson+Leibman).

1. Background

In this talk I only focus on the corresponding topological aspects related to the above question.

The way Host-Kra solved the question for a single transformations (L^2 convergence) is that for each $d \in \mathbb{N}$

- For any (X, μ, T) construct a factor Z_d.
- Show that Z_d is a characteristic factor.
- Show that Z_d is an inverse limit of d-step nilsystems (system of order d).

Then a natural question is:

For a minimal system (X, T), how to define an equivalence relation R (closed, invariant) such that X/R is the maximal factor which is a system of order d?

Difficulty: (1) Find a suitable definition of R. (2) Show it an equivalence relation. (3) Prove X/R is a system of order d.

1. Background

Let G be a group. For $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h and we write $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$.

The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting

$$G_1 = G, \text{ and } G_{j+1} = [G_j, G].$$

Let $k \geq 1$ be an integer. We say that G is k-step nilpotent if $G_{k+1} = \{e_G\}$ is the trivial subgroup.

1. Background

Let G be a k-step nilpotent Lie group and Γ a discrete cocompact subgroup of G. The compact manifold $X = G/\Gamma$ is called a k-step nilmanifold.

The group G acts on X by left translations and we write this action as $(g, x) \mapsto gx$. The Haar measure μ of X is the unique probability measure on X invariant under this action. Let $\tau \in G$ and T be the transformation

$$g\Gamma \mapsto (\tau g)\Gamma, g \in G$$

of X. Then (X, T, μ) is called a k-step nilsystem. When the measure is not needed for results, we omit and write that (X, T) is a k-step nilsystem.

1. Background

Definition (Host-Kra-Maass, 2010)

A t.d.s. is a system of order d ($d \in \mathbb{N}$) if it is an inverse limit of d-step nilsystems.

Definition (Dong-Donoso-Maass-Shao-Ye, 2013)

A t.d.s. is a system of order ∞ if it is an inverse limit of d_i-step nilsystems.

2. $\mathbb{Z}$-actions

Now return to the 'natural question'. As I said before, it is a hard one.

For $d = 1$, one defined the regionally proximal relation Q and showed for a minimal system, it is an equivalence relation, and X/Q is the maximal equicontinuous factor. (Ellis-Gottschalk, Veech, Ellis-Keynes, McMahon).

Now we state the definition of regionally proximal relation of order d introduced by Host-Kra-Maass.

Definition

Let (X, T) be a t.d.s. and let $d \in \mathbb{N}$. The points $x, y \in X$ are said to be **regionally proximal of order** d *if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta, \rho(y, y') < \delta$, and*

$$\rho(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta$$

for any $\epsilon \in \{0,1\}^d \setminus \{0, \ldots, 0\}$.

The set of all such pairs is denoted by $\mathbf{RP}^{[d]}$, and is called **the regionally proximal relation of order** d.

2. $\mathbb{Z}$-actions

Host-Kra-Maass (2010) showed that

Theorem

Let (X, T) be a minimal distal system [a]. Then $\mathbf{RP}^{[d]}$ is an equivalence relation, and $X/\mathbf{RP}^{[d]}$ is the maximal factor of system of order d.

[a] (X, T) is distal if for any $x \neq y$, $\inf_{n \in \mathbb{Z}} \rho(T^n x, T^n y) > 0$

• To show $\mathbf{RP}^{[d]}$ is an equivalence relation they passed to $\mathbb{Z}^d$ and $\mathbb{Z}^{d+1}$ actions on $X^{2^d} = X^{[d]}$.

• To prove $X/\mathbf{RP}^{[d]}$ is the maximal factor of system of order d they used ergodic method.

2. $\mathbb{Z}$-actions

Now we introduce $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ actions defined by H-K-M.

For $d = 1$, $\mathcal{F}^{[1]}$ is generated by $id \times T$, and thus

$$\mathcal{F}^{[1]} = \{id \times T^n : n \in \mathbb{Z}\},$$

acting on $X^2 = X^{[1]}$.

