
Weak model sets and dynamical systems of number-theoretic origin

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Square-free integers, visible lattice points and their generalisations give rise to dynamical systems with interesting spectral properties. Many of them have recently found a systematic description in terms of weak model sets. Under a mild extra condition, the latter display pure point spectrum, both in the diffraction and the dynamical sense. This talk will introduce some of the concepts and results (based on joint work with Robert Moody, Peter Pleasants, Christian Huck and Nicolae Strungaru), while the related talk by Christoph Richard (based on joint work with Gerhard Keller) will present an alternative view from a different perspective.

Ergodic theory for Hénon-like maps

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Ergodic theory in the uniformly hyperbolic setting is well established through the fundamental work by Sinai, Ruelle and Bowen and others. One fundamental tool is here Markov partitions. In the non-uniform hyperbolic setting such as quadratic maps or Hénon maps the method of Markov partions can be replaced by Markov extensions (L.S. Young towers). I will describe how these can be used to prove existence of SRB-measures and to prove decay of correlation and the Central Limit Theorem. More general Gibbs states and the thermodynamical formalism in this setting will also be discussed (ongoing work with Sam Senti based on recent work by Pesin, Senti and Zhang).

Some open problems and conjectures at the interface of Ergodic Theory, Number Theory and Combinatorics

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We will formulate and discuss some natural open problems and conjectures which deal with various aspects of the theory of multiple recurrence.

Nice Banach limits associated with Hardy fields

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The collection $\mathcal{P} = 2^{\mathcal{N}}$ of subsets of the set $\mathcal{N}$ of natural numbers is equipped with the following distance function

$$\operatorname{dist}^*(U,V) = d^*(U \triangle V) \qquad (U, V \in \mathcal{P})$$

where $d^*(W) = \limsup_{n\to\infty} \frac{\left|[1,n]\cap W\right|}{n}$ stands for the upper (asymptotic) density of a set $W \in \mathcal{P}$, and $U \triangle V := (U \setminus V) \cup (V \setminus U)$ denotes the symmetric difference of U, $V \in \mathcal{P}$. We discuss the following observation.

Proposition. $(\mathcal{P}, \operatorname{dist}^*)$ is a *complete* pseudometric space. (Every Cauchy sequence of sets in $\mathcal{P}$ must converge to a set in $\mathcal{P}$).

The above claim fails if, instead of the asymptotic density d^*, some other "densities" (e.g., the Banach density

$d_B^*(W) := \limsup_{\substack{m,n\in\mathcal{N}\\ n\to+\infty}} \frac{|[m,m+n]\cap W|}{n}$ is used). In particular, the pseudemetric space $(\mathcal{P}, \mathrm{dist}_B^*)$ is not complete (where $\mathrm{dist}_B^*(U,V) := d_B^*(U\triangle V)$).

Denote by $\mathcal{P}_p \subset \mathcal{P}$ the subalgebra of (ultimately) periodic subsets in $\mathcal{N}$ and by $\overline{\mathcal{P}}_p$ its closure in $(\mathcal{P}, \mathrm{dist}^*)$. The sets $U \in \overline{\mathcal{P}}_p$ will be called *pseudoperiodic*; they form a subalgebra in $\mathcal{P}$. One verifies that for $U \in \overline{\mathcal{P}}_p$ every finite block in the characteristic sequence $1_U(n) \in \{0,1\}^{\mathcal{N}}$ appears with certain density; in particular, the asymptotic density $d(U)$ exists.

Note that every non-decreasing sequence (U_k) of periodic subsets $U_k \in \mathcal{P}_p$, $U_k \subseteq U_{k+1}$, must be Cauchy in both pseudometrics dist^* and dist_B^*. Thus the limit $\lim_k U_k \overset{\mathrm{dist}^*}{=} U \in \overline{\mathcal{P}}_p$ always exists, while the same limit relative dist_B^* does not need to exist in $\mathcal{P}$.

In general, the equality $\lim_n U_n \overset{\mathrm{dist}^*}{=} \bigcup_{k=1}^{\infty} U_k$ does not need to hold (even though the limit on the left must exist). This equality takes place only under some additional conditions.

