Counting solutions in Intrinsic Diophantine approximation

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Based on joint work with

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- and call $\mathbf{x}$ ψ-approximable if there are infinitely many solutions $(\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N}_+$ to the Diophantine inequality:

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- Dirichlet's theorem. For $\psi(q) = \frac{1}{q^{1/m}}$, every $\mathbf{x} \in \mathbb{R}^d$ is ψ-approximable.

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- then the Borel-Cantelli lemma shows that almost every $\mathbf{x}$ is not ψ-approximable.

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- This sharp dichotomy, giving rise to infinitely many solutions in the divergence case, is very satisfying. Naturally, it raises the following
- Question: If $\mathbf{x}$ is ψ-approximable, so that there are infinitely many solutions to the Diophantine inequality, how many solutions are there of a given bounded size ?

W. Schmidt's theorem in Euclidean space (1960)

- Define the solution counting function at $\mathbf{x}$ (with gauge ψ)

$$N_T(\mathbf{x}) = \left|\left\{(\mathbf{p}, q) \in \mathbb{Z}^d \times N_+ \ ; \ 1 \leq q \leq T \text{ and } \left\|\mathbf{x} - \frac{\mathbf{p}}{q}\right\| \leq \frac{\psi(q)}{q}\right\}\right|$$

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- W. Schmidt's theorem: If $V_T \to \infty$, namely if the divergence case of Khinchin's theorem holds, then for almost every $\mathbf{x} \in \mathbb{R}^d$

$$N_T(\mathbf{x}) = V_T + O_{\mathbf{x},\epsilon}\left(V_T^{\frac{1}{2}+\epsilon}\right)$$

(for all $\epsilon > 0$).

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- Recent (and not so recent) contributors to this theory include Beresnevich, Bernik, Dani, Dickinson, Dodson, Drutu, Kleinbock, Margulis, Paulin, Sprindzuk, Velani....and many others.
- We emphasize that the approximation process here is allowed to utilize all rational points in $\mathbb{R}^d$.

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- The challenge here, however, is that the approximation process is allowed to use only rational points on the variety itself rather than all rational points in $\mathbb{R}^d$.
- Previously, the problem of intrinsic Diophantine approximation has been studied mainly when the variety in question is a commutative algebraic group or an Abelian variety.

Diophantine approximation on algebraic varieties

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- Denote by $\mathrm{X}(\mathbb{R}) \subset \mathbb{R}^N$ the set of real solutions, and by $\mathrm{X}(\mathbb{Q})$ the set of rational solutions. We would like to analyze the system of intrinsic Diophantine inequalities :

$$\|x - r\| \le \epsilon \quad \text{and} \quad D(r) \le \epsilon^{-\kappa}$$

with $x \in X(\mathbb{R})$ and $r \in \mathrm{X}(\mathbb{Q})$, and $D(r)$ the reduced denominator.

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- Our goal is to establish a value of κ that gives an exponent of Diophantine apporoximation, to establish the existence of infinitely many solutions almost surely as in Khintchin's theorem, and then to count their number, as in Schmidt's theorem.

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- and we have also established an effective almost sure approximation exponent, and a Khinchin type result, which are best possible in some cases.
- Main new point : We prove an analog of Schmidt's theorem, establishing an effective uniform asymptotics for the solutions counting functions on group varieties.

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- and we attach to X, G and p two parameters, as follows.

The parameters of diophantine exponents

- Growth parameter. For a general algebraic variety X, we consider the empirical distribution of $\mathrm{X}(\mathbb{Z}[\frac{1}{p}])$ points, and set :

$$a_p(\mathrm{X}) = \sup_{\text{compact } \Omega \subset \mathrm{X}(\mathbb{R})} \limsup_{R \to \infty} \frac{\log |\{r \in \Omega \cap \mathrm{X}(\mathbb{Z}[1/p]) : D(r) \leq R\}|}{\log(R)}.$$

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- Spectral parameter. $q_p(\mathrm{G})$ will denote the integrability parameter of spherical functions appearing in a suitable automorphic representation associated with the group G, as will be explained below.

Uniform and almost sure approximation exponents on homogeneous varieties

We can now state :

- for almost every $x \in \mathrm{X}(\mathbb{R})$, every $\kappa > \frac{\dim \mathrm{G}}{a_p(\mathrm{G})} \frac{q_p(\mathrm{G})}{2}$, and $\epsilon \in (0, \epsilon_0(x, \kappa))$, the system of inequalities

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- Note that usually $q_p(\mathrm{G}) \geq 2$, and equality provides the best possible result, namely a sharp threshold in Khinchin's theorem. This occurs, for example, for $\mathbb{Z}[\frac{1}{p}]$-approximations on S^2 and S^3.

Uniform version of Schmidt's theorem

- Define the solution counting function (with gauge $\psi(h) = h^{-b}$ for suitable $b > 0$) :

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- then there exist $\theta = \theta(b) \in (0,1)$ such that for every $x \in \mathrm{X}(\mathbb{R})$

$$N_T(x) = V_T + O_x\left(V_T^\theta\right)$$

Discrepancy of $\mathbb{Z}[\frac{1}{p}]$ points on the variety

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- The uniform analog of Schmidt's theorem stated above obviously yields an effective uniform bound on the discrepancy of $\mathbb{Z}[\frac{1}{p}]$ points on the variety, for suitable scales $\delta \sim h^{-b}$ for $0 \leq b \leq b_0$.