$\mathcal{G}^{[1]}$ is generated by $\mathcal{F}^{[1]}$ and $T \times T$,

$$\mathcal{G}^{[1]} = \{(T^n, T^{n+n_1}) : n, n_1 \in \mathbb{Z}\}.$$

2. $\mathbb{Z}$-actions

For $d = 2$, $\mathcal{F}^{[2]}$ is generated by

$$id \times id \times T \times T = id^{[1]} \times T^{[1]}, \quad id \times T \times id \times T = (id \times T)^{[1]}$$

and thus

$$\mathcal{F}^{[2]} = \{(id, T^{n_1}, T^{n_2}, T^{n_1+n_2}) : n_1, n_2 \in \mathbb{Z}\}$$

acting on $X^4 = X^{[2]}$. $\mathcal{G}^{[2]}$ is generated by $\mathcal{F}^{[2]}$ and $T \times T \times T \times T$. Hence

$$\mathcal{G}^{[2]} = \{(T^n, T^{n+n_1}, T^{n+n_2}, T^{n+n_1+n_2}) : n, n_1, n_2 \in \mathbb{Z}\}.$$

2. $\mathbb{Z}$-actions

Generally, if $\mathcal{F}^{[d]}$ is defined, then $\mathcal{F}^{[d+1]}$ is generated by

$$id^{[d]} \times T^{[d]}, F \times F, F \in \mathcal{F}^{[d]}.$$

$\mathcal{G}^{[d+1]}$ is generated by

$$\mathcal{F}^{[d+1]} \text{ and } T^{[d+1]}.$$

Note that $\mathcal{F}^{[d]}$ is generated by d elements, $\mathcal{G}^{[d]}$ is generated by $d+1$ elements. They act on $X^{2^d} = X^{[d]}$.

2. $\mathbb{Z}$-actions

Now let (X,T) be a t.d.s., $x \in X$.

$$\overline{\mathcal{F}^{[d]}}(\mathbf{x}) = \overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$$

for $\mathbf{x} \in X^{2^d}$, and

$$\mathbf{Q}^{[d]}(X) = \overline{\mathcal{O}(\mathbf{x}, \mathcal{G}^{[d]})},$$

$$\mathbf{Q}^{[d]}[x] = \{\mathbf{z} \in \mathbf{Q}^{[d]}(X) : z_\emptyset = x\}.$$

2. $\mathbb{Z}$-actions

Host-Kra-Maass showed that $(x, y) \in \mathbf{RP}^{[d]}$ if and only if there is $a_* \in X^{2^d-1}$ such that

$$(x, a_*, y, a_*) \in \mathbf{Q}^{[d+1]}(X)$$

for a minimal system.

They showed that for a minimal distal system

1. $\mathbf{Q}^{[d]}[x] = \overline{\mathcal{F}^{[d]}}(x^{[d]})$ which is minimal for $\mathcal{F}^{[d]}$.
2. $(x, y) \in \mathbf{RP}^{[d]}$ iff $(x, y, \ldots, y) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.
3. $\mathbf{RP}^{[d]}$ is an equivalence relation.

2. $\mathbb{Z}$-actions

For a general minimal system (X, T), Shao-Ye (2012) showed

1. $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for any $x \in X$.
2. $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique minimal set in $\mathbf{Q}^{[d]}[x]$.
3. $(x, y) \in \mathbf{RP}^{[d]}$ iff $(x, y, \ldots, y) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.
4. $\mathbf{RP}^{[d]}$ is an equivalence relation.
5. if $\pi : X \longrightarrow Y$ is a factor map , then $\pi \times \pi(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$.

By (5) and Host-Kra-Maass we know that $X/\mathbf{RP}^{[d]}$ is the maximal factor of system of order d.

2. $\mathbb{Z}$-actions

Veech (1968) showed that for a miniaml t.d.s. $(x, y) \in \mathbf{RP}^{[1]}$ if and only if $N(x, U)$ contains a Δ-set for any neighborhood U of y. Note that

$$N(x, U) = \{n \in \mathbb{Z} : T^n x \in U\}.$$

Huang-Lu-Ye (2011) showed that for a miniaml t.d.s. $(x, y) \in \mathbf{RP}^{[1]}$ if and only if $N(x, U)$ contains a Poincare-set for any neighborhood U of y.

In fact $\mathbf{RP}^{[d]}$ can be characterized by interesting subsets of $\mathbb{Z}$.