For example, if $U_n = \bigcup_{k=1}^{n} V_k$ are consecutive unions of infinite arithmetical progression $V_k = \{m_k, m_k + d_k, m_k + 2d_k, \ldots\} = \{m_k + rd_k \mid 0 \le r < \infty\}$ (with the first terms $m_k \ge 1$ and the steps $d_k \ge 1$) such that $\sum_k \frac{1}{d_k} < \infty$ and $d(\{m_k \mid k \ge 1\}) = 0$, then one can show that $\lim_k U_k \overset{\mathrm{dist}^*}{=} \bigcup_{k=1}^{\infty} U_k \in \overline{\mathcal{P}}_p$.

We discuss the properties of pseudoperiodic sets $U \in \overline{\mathcal{P}}_p$.

Normal Subsequences of Automatic Sequences

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Joint work with Christian Maudiut and Joël Rivat.

Recently Drmota, Mauduid and Rivat [1] observed that the subsequence along the squares $(t(n^2))_{n\ge 0}$ of the Thue-Morse sequence $(t(n))_{n\ge 0}$ (that can be defined by $t_0 = 0$, $t(2n) = t(n)$, and $t(2n+1) = 1 - t(n)$, $n \ge 0$, or equivalently by $t(n) = s_2(n) \bmod 2$, where $s_2(n)$ denotes the binary sum-of-digites function) is a normal sequence on the alphabet $\{0,1\}$.

The purpose of this talk is to discuss this result also from a more general point of view. The Thue-Morse sequence is a special case of an q-automatic sequence $T(n)$, that is, a sequence that is the output of a finite state automaton, where the input sequence is the q-ary expansion of n. Automatic sequences have a sub-linear subword complexity, so they are far from being normal. Even every linear subsequence $T(an+b)$ is automatic and has therefore the same properties. However, if we consider a subsequence $T(\phi(n))$, where $\phi(n)/n \to \infty$ the situation can change – as the example $t(n^2)$ shows – and one is led to ask for which subsequences we might expect a normal sequence. First results into this direction by Lukas Spiegelhofer and Clemens Müller [2] (who considered $\phi(n) = \lfloor n^c \rfloor$ for $1 < c < 3/2$) and by Clemens Müller [3] (who considered block-additive functions like the Rudin-Shapiro sequence and $\phi(n) = n^2$) indicate that there might be a more general property behind.

References

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Measure Rigidy and Quantitative Recurrence

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This is ongoing joint work with Elon Lindenstrauss.

We will discuss rigidity of positive entropy measures for higher rank diagonalisable actions. Using a quantitative form of recurrence along unipotent directions we prove a complete classification of positive entropy measures for any higher rank action on any irreducible quotients of SL_2^k as well as some other cases of interest.

Ergodicity of the Liouville system implies the Chowla conjecture

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The Chowla conjecture asserts that the signs of the Liouville function are distributed randomly on the integers. Reinterpreted in the language of ergodic theory this conjecture asserts that the Liouville dynamical system is a Bernoulli system. We prove that ergodicity of the Liouville system implies the Chowla conjecture. Our argument has an ergodic flavor and combines recent results in analytic number theory, finitistic and infinitary decomposition results involving uniformity norms, and equidistribution results on nilmanifolds.

Ramsey Theory for Non-amenable Groups

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Ergodic-theoretic methods have been found to be useful for certain combinatorial questions on the integers - notably, proving Szemerédi's theorem on arithmetic progressions. This has recently been extended to amenable groups where the notion of an invariant mean is available, and with it the notion of invariant density. For more general groups one can define the notion of a "stationary" density corresponding to "stationary group actions". Using a theory developed by S. Glasner and myself, one can derive similar results for a more general class of groups. For instance one can identify subsets of a finitely generated free group which possess arbitrarily long geometric progressions.

Convergence of measures and arithmetics of Shimura varieties

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Joint work with Christopher Daw, Emmanuel Ullmo.

In this talk we will be interested in behaviour of certain arithmetically defined families of subvarieties of Shimura varieties that appear in the André-Oort conjecture and its generalisations. We give a gentle introduction to this topic assuming no previous knowledge of Shimura varieties. It had been previously observed by Clozel and Ullmo that distribution of special subvarieties of Shimura varieties can be studied using ergodic-theoretic techniques though analysing limits of sequences of probability measures. However, in general these sequences may diverge to infinity. We generalise the Clozel-Ullmo approach and prove that the sequences of measures converge in compactifications of Shimira varieties.

