On two dimensional gravity water waves with angled crests

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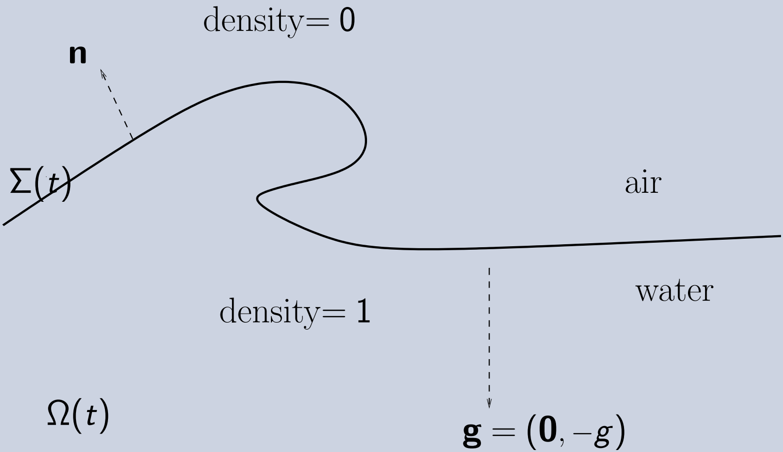

density$= 0$
$\mathbf{n}$
$\Sigma(t)$
air
water
density$= 1$
$\Omega(t)$
$\mathbf{g} = (\mathbf{0}, -g)$

The water wave equations

We assume that

- the air density is 0, the fluid density is 1. $(0, -g)$ is the gravity.
- the fluid is inviscid, incompressible, irrotational,
- the surface tension is zero.

Let $\Omega(t)$ be the fluid domain, $\Sigma(t)$ be the interface at time t.

The motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = (\mathbf{0}, -g) - \nabla P & \text{in } \Omega(t) \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \Sigma(t) \end{cases} \tag{1}$$

$\mathbf{v}$ is the fluid velocity, P is the fluid pressure.

When surface tension is zero, the motion can be subject to the Taylor instability

- Taylor sign condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq 0; \tag{2}$$

  on the interface $\Sigma(t)$. $\mathbf{n}$ is the unit outward normal to the fluid domain $\Omega(t)$.

- G. I. Taylor (1949)
- If the Taylor sign condition (2) fails, the water wave equation (1) is ill-posed (Ebin 1987).
- Strong Taylor sign condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0. \tag{3}$$

Local wellposedness in Sobolev spaces

- Nalimov (1974): infinite depth, 2D, assume initial interface flat, initial velocity small
- Yoshihara (1982): finite depth, 2D, assume initial data small
- W. Craig (1985): finite depth, 2D, assume initial data small, KdV asymptotics
- T. Beale, T. Hou & Lowengrub (1992).
  Linear wellposedness assuming the presumed solution satisfies the strong Taylor sign condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0.$$

Local wellposedness in Sobolev spaces continues...

- S. Wu (1997, 99): 2D and 3D. Local wellposedness for arbitrary data in Sobolev spaces H^s, for $s \geq 4$.
- proved the strong Taylor sign condition always holds, i.e.

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0 \tag{4}$$

  as long as the interface is non-selfintersecting and smooth ($C^{1,\gamma}$, $\gamma > 0$).
- in 2D: derived the quasilinear equation for the water waves in the Riemann mapping variable;
- in 3D: we used Lagrangian coordinates, in Clifford algebra framework.

local wellposedness in Sobolev spaces continues....

Local wellposedness with additional effects: nonzero surface tension, finite depth, nonzero vorticity, assuming the strong Taylor sign condition holds.