2. $\mathbb{Z}$-actions

Theorem (Huang-Shao-Ye, 2016, M. AMS)

Let (X,T) be a minimal t.d.s. and $x,y \in X$. Then the following statements are equivalent for $d \in \mathbb{N} \cup \{\infty\}$:

1. $(x,y) \in \mathbf{RP}^{[d]}$.
2. $N(x,U) \in \mathcal{F}^*_{d,0}$ *for each neighborhood U of y.*
3. $N(x,U) \in \mathcal{F}_{Poi_d}$ *for each neighborhood U of y.*
4. $N(x,U) \in \mathcal{F}_{Bir_d}$ *for each neighborhood U of y.*
5. $N(x,U) \in \mathcal{F}_{SG_d}$ *for each neighborhood U of y.*
6. $N(x,U) \in \mathcal{F}_{B_d}$ *for each neighborhood U of y.*
7. $N(x,U) \in \mathcal{F}_{P_d}$ *for each neighborhood U of y.*

2. $\mathbb{Z}$-actions

Definition

A subset A of $\mathbb{Z}$ is a Nil$_d$ Bohr$_0$-set if there exist a d-step nilsystem (X,T), $x_0 \in X$ and an open neighborhood U of x_0 such that $N(x_0,U) =: \{n \in \mathbb{Z} : T^n x_0 \in U\}$ is contained in A. Denote by $\mathcal{F}_{d,0}$ the family consisting of all Nil$_d$ Bohr$_0$-sets. $\mathcal{F}^*_{d,0}$ the family of sets intersecting all Nil$_d$ Bohr$_0$-sets.

A collection $\mathcal{F}$ of subsets of $\mathbb{Z}$ (or $\mathbb{N}$) is a family if it is hereditary upward, i.e. $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$.

Any nonempty collection $\mathcal{A}$ of subsets of $\mathbb{Z}$ generates a family

$$\mathcal{F}(\mathcal{A}) := \{F \subset \mathbb{Z} : F \supset A \text{ for some } A \in \mathcal{A}\}.$$

2. $\mathbb{Z}$-actions

Definition (Host-Kra)

Let $d \geq 1$ be an integer and let $P = \{p_i\}_i$ be a (finite or infinite) sequence in $\mathbb{Z}$. The set of sums with gaps of length less than d of P is the set $SG_d(P)$ of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \ldots + \epsilon_n p_n$$

where $n \geq 1$ is an integer, $\epsilon_i \in \{0, 1\}$ for $1 \leq i \leq n$, the ϵ_i are not all equal to 0, and the blocks of consecutive 0's between two 1 have length less than d.

2. $\mathbb{Z}$-actions

A subset A of $\mathbb{Z}$ is an SG_d-set if $A = SG_d(P)$ for some infinite sequence in $\mathbb{Z}$; and it is an SG_d^*-set if $A \cap SG_d(P) \neq \emptyset$ for every infinite sequence P in $\mathbb{Z}$.

Let $\mathcal{F}_{SG_d}$ be the family generated by all SG_d-sets.

Note that each SG_1-set is a Δ-set, and each SG_1^*-set is a Δ^*-set.

2. $\mathbb{Z}$-actions

Theorem (Host-Kra (2010))

Every SG_d^*-set is a PW-Nil$_d$ Bohr$_0$-set.

Question (Host and Kra)

Is it true $\mathcal{F}_{d,0} \subset \mathcal{F}^*_{SG_d}$?

By Huang-Shao-Ye we know that $\mathcal{F}_{d,0}$ and $\mathcal{F}^*_{SG_d}$ are the 'same' when considering $\mathbf{RP}^{[d]}$. The best answer for the question was obtained

Theorem (Konieczny, arXiv:1507.07370)

$\mathcal{F}_{d,0} \subset \mathcal{F}^*_{SG_k}$ provided that $k \geq \frac{1}{2}d(d+1)$. ($k = 4b$ in the published version.)

2. $\mathbb{Z}$-actions

To state another question we give some definitions. Let $d \in \mathbb{N}$.

1. We say that $S \subset \mathbb{Z}$ is a set of d-recurrence if for every measure preserving system $(X, \mathcal{X}, \mu, T)$ and for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $n \in S$ such that
$$\mu(A \cap T^{-n}A \cap \ldots \cap T^{-dn}A) > 0.$$
2. We say that $S \subset \mathbb{Z}$ is a set of d-topological recurrence if for every minimal t.d.s. (X, T) and for every nonempty open subset U of X, there exists $n \in S$ such that
$$U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset.$$

2. $\mathbb{Z}$-actions

Let $\mathcal{F}_{Poi_d}$ (resp. $\mathcal{F}_{Bir_d}$) be the family consisting of all sets of d-recurrence (resp. sets of d-topological recurrence).