The diameter of the symmetric group: ideas and tools

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Given a finite group G and a set A of generators, the diameter $\mathrm{diam}(\Gamma(G,A))$ of the Cayley graph $\Gamma(G,A)$ is the smallest ℓ such that every element of G can be expressed as a word of length at most ℓ in $A \cup A^{-1}$. We are concerned with bounding $\mathrm{diam}(G) := \max_A \mathrm{diam}(\Gamma(G,A))$.

It has long been conjectured that the diameter of the symmetric group of degree n is polynomially bounded in n. In 2011, Helfgott and Seress gave a quasipolynomial bound, namely, $O\left(e^{(\log n)^{4+\epsilon}}\right)$. We will discuss a recent, much simplified version of the proof.

Dimension of Furstenberg measure of $SL_2(R)$ random matrix products

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Given a probability measure mu on the space of 2×2 matrices, there is, under mild conditions, a unique measure nu on the space of lines which is stationary for mu. This measure is called the Furstenberg measure of mu, and is important in many contexts, from the study of random matrix products to recent work on self-affine sets and measures. Of particular importance are the smoothness and dimension of the Furstenberg measure. In this talk I will discuss joint work with Boris Solomyak in which we adapt methods from additive combinatorics and the theory of self-similar measures to compute its dimension in many cases.

Correlations sequences are nilsequences

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Joint work with N. Frantzikinakis.

Many sequences defined as correlations appear to be nilsequences, up to the addition of a small error term. Results of this type hold both in the ergodic and in the finite settings. The proofs follow the same general strategy, although the context and the tools are completely different. I'll try to explain the ideas behind these results. This is a common work with Nikos Frantzikinakis.

References

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Shifts of low complexity

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For subshifts of zero entropy, the behavior is quite different than general subshifts. We illustrate these differences, describing the relations between dynamical features of the system, algebraic properties of the system, and combinatorial characterizations.

Primes with missing digits

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We will talk about recent work showing there are infinitely many primes with no 7 in their decimal expansion. (And similarly with 7 replaced by any other digit.) This shows the existence of primes in a 'thin' set of numbers (sets which contain at most X^{1-c} elements less than X) which is typically vey difficult.

The proof relies on a fun mixture of tools including Fourier analysis, Markov chains, Diophantine approximation, combinatorial geometry as well as tools from analytic number theory.

On the natural extensions of some complex continued fraction transformations

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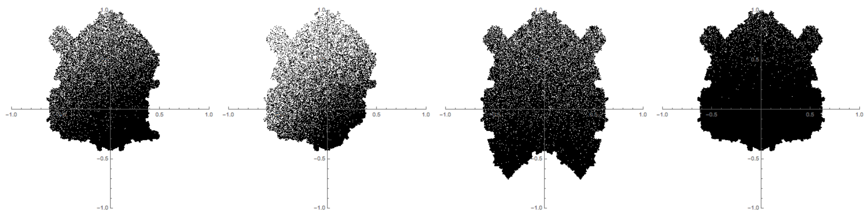
Joint work with Hiromi Ei and Rie Natsui.

We consider the nearest integer type complex continued fraction transformations T_m associated to Euclidean imaginary quadratic fields $\mathbb{Q}(\sqrt{-m})$ where $m = 1, 2, 3, 7$, and 11. For each m, it is easy to show that there exists an absolutely continuous ergodic invariant measure for T_m. We are interested in finding its density function. For this purpose we construct an invertible map $\tilde{T}_m$ of the form $\tilde{T}_m(z_1, z_2) = (1/z_1 - a, 1/z_2 - a)$ on a subset of of $\mathbb{C}^2$, equivalently on a subset of geodesics over $\mathbb{H}^3$. Then the measure $\frac{d(m_{\mathbb{C}} \times m_{\mathbb{C}})(z_1 z_2)}{|z_1 - z_2|^4}$ is an invariant measure for $\tilde{T}_m$, where $m_{\mathbb{C}}$ denotes the Lebesgue measure of $\mathbb{C}$. from this measure we get the density function of the absolutely continuous invariant measure for T_m as the marginal density function. We see that $\tilde{T}_m$ is the natural extension of T_m in the sense of Rokhlin. We characterize the domain of $\tilde{T}_m$ as follows:
There exists a finite partion $\{V_\ell\}$ of the domain of T_m and (unbounded) closed subsets V_ℓ^* of $\mathbb{C}$ such that the domain of $\tilde{T}_m$ is $\cup V_\ell \times V_\ell^*$. Moreover every $X_\ell = \{z : 1/z \in V_\ell^*\}$ is simply connected and its boundary is a Jordan curve. For example, the following are some of X_ℓ in the case of $m = 2$.