- Christodoulou & Lindblad (2000), Iguchi(2001), Ogawa & Tani (2002), Ambrose & Masmoudi(2005), D. Lannes (2005), Lindblad (2005), Coutand & Shkoller (2007), P. Zhang & Z. Zhang (2007), Shatah & Zeng (2008)

Global behavior for small, smooth and localized data

- S. Wu (2009): almost global well-posedness for 2-D,
- S. Wu (2011): global well-posedness for 3-D
- Germain, Masmoudi & Shatah (2012): global well-posedness for 3-D
- Ionescu & Pusateri (2013): 2-D water waves, global existence and modified scattering
- Alazard & Delort (2013): similar result
- Hunter, Ifrim & Tataru (2014): 2-D water wave, almost global, global existence, modified energy

Global behavior for small, smooth and localized data

- Xuecheng Wang (2015): global existence for 3-D water waves with flat bottom
- Harrop-Griffiths, Ifrim & Tataru (2016): 2-D with flat bottom, longer life span
- Deng, Ionescu, Pausader & Pusateri (2016): 3-D gravity-capillary water waves, global existence
- Bieri, Miao, Shahshahani & Wu(2015): self gravitating fluid with a free boundary; longer lifespan

local wellposedness in low regularity Sobolev spaces:

- Alazard, Burq & Zuily (2012): Local wellposedness in low regularity Sobolev space–the interface is $C^{3/2+\epsilon}$.
- Alazard, Burq & Zuily (2014): Local wellposedness in low regularity Sobolev space–the interface is $C^{3/2-\epsilon}$.
- Hunter, Ifrim & Tataru (2014): low regularity local wellposedness in 2-D.

- Th. De Poyferré (2016): a priori estimates, with a bottom that interacts with the free surface

In all these work,

- it is either proved or assumed that the strong Taylor sign condition holds:
$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$$
- The lowest regularity of the interface considered is $C^{3/2}$.

Singular interfaces:

Questions:

- Does the water wave equation admits such solutions?
- Once waves with angled crests form, does it persist?
- Is the water wave equation well-posed in any class that includes non-C^1 interfaces with angled crests?

The first question:

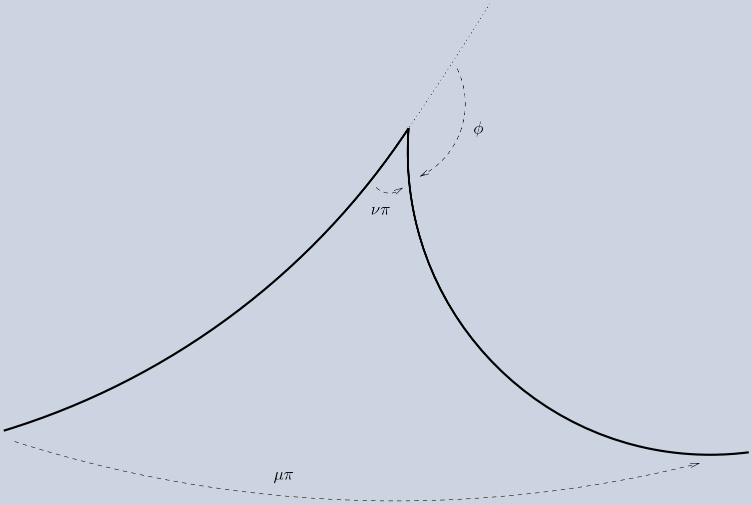
- What are some typical singular bebaviors?

Self-similar solutions:

S. Wu (2012): construction of self-similar solution for 2-D water waves in the regime where convection is in dominance:

- $z \sim t$, or in hyperbolic scaling: $s = 1$.
- neglecting gravity and surface tension.
- satisfies the Taylor sign condition $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$.