Question (Huang-Shao-Ye)

Is it true that $\mathcal{F}_{Bir_d} = \mathcal{F}^*_{d,0}$?

We remark that by Huang-Shao-Ye:

$$\mathcal{F}_{Bir_d} \subset \mathcal{F}^*_{d,0}.$$

We also remark that for $d = 1$ this is related to Katznelson's question.

2. $\mathbb{Z}$-actions

Veech showed that $(x,y) \in \mathbf{RP}^{[1]}$ if and only if there exist $\{n_i\}$ and $z \in X$ such that

$$T^{n_i}x \longrightarrow z \text{ and } T^{-n_i}z \longrightarrow y.$$

Question (HSY, Auslander)

How can we generalize Veech's result for $\mathbf{RP}^{[d]}$, $d \geq 2$?

3. High order AA

A minimal system is almost automorphy if it is an almost one-to-one extension of its maximal equicontinuous factor. Generalizing the notion we have

Definition

Let $d \in \mathbb{N} \cup \{\infty\}$. A minimal system is d-step almost automorphy if it is an almost one-to-one extension of its maximal factor of system of order d.

3. High order AA

Theorem (HSY)

Let (X,T) be a minimal t.d.s., $x \in X$ and $d \in \mathbb{N} \cup \{\infty\}$. Then the following statements are equivalent:

1. *x is a d-step AA point.*
2. *$N(x,V) \in \mathcal{F}_{d,0}$ for each neighborhood V of x.*
3. *$N(x,V) \in \mathcal{F}^*_{Poi_d}$ for each neighborhood V of x.*
4. *$N(x,V) \in \mathcal{F}^*_{Bir_d}$ for each neighborhood V of x.*

3. High order AA

As a special case of the above theorem we have

Theorem (HSY, Theorem 8.1.7)

Let (X, T) be a minimal t.d.s. Then (X, T) is ∞-step AA if and only if there is $x \in X$ such that $N(x, V) \in \mathcal{F}_{fip}^*$ for each neighborhood V of x.

Note that $\mathcal{F}_{fip}$ is the family consisting of all sequences containing arbitrarily long finite IP-sets.

3. High order AA

To show the above theorem we used the Ramsey property of the families. Moreover, we showed that the family $\mathcal{F}_{SG_d}$ does not have the Ramsey property.

Question (Huang-Shao-Ye)

Let (X, T) be a minimal t.d.s., $x \in X$ and $d \in \mathbb{N} \cup \{\infty\}$. Is it true that the following statements are equivalence?

1. *x is a d-step AA point.*
2. *$N(x, V) \in \mathcal{F}^*_{SG_d}$ for each neighborhood V of x.*

Now we present a recent result by Glasner-Gutman-Ye.

Let G be a group and (X, G) be minimal. Assume that G is abelian. For $d = 2$, $\mathcal{F}^{[2]}$ is generated

$$id \times id \times h_2 \times h_2, \;\; id \times h_1 \times id \times h_1, \;\; h_1, h_2 \in G.$$

Thus,

$$\mathcal{F}^{[2]} = \{(id, h_1, h_2, h_1 h_2) : h_1, h_2 \in G\}.$$

$\mathcal{G}^{[2]}$ is generated by $\mathcal{F}^{[2]}$ and $h \times h \times h \times h, \;\; h \in G$. Hence

$$\mathcal{G}^{[2]} = \{(h, hh_1, hh_2, hh_1 h_2) : h, h_1, h_2 \in G\}.$$

The proof of Shao-Ye works similarly.