To show our result, we need the existence of the Legendre constant for the complex continued fractions and then we use the ergodicity of the geodesic flow over $\mathbb{H}^3/\mathrm{PSL}(2, \mathfrak{o}(\sqrt{-\mathrm{m}}))$, where $\mathfrak{o}(\sqrt{-m})$ denotes the integer ring of $\mathbb{Q}(\sqrt{-m})$.

Intrinsic Diophantine approximation

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Based on joint work with Anish Ghosh and Alex Gorodnik.

We will consider intrinsic Diophantine approximation on a semisimple group variety, by rational points satisfying integrality constraints. We will describe a recently established analog of Schmidt's classical result on the number of solutions to Diophantine inequalities in Euclidean space, and note some of its applications to discrepancy bounds for rational points on the variety.

Patterns of primes in arithmetic progressions

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We plan to sketch the proof of the following common generalization of the theorems of Green–Tao and Maynard (and Tao):
Theorem. *Let $m > 0$ and $\mathcal{A} = \{a_1, \ldots, a_r\}$ be a set of r distinct integers with r sufficiently large depending on m. Let $N(\mathcal{A})$ denote the number of integer m-tuples $\{h_1, \ldots, h_m\} \subseteq \mathcal{A}$ such that there exist for every ℓ infinitely many ℓ-term arithmetic progressions of integers $\{n_i\}_{i=1}^{\ell}$ where $n_i + h_j$ is the jth prime following $n_i + h_j$ for each pair i, j. Then*

$$N(\mathcal{A}) \gg_m \#\big\{(h_1, \ldots, h_m) \in \mathcal{A}\big\} \gg_m |A|^m = r^m.$$

Quantum Unique Ergodicity for half-integral weight forms

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Joint work with Steve Lester.

I will discuss joint work with Steve Lester in which we establish Quantum Unique Ergodicity for half-integral weight forms (both holomorphic and non-holomorphic) on the assumption of the Generalized Riemann Hypothesis.

Dynamical properties of weak model sets

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Joint work with Gerhard Keller.

Model sets arose in the context of commutative harmonic analysis in the 1970s in works by Meyer and Schreiber. So-called regular model sets were advocated by Moody in the 1990s for modelling physical quasicrystals. They have been intensely studied since then, in particular by dynamical methods. A description of squarefree integers and visible lattice points as so-called weak model sets was put forward by Baake, Moody and Pleasants in the late 1990s. In fact general B-free sets fall into that class. Dynamical properties of general weak model sets have been studied systematically only lately. Here we give an overview of recent results obtained together with Gerhard Keller. In this context major structural results for B-free systems reappear, by proofs of topological or measure-theoretic nature. Our approach to weak model sets complements that of Baake, Huck and Strungaru, which will be presented in a talk by Michael Baake.

Integral points on Markoff type cubic surfaces and dynamics

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Joint work with Bourgain/Gamburd and Ghosh.

Cubic surfaces in affine three space tend to have few integral points .However certain cubics such as $x^3 + y^3 + z^3 = m$,may have many such points but very little is known .We discuss these questions for Markoff type surfaces: $x^2 + y^2 + z^2 - x \cdot y \cdot z = m$ for which a (nonlinear) descent allows for a study. Specifically that of a Hasse Principle and strong approximation ,together with "class numbers" and their averages for the corresponding nonlinear group of morphims of affine three space.

Nonmixing sets of algebraic $\mathbf{Z}^d$-actions

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In 1979, Ledrappier gave an example of a mixing, but not 3-mixing, $\mathbf{Z}^2$-action α by automorphisms of a compact abelian group X. In this example, higher order mixing fails due to the existence of a finite 'nonmixing' set $S \subset \mathbf{Z}^2$ for which there exist Borel sets $\{B_{\mathbf{n}} : \mathbf{n} \in S\}$ in X such that $\{\alpha^{k\mathbf{n}} B_{\mathbf{n}} : \mathbf{n} \in S\}$ fail to become independent as $k \to \infty$.