- Ansatz: fluid interface at time t: $z = z(\alpha, t)$, $\alpha \in \mathbb{R}$
- $$z(\alpha, t) = t\zeta\left(\frac{\alpha}{t}\right)$$
- $$\zeta'(\beta) = e^{i(b(\beta)+\phi\chi(\beta))}$$
  $$b'(\beta) = \frac{a(\beta)}{\beta^2}$$
- $a(\beta)$ is a quantity related to $-\frac{\partial P}{\partial \mathbf{n}}$; $a(\beta) \geq 0$.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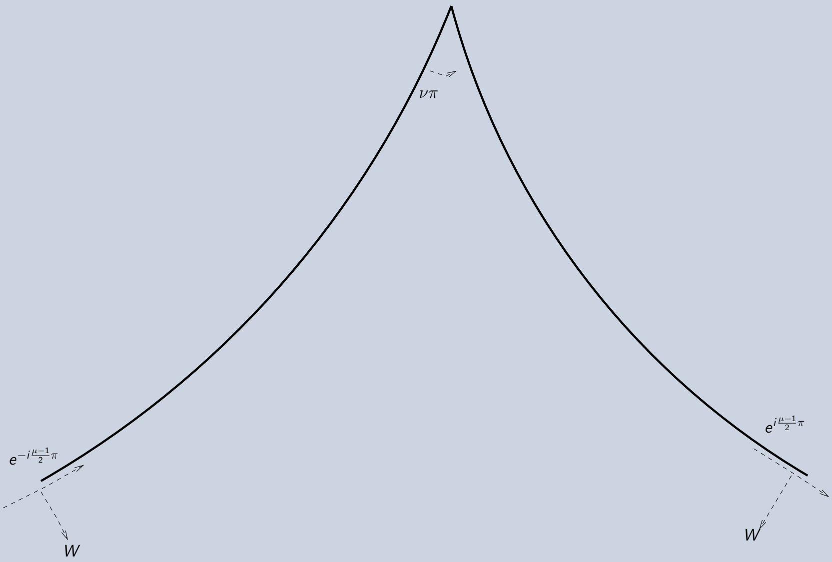
$\nu\pi$
$\mu\pi$

- $\nu < \frac{1}{2}, \qquad \mu > \frac{1}{2}$
- concave up on both sides, the concavity is due to the Taylor stability condition.



$\nu\pi$
$e^{-i\frac{\mu-1}{2}\pi}$
W
$e^{i\frac{\mu-1}{2}\pi}$
W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Main Goal:

Q: Well-posedness of the water wave equation (1) in any class that includes non-C^1 interfaces?

How do we solve the water wave equation (1)?

- A usual approach is to reduce from (1) to an equation on the interface, and study the interface equation.
- Here we describe the approach of Wu (1997), Wu (1999)
- We consider the 2d case. We used Riemann mapping in Wu (1997).

surface equation in Lagrangian coordinates

- We identify $(x, y) = x + iy$; the gravity $(0, -g) = (0, -1) = -i$.
- Let the free surface
$$\Sigma(t) : z = z(\alpha, t),$$
$\alpha \in \mathbb{R}$; α is the Lagrangian coordinate.
- $z_t = z_t(\alpha, t)$ velocity,
- $z_{tt} = z_{tt}(\alpha, t)$ acceleration,
- $P = 0$ on $\Sigma(t)$ implies $\nabla P \perp \Sigma(t)$,
- $-\nabla P = i\mathfrak{a} z_\alpha$,
- $$\mathfrak{a} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|};$$
- $\bar{z}_t$ boundary value of the holomorphic function $\bar{\mathbf{v}}$.

$$z_{tt} + i = i\mathfrak{a} z_\alpha$$

$$\bar{z}_t = \mathfrak{H}\bar{z}_t$$

Equation of the free surface:

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \tag{5}$$

where $\mathfrak{H}$ is the Hilbert transform,

$$\mathfrak{H} f(\alpha) = \frac{1}{\pi i} \int \frac{z_\beta(\beta, t)}{z(\alpha, t) - z(\beta, t)} f(\beta)\, d\beta$$

surface equation in Riemann mapping framework

Let

$$\Phi = \Phi(z;t) : \Omega(t) \to P_-$$

be the Riemann mapping satisfying $\lim_{z\to\infty} \Phi_z(z;t) = 1$; here P_- is the lower half plane.