4. Group-actions

For a general group, and $d = 2$, $\mathcal{F}^{[2]}$ is generated by

$$id \times id \times h \times h, \;\; id \times t \times id \times t, \quad t, s \in G.$$

Thus

$$\mathcal{F}^{[2]} = \{(id, t_1 \cdots t_l, h_1 \cdots h_j, t_1 h_1 \cdots t_l h_l) : t_l, h_l \in G, l \in \mathbb{N}\}.$$

$\mathcal{G}^{[2]}$ is generated by $\mathcal{F}^{[2]}$ and $\Delta_2 =: \{h \times h \times h \times h : h \in G\}$. We may define $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ similarly. We have that:

$$\mathcal{G}^{[d]} = \mathcal{F}^{[d]} \Delta_d.$$

4. Group-actions

Definition

Let $d \geq 1$ be an integer and assume that L is a nilpotent Liegroup of nilpotency class d and $\Gamma \subset L$ a discrete, cocompact subgroup of L. Denote $X = L/\Gamma$. Notice that L acts naturally on X by left translations: $l\Gamma \to gl\Gamma$ for $g \in L$.

*Let G be a topological group and let $\phi : G \to L$ be a continuous homomorphism, then the induced action (G, X) is called a d**-step nilsystem.***

$$g(l\Gamma) \to (\phi(g)l)\Gamma.$$

The inverse limit of d-step nilsystems is called a system of order d or a pro-nilsystem of order d.

4. Group-actions

Glasner-Gutman-Ye (2016) define a relation $\mathbf{NRP}^{[d]}$ for any G.

Definition

Let G be a topological group, (X, G) be a t.d.s. $(x, y) \in \mathbf{NRP}^{[d]}$ if

$$(x, x, \dots, x, y) \in \mathbf{Q}^{[d+1]}$$

If G is abelian and (X, G) is minimal, then $\mathbf{NRP}^{[d]} = \mathbf{RP}^{[d]}$ by Shao-Ye.

4. Group-actions

Glasner-Gutman-Ye showed

1. $(\overline{\mathcal{F}^{[d]}(x^{[d]})}, \mathcal{F}^{[d]})$ is minimal for any $x \in X$.
2. $(\overline{\mathcal{F}^{[d]}(x^{[d]})}, \mathcal{F}^{[d]})$ is the unique minimal set in $\mathbf{Q}^{[d]}[x]$.
3. $(x, y) \in \mathbf{NRP}^{[d]}$ iff $(x, y, \ldots, y) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.
4. $\mathbf{RP}^{[d]}$ is an equivalence relation.
5. if $\pi : X \longrightarrow Y$ is a factor map , then $\pi \times \pi(\mathbf{NRP}^{[d]}(X)) = \mathbf{NRP}^{[d]}(Y)$.

4. Group-actions

Theorem (GGY)

When G is compactly generated, $X/\mathbf{NRP}^{[d]}$ is a system of order d (or pro-nilsystem of order d). (lifting property+ Gutman-Manners-Varju's work for a minimal distal system).

• Host-Kra-Maass used ergodic method for $\mathbb{Z}$-actions.

• Gutman-Manners-Varju used topological and combinatorial method with more details for the former work by Camarena and Szegedy.

• For $d = 1$, $X/\mathbf{NRP}^{[1]}$ is the maximal abelian factor.

4. Group-actions

Let (X, G) be minimal. In fact we also have

1. There is a dense G_δ subset X_0 of X such that for each $x \in X_0$, $\mathbf{Q}^{[d]}[x] = \overline{\mathcal{F}^{[d]}}(x^{[d]})$. (This is new even for $\mathbb{Z}$-actions).

   There is an example (Tu-Ye, 2013) for some $x \in X$, $\mathbf{Q}^{[d]}[x] \neq \overline{\mathcal{F}^{[d]}}(x^{[d]})$.

2. If in addition (X, G) is distal, then $\mathbf{Q}^{[d]}[x] = \overline{\mathcal{F}^{[d]}}(x^{[d]})$ which is minimal under $\mathcal{F}^{[d]}$ for any $x \in X$.

4. Group-actions

We can define another relation.

Definition

Let G be a group and (X, G) be a t.d.s. $(x, y) \in \mathbf{RP}^{[d]}$ if for any $\epsilon > 0$ and for any neighborhood $U \times V$ of (x, y) there are $(x', y') \in U \times V$ and $F = (id, F_1, \ldots, F_{2^d-1}) \in \mathcal{F}^{[d]}$ such that

$$\rho(F_i x', F_i y') < \epsilon, \ \ 1 \le i \le 2^d - 1.$$

It is not hard to show that

$$\mathbf{RP}^{[d]} \subset \mathbf{NRP}^{[d]}.$$

4. Group-actions

Again if G is abelian, then $\mathbf{RP}^{[d]} = \mathbf{NRP}^{[d]}$ by Shao-Ye. The open question is

Question (Glasner-Gutman-Ye)

Let (X, G) be minimal. For which G we have $\mathbf{RP}^{[d]}$ is an equivalence relation?