Some previous results

- Uniform discrepancy bounds on the spheres S^2 and S^3 for approximation by $\mathbb{Z}[\frac{1}{p}]$-points appears in the celebrated work of Lubotzky-Phillips-Sarank. Diophantine exponents can also be derived from their work, and in higher dimensional spheres from the work of Clozel, Ullmo and Oh. The intersection with group varieties amounts to S^3 only.

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- The methods they use involve homogeneous dynamics, particularly Dani-Margulis arguments regarding visiting times to shrinking neighborhoods of cusps, and a method previously introduced by Drutu to reduce the problem to this context. It is not clear whether this approach can yield an effective analog of Schmidt's theorem.

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- for arbitrary simply transitive affine homogenous varieties of all semisimple groups,
- defined over an arbitrary number field K,
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It is possible to derive similar results on uniform and almost sure Diophantine approximation :

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- with the approximation rate being given as an explicit exponent.

Method of proof

Our approach to the proof of the analogs of Schmidt's theorem is based on the following ingredients :

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- Reduction of the lattice point counting problem to an effective mean ergodic theorem for the action of the group on the probability space G/Γ.
- Utilizing spectral estimates in the automorphic representation of G in $L^2(G/\Gamma)$ to bound the averages appearing in the mean ergodic theorem.

From Diophantine approximation to dynamics

- In order to study the density of the set $\mathrm{X}(\mathbb{Z}[1/p])$ in $\mathrm{X}(\mathbb{R})$ when $\mathrm{X} = \mathrm{G}$ is a group variety, note first that this set coincides with the subgroup $\Gamma = \mathrm{G}(\mathbb{Z}[1/p]) \subset \mathrm{G}(\mathbb{R})$.

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- Consider therefore the group $\mathrm{G}(\mathbb{R}) \times \mathrm{G}(\mathbb{Q}_p)$, where the group $\Gamma = \mathrm{G}(\mathbb{Z}[1/p])$ embeds diagonally as a lattice subgroup, namely as a discrete subgroup with finite covolume, (Borel-Harish Chandra 1960).

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- We can then consider the finite-measure homogeneous space

$$Y = (\mathrm{G}(\mathbb{R}) \times \mathrm{G}(\mathbb{Q}_p))/\mathrm{G}(\mathbb{Z}[1/p]).$$

  on which $\mathrm{G}(\mathbb{R}) \times \mathrm{G}(\mathbb{Q}_p)$ acts (transitively) as a group of probability measure preserving transformations.

Counting lattice points

- Consider the increasing sequence of height balls $\mathcal{B}_h$ of $\mathrm{G}(\mathbb{Q}_p)$

$$\mathcal{B}_h = \{g \in \mathrm{G}(\mathbb{Q}_p) \; ; \; H_p(g) \leq h\}$$

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- Clearly, this problem is identical to counting the number of points in the (diagonally embedded) lattice $\Gamma = \mathrm{G}(\mathbb{Z}[1/p])$ which fall in the set

$$B(x, \delta) \times \mathcal{B}_h \subset \mathrm{G}(\mathbb{R}) \times \mathrm{G}(\mathbb{Q}_p)$$

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- Let $\beta_{\delta,h}$ denote the Haar-uniform probability measure supported on the sets $B(e,\delta) \times \mathcal{B}_h \subset \mathrm{G}(\mathbb{R}) \times \mathrm{G}(\mathbb{Q}_p)$. Consider the operators

$$\pi_Y(\beta_{\delta,h}) : L^2(Y) \to L^2(Y)$$

defined by averaging over these sets, namely

$$\pi_Y(\beta_{\delta,h})\phi((x,y)) =$$

$$= \frac{1}{m(B(e,\delta))m(\mathcal{B}_h)} \int_{(u,v)\in(B(e,\delta)\times\mathcal{B}_h} \phi((u^{-1}x, v^{-1}y))dudv\,.$$

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- Then the following effective mean ergodic theorem holds :

$$\left\| \pi_Y(\beta_{\delta,h})(\phi) - \int_Y \phi dm_Y \right\|_2 \le C_\eta \cdot \mathrm{vol}(\mathcal{B}_h)^{-\frac{1}{q_p(\mathrm{G})}+\eta} \|\phi\|_2.$$

- The rate of convergence amounts to an operator norm estimate of the averaging operators acting in the automorphic representation, which can be deduced from the spectrum of the unitary representation of the group $\mathrm{G}(\mathbb{Q}_p)$ in $L^2(Y)$.

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- We note that in order to derive the optimal Diophantine exponent and threshold in Khinchin's theorem using this method, optimality of the operator norm estimate in L^2 is crucial. This amounts to $q_p(\mathrm{G}) = 2$, or equivalently, the representation of $\mathrm{G}(\mathbb{Q}_p)$ being tempered.

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- For a locally compact second countable group G, a lattice subgroup Γ, and a family of sets B_t let us assume the following :

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- where $\mathcal{O}_\varepsilon$ is a family of decreasing neighborhoods of $e \in G$ satisfying : $m(\mathcal{O}_\varepsilon) \geq C\varepsilon^d$.

Solution of the lattice point counting problem

- **Theorem** (Gorodnik+N, 2008.) For any lcsc group G and any lattice Γ, under conditions A and B, the lattice point counting problem in the domains B_t has the solution

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- We establish such an extension under suitable restrictions limiting the speed at which δ can converge to 0, as a function of h^{-1}. This concludes the outline of the proof of the uniform analog Schmidt's theorem in this context.