Define

- $h(\alpha, t) = \Phi(z(\alpha, t); t)$
- $h(h^{-1}(\alpha', t); t) = \alpha'; \quad f \circ g := f(g(\cdot, t); t) := U_g f$
- $Z(\alpha'; t) := z(h^{-1}(\alpha', t); t) = \Phi^{-1}(\alpha'; t)$, $\partial_{\alpha'} Z = Z_{,\alpha'}$
- $Z_t = z_t(h^{-1}(\alpha', t); t); \quad Z_{tt} = z_{tt}(h^{-1}(\alpha', t); t);$
- $Z_{tt,\alpha'} = \partial_{\alpha'}\{z_{tt}(h^{-1}(\alpha', t); t)\};$

we have

$$\lim_{\alpha'\to\pm\infty} Z_{,\alpha'} = \lim_{\alpha'\to\infty} (\Phi^{-1})_z(\alpha'; t) = 1$$

surface equation in Riemann mapping framework

$$\bar{Z}_t(\alpha'; t) = \bar{\mathbf{v}} \circ \Phi^{-1}(\alpha', t)$$

Precomposing (5) with h^{-1} we get

Equation of the free surface:

$$\begin{cases} Z_{tt} + i = iAZ_{,\alpha'} \\ \bar{Z}_t = \mathbb{H}\bar{Z}_t \end{cases} \tag{6}$$

where $\mathbb{H}$ is the Hilbert transform for P_-:

$$\mathbb{H}f(\alpha') = \frac{1}{\pi i}\int \frac{1}{\alpha' - \beta'} f(\beta')\, d\beta'$$

$$A \circ h = \mathfrak{a} h_\alpha.$$

$$Z_{tt} = (\partial_t + b\partial_{\alpha'})Z_t, \quad Z_t = (\partial_t + b\partial_{\alpha'})Z,$$

$$b = h_t \circ h^{-1}.$$

A and b were analyzed in Wu (1997).

$$A = (\mathfrak{a} h_\alpha) \circ h^{-1} = -\frac{\partial P}{\partial \mathbf{n}} \frac{h_\alpha}{|z_\alpha|} \circ h^{-1} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|Z_{\alpha'}|}.$$

Proposition (Wu, 1997)

Let $A_1 := A|Z_{,\alpha'}|^2 := iZ_{,\alpha'}(\bar{Z}_{tt} - i)$. *Then*

$$A_1 = 1 - \Im[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi} \int \frac{|Z_t(\alpha'; t) - Z_t(\beta'; t)|^2}{(\alpha' - \beta')^2}\, d\beta'$$

consequently

- $$A_1 \geq 1;$$
- $$-\frac{\partial P}{\partial \mathbf{n}} = \frac{A_1}{|Z_{,\alpha'}|} \geq 0; \quad A \geq c_0 > 0, \text{ if } \Sigma(t) \text{ is } C^{1,\gamma}, \gamma > 0.$$

Proof: We know $\bar{z}_t = F(z(\alpha, t); t)$, F is holomorphic in $\Omega(t)$.

- $\bar{z}_{tt} = F_z z_t + F_t$,
- $\bar{z}_{t\alpha} = F_z z_\alpha$
- $\bar{z}_{tt} = \frac{\bar{z}_{t\alpha}}{z_\alpha} z_t + F_t$
- $\bar{Z}_{tt} = \frac{\bar{Z}_{t,\alpha'}}{Z_{,\alpha'}} Z_t + F_t \circ \Phi^{-1}$
- $\bar{Z}_{tt} - i = -iAZ_{,\alpha}$,
- $-iA_1 := -iA|Z_{,\alpha'}|^2 = Z_{,\alpha'}(\bar{Z}_{tt} - i) = Z_t \bar{Z}_{t,\alpha'} + F_t \circ \Phi^{-1} Z_{,\alpha'} - iZ_{,\alpha'}$
- we know $F_t \circ \Phi^{-1}$, $Z_{,\alpha'} = (\Phi^{-1})_{z'}(\alpha', t)$ are holomorphic in P_-,
- $(I - \mathbb{H})(F_t \circ \Phi^{-1} Z_{,\alpha'}) = 0$, $(I - \mathbb{H})(Z_{,\alpha'} - 1) = 0$.
- $-i(I - \mathbb{H})A_1 = (I - \mathbb{H})(Z_t \bar{Z}_{t,\alpha'} - i)$.
- $A_1 = 1 - \Im(I - \mathbb{H})(Z_t \bar{Z}_{t,\alpha'}) = 1 - \Im[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'}$.