We remark that when G is a finite non-abelian group, the for the minimal system (G, G),

$$\{(g, g) : g \in G\} = \mathbf{RP}^{[1]} \neq \mathbf{NRP}^{[1]}.$$

4. Group-actions

We guess that $\mathbf{RP}^{[d]}$ is an equivalence relation when G is amenable or G has the property that $(X \times X, G)$ has a dense set of minimal points (this is true when $d = 1$).

It is a very interesting question to determine the structure $X/\mathbf{RP}^{[d]}$ when $\mathbf{RP}^{[d]}$ is an equivalence relation for a non-abelian group G.

5. Related results

Theorem

Let (X, T) be weakly mixing and minimal. Then there is $x \in X$ such that

$$\{T^{n^2}x : n \in \mathbb{Z}\}$$

is dense in X.

In fact it is a special case proved by Huang-Shao-Ye (2016) for nilpotent minimal weakly mixing actions.

5. Related results

Condition $*$:

Let (X, Γ) be a topological system, where Γ is a nilpotent group such that for each $T \in \Gamma$, $T \neq e_\Gamma$, is weakly mixing and minimal. For $d, k \in \mathbb{N}$ let $T_1, \ldots, T_d \in \Gamma$, $\{p_{i,j}(n)\}_{1 \le i \le k, 1 \le j \le d} \in \mathcal{P}_0$ such that the expression

$$g_i(n) = T_1^{p_{i,1}(n)} \ldots T_d^{p_{i,d}(n)}$$

depends nontrivially on n for $i = 1, 2, \ldots, k$, and for all $i \neq j \in \{1, 2, \ldots, k\}$ the expressions $g_i(n) g_j(n)^{-1}$ depend nontrivially on n.

5. Related results

Up till now we could not prove the pointwise convergence of MEA, but we now know the topological analogue holds.

Theorem (Huang-Shao-Ye, 2016)

Assume the condition $*$. Then there is a residual set X_0 of X such that for all $x \in X_0$

$$\{(g_1(n)x, \dots, g_k(n)x) : n \in \mathbb{Z}\}$$

is dense in X^k.

5. Related results

The following question is still open.

Question (Bergelsen)

Let (X,T) be totally minimal [a]. Is there an $x \in X$ such that

$$\{T^{n^2}(x) : n \in \mathbb{Z}\}$$

is dense in X?

[a] (X,T) is totally minimal if (X,T^n) is minimal for $n \neq 0$.

Glasner (1994), Frantzikinakis-Kra (2005), Bergelson and Leibman(1996), Leibman (2005).

5. Related results

Now we consider a relation, denoted by $\mathbf{AP}^{d]}$.

Definition

Let (X,T) be a t.d.s. and $d \in \mathbb{N}$. We say $(x,y) \in X \times X$ is a *regionally proximal pair of order d along arithmetic progressions* *if for each $\delta > 0$ there exist $x', y' \in X$ and $n \in \mathbb{Z}$ such that $\rho(x,x') < \delta, \rho(y,y') < \delta$ and*

$$\rho(T^{in}(x'), T^{in}(y')) < \delta \text{ for each } 1 \leq i \leq d.$$

The set of all such pairs is denoted by $\mathbf{AP}^{[d]}(X)$ and is called the *regionally proximal relation of order d along arithmetic progressions.*

5. Related results

We only have some partial results.

Theorem

Let (X, T) be a unique ergodic minimal distal system such that for each $d \geq 1$, Z_d is isomorphic to X_d. Then for $d \geq 1$, $\mathbf{AP}^{[d]} = \mathbf{RP}^{[d]}$.

Consequently, for a minimal ∞-nilsystem, we have for $d \geq 1$, $\mathbf{AP}^{[d]} = \mathbf{RP}^{[d]}$.

Question (Huang-Shao-Ye)

Is it true that for a minimal distal system, we have for $d \geq 1$, $\mathbf{AP}^{[d]} = \mathbf{RP}^{[d]}$?

Thank you for the attention!