Nonmixing sets of algebraic $\mathbf{Z}^d$-actions (i.e., of $\mathbf{Z}^d$-actions by automorphisms of compact groups) were investigated further by Einsiedler, Kitchens, Ward, myself, and others, and in 2004 Masser proved that the order of mixing of such an action is determined by the minimal size of its nonmixing sets.

Finding the minimal nonmixing sets of an algebraic $\mathbf{Z}^d$-action is an interesting arithmetical problem with dynamical consequences. In three recent papers Masser and Derksen effectively determine the smallest order n of non-mixing of such an action: if $n \geq 3$, there are only finitely many inequivalent classes of n-nonmixing sets, and these can be effectively determined.

The Unsolved Problems of Halmos

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Sixty years ago Paul Halmos concluded his *Lectures on Ergodic Theory* with a chapter **Unsolved Problems** which contained a list of ten problems. I will discuss some of these and some of the work that has been done on them. He considered actions of $\mathbb{Z}$ but I will also widen the scope to actions of general countable groups.

Random differences of arithmetic progressions in the primes

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Researchers have been looking at differences between primes numbers for a long time. If we just looking at the odd primes, only the even numbers occur as differences. Can the set of differences restricted more, or be described more precisely?

The twin prime conjecture says that there are infinitely many prime pairs p_1, p_2 so that the difference $p_1 - p_2$ is from the singleton $\{2\}$. Green, Tao and Ziegler prove, for example, that there are infinitely many prime pairs p_1, p_2 so that the difference $p_1 - p_2$ is a cube, so it is from the set $\{1, 8, 27, \ldots, n^3, \ldots\}$. In fact, they prove more: given any positive density subset A of the primes, there are infinitely many prime pairs p_1, p_2 in A so that the difference $p_1 - p_2$ is a cube.

In this talk we are interested in possible difference sets for the primes which are randomly generated. The simplest example would be to generate this random set by repeatedly flipping a fair coin. But we will look at sets that are random versions of known sets of 0 density. For example, we get the set of "random squares" by taking the integer n into the random set with probability $\frac{1}{\sqrt{n}}$ or we get the set of "random cubes" by taking n into the random set with probability $\frac{1}{n^{2/3}}$.

We will discuss two generalizations: one, when we consider not the whole set of primes but positive density subsets of it, and two, when we look at not just differences between primes, but differences of three or longer term arithmetic progressions of primes.

As can be suspected, there are more questions than answers.

Regionally proximal relation of higher order: results and questions

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In a pioneer work Host-Kra-Maass introduced and studied the regionally proximal relation of order d. We will review results along the line and state unsolved questions. This talk is based on several joint works with Huang-Shao, and one with Glasner-Gutman.

Jump inequalities for polynomial ergodic averages

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Joint work with M. Mirek and E. Stein.

The *jump counting function* $N_\lambda(a_N : N \in \mathbb{N})$ is the maximal length J of a sequence of increasing times $N_0 < \cdots < N_J$ such that $|a_{N_j} - a_{N_{j-1}}| \geq \lambda$ for all $1 \leq j \leq N$. The jump counting function is finite for all $\lambda > 0$ if and only if the sequence (a_N) is Cauchy, so estimates on the jump counting function provide quantitative information about convergence. We consider the polynomial ergodic averages $A_N f = \frac{1}{N}\sum_{n=1}^{N} T^{p(n)} f$ and prove the jump inequality

$$\sup_{\lambda>0} \|\lambda\sqrt{N_\lambda(A_N f(x) : N \in \mathbb{N})}\|_{L^p_x} \lesssim \|f\|_{L^p}, \quad 1 < p < \infty.$$

This is an endpoint for the r-variational estimates for $r > 2$ recently obtained by Mirek, Stein, and Trojan [1]. An important building block is an interpretation of the jump inequality as an estimate in a certain real interpolation space introduced by Pisier and Xu in the case $p = 2$ [2]. I will focus on basic properties of this interpolation space and on how it fits into the Fourier analytic approach to polynomial ergodic averages initiated by Bourgain.

References

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