we have

Proposition (Wu, 1997)

$$\begin{aligned} b := h_t \circ h^{-1} &= \Re(I - \mathbb{H})\frac{Z_t}{Z_{,\alpha'}} \\ &= 2\Re Z_t + \Re[Z_t, \mathbb{H}]\Big(\frac{1}{Z_{,\alpha'}} - 1\Big) \end{aligned} \tag{7}$$

Deriving the quasi-linear equation

Taking derivative to t to the first equation in (2), we get

$$\begin{cases} \bar{z}_{ttt} + i\mathfrak{a}\bar{z}_{t\alpha} = -i\mathfrak{a}_t\bar{z}_\alpha \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \tag{8}$$

$$-i\mathfrak{a}_t\bar{z}_\alpha = \frac{\mathfrak{a}_t}{\mathfrak{a}}(\bar{z}_{tt} - i).$$

Precomposing with h^{-1} yields

$$\begin{cases} \bar{Z}_{ttt} + iA\bar{Z}_{t,\alpha'} = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \\ \bar{Z}_t = \mathbb{H}\bar{Z}_t \end{cases} \tag{9}$$

where

$$\bar{Z}_{tt} = (\partial_t + b\partial_{\alpha'})\bar{Z}_t; \quad \bar{Z}_{ttt} = (\partial_t + b\partial_{\alpha'})^2\bar{Z}_t$$

$$(\partial_t + b\partial_{\alpha'})^2\bar{Z}_t + iA\partial_{\alpha'}\bar{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \tag{10}$$

Let $D_{\alpha'} f = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} f$; $D_\alpha g = \frac{1}{z_\alpha} \partial_\alpha g$; $\Im(x+iy) = y$.

Proposition (Wu, 1997)

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{-\Im(2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\bar{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'}\bar{Z}_t])}{A_1}. \quad (11)$$

where

$$[Z_t, Z_t; D_{\alpha'}\bar{Z}_t] = \frac{1}{\pi i} \int \left(\frac{Z_t(\alpha'; t) - Z_t(\beta'; t)}{\alpha' - \beta'}\right)^2 D_{\beta'}\bar{Z}_t(\beta'; t)\, d\beta'$$

the quasilinear equation in Riemann mapping coordinate:

$$(\partial_t + b\partial_{\alpha'})^2 \bar{Z}_t + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\bar{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \tag{12}$$

where

$$\begin{aligned}
&\frac{1}{Z_{,\alpha'}} = i\frac{\bar{Z}_{tt} - i}{A_1} \\
&A_1 = 1 - \Im[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi}\int \frac{|Z_t(\alpha';t) - Z_t(\beta';t)|^2}{(\alpha' - \beta')^2}\, d\beta' \\
&b := h_t \circ h^{-1} = 2\Re Z_t + \Re[Z_t, \mathbb{H}]\Big(\frac{1}{Z_{,\alpha'}} - 1\Big) \\
&\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{-\Im(2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\bar{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'}\bar{Z}_t])}{A_1}.
\end{aligned} \tag{13}$$

- $$i\partial_{\alpha'}\bar{Z}_t = i\partial_{\alpha'}\mathbb{H}\bar{Z}_t = |\partial_{\alpha'}|\bar{Z}_t$$
- The Dirichlet-Neumann operator: $\nabla_{\mathbf{n}} = i\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'} = \frac{1}{|Z_{,\alpha'}|}|\partial_{\alpha'}|$
- $-\frac{\partial P}{\partial \mathbf{n}} = \frac{A_1}{|Z_{,\alpha'}|}$.

Conclusion:

- In the regime of smooth ($C^{1,\gamma}$, $\gamma > 0$) interfaces, $Z_{,\alpha'} \in L^\infty$, then $A = \frac{A_1}{|Z_{,\alpha'}|^2} \geq c_0 > 0$. In this case, equation (12) is of the hyperbolic type,
- Local wellposedness for arbitrary initial data $(Z_t(0), Z_{tt}(0)) \in H^{s+1/2} \times H^s$, $s \geq 4$; — Wu 1997.

- Equation (12) is an equation of velocity Z_t and acceleration Z_{tt}.
- solution may have a self-intersecting interface, but only the non-self-intersecting ones give rise to a solution of the water wave equation (1).
- In (Wu 1997), we used the chord-arc condition to handle this issue.
- This idea was used in the existence of splash and splat singularities by Castro, Cordoba, Fefferman, Gancedo, Gomez-Serrano (2011)
- Coutand & Shkoller (2012), 3d

water waves with angled crests

- Is the water wave equation (1) wellposed in any class that includes interfaces with angled crests?

Difficulty:

$$\frac{1}{Z_{,\alpha'}} = 0$$

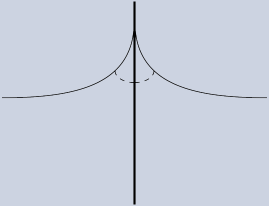
at the singularity where it has an interior angle $< \pi$.

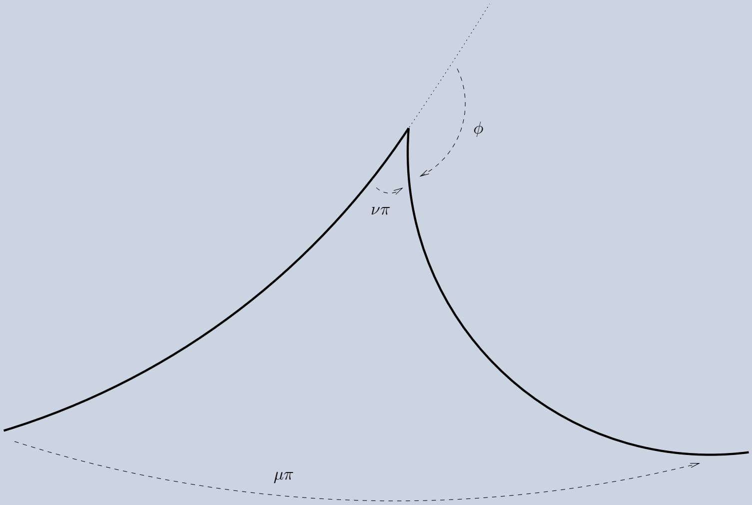
If the fixed rigid boundary Υ is a vertical wall $\{x = 0\}$, and the fluid domain $\Omega(t)$ is the domain to the right of $\{x = 0\}$. Then the velocity field $\mathbf{v} = (v_1, v_2)$ satisfies $v_1(0, y; t) = 0$.

By Schwarz reflection:

$\mathbf{v}(-x, y; t) = (-v_1(x, y; , t), v_2(x, y; t))$; $P(-x, y; t) = P(x, y, ; t)$

we can reduce the problem to the one on the symmetric domain without a fixed wall.

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Basic energy estimate (Wu 2009 version):

Consider equation

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\theta = G.$$

Let

$$E(t) := \int \frac{1}{\mathfrak{a}}|\partial_t\theta|^2 + i\bar{\theta}\partial_\alpha\theta \, d\alpha$$

Then

$$E'(t) \leq 2\left(\int \frac{1}{\mathfrak{a}}|G|^2 \, d\alpha\right)^{1/2} E(t)^{1/2} + \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty} E(t).$$

Remarks:

- $\int i\partial_\alpha\theta\bar{\theta} \, d\alpha = \int_{\partial\Omega(t)} \theta \cdot \nabla_{\mathbf{n}}\theta \, dS \geq 0.$
- only need to control $\int \frac{1}{\mathfrak{a}}|G|^2 \, d\alpha$ and $\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\|_{L^\infty}$ by $E(t)$.

change of coordinates (to Riemann mapping variable):

$$(\partial_t + b\partial_{\alpha'})^2\Theta + iA\partial_{\alpha'}\Theta = G \circ h^{-1}$$

$$E(t) = \int \frac{1}{A}|(\partial_t + b\partial_{\alpha'})\Theta|^2 + i(\partial_{\alpha'}\Theta)\bar{\Theta}\, d\alpha'$$

$$E'(t) \leq 2\left(\int \frac{1}{A}|G \circ h^{-1}|^2\, d\alpha\right)^{1/2} E(t)^{1/2} + \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty} E(t)$$

$$A = \frac{A_1}{|Z_{,\alpha'}|^2}.$$

Results

- The interior angle is $\leq \pi$, provided the acceleration Z_{tt} is finite;
- this is a consequence of

$$\frac{1}{Z_{,\alpha'}} = i\frac{\bar{Z}_{tt} - i}{A_1}, \quad A_1 \geq 1$$

Let

$$\mathcal{E}(t) = \|\bar{Z}_{t,\alpha'}\|_{L^2}^2 + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2 + \|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\|_{L^2}^2 + \|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\|_{L^2}^2 + \|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t\|_{\dot{H}^{1/2}}^2 + \|D_{\alpha'} \bar{Z}_t\|_{\dot{H}^{1/2}}^2 + \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|^2 \tag{14}$$

- $D_{\alpha'} := \frac{1}{Z_{,\alpha}} \partial_{\alpha'}$.
- This is the sum of the basic energies of two weighted derivatives $D_\alpha \bar{Z}_t$ and $\frac{1}{Z_{,\alpha}} D_\alpha^2 \bar{Z}_t$.
- For $(Z_t(t), Z_{tt}(t)) \in H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 2$,

$$\mathcal{E}(t) < \infty$$

- It allows $\frac{1}{Z_{,\alpha}} = 0$, hence allows water waves with angled crest type singularities.
- No requirement on lower order derivatives.

The main results

Theorem

The water wave equation (1) *is wellposed in the class* $\mathcal{E}(t) < \infty$*. That is, for any data with* $\mathcal{E}(0) < \infty$*, there is a* $T > 0$*, depending only on* $\mathcal{E}(0)$*, such that there is a unique solution of the water wave equation* (1)*, satisfying the initial condition, and the solutions are stable in the sense that the "norm" of the difference of the solutions at later times is controlled by the "norm" of the difference in the initial time.*

Proved via three steps.

Theorem (A priori estimate, R. Kinsey & S. Wu, 2014)

Assume $\mathcal{E}(0) < \infty$. There exists $T > 0$, depending only on $\mathcal{E}(0)$, such that for any solution of water wave equations,

$$\mathcal{E}(t) \leq C(\mathcal{E}(0)) < \infty, \quad \textit{for all } t \in [0, T]. \tag{15}$$

Remarks:

- The self-similar solution has finite energy $\mathcal{E}$.
- In general, surfaces that have angled crests of interior angle $< \frac{\pi}{2}$, and the angle ν of the wave with the vertical wall $\nu < \frac{\pi}{4}$ have finite energy.
- The Stokes wave of maximal height has infinite energy $\mathcal{E}$

Theorem (local existence, S.Wu (2015))

For any initial data satisfying $\mathcal{E}(0) < \infty$, there exists $T > 0$, depending only on $\mathcal{E}(0)$, such that the water wave equation is solvable for time $t \in [0, T]$, with $\mathcal{E}(t) < \infty$ for $t \in [0, T]$.

Theorem (blow-up criteria, S. Wu (2015))

Given smooth data, there is a unique smooth solution exist for a positive time period $[0, T]$. Let T^* be the maximum existence time for the smooth solution. Then either $T^* = \infty$, or $T^* < \infty$, but the interface $z = z(\cdot, t)$ becomes self-intersecting at time T^*, or $\sup_{[0,T^*)} \mathcal{E}(t) = \infty$.

Theorem (uniqueness, S. Wu (2017))

For any data satisfying $\mathcal{E}(0) < \infty$, the solution in the class $\mathcal{E}(t) < \infty$ is unique.

- Idea of proof: construct an appropriate energy functional, show that the "norm" of "the difference of any two solutions" at time t is bounded by the "norm" of the difference of the initial datas.
- $$(\partial_t + b\partial_{\alpha'})^2 \bar{Z}_t + i\frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\bar{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i)$$
- Difficulty: the quasilinear equation has a solution dependent possibly degenerative coefficient.
- Question: how to compare two solutions with their singularities propagate differently?
- Key: understand in precise terms the propagation of the singularities and how it manifests in the solutions.

Conclusion: we have found a framework, which allows non-C^1 interfaces, in which the quasilinear equation

$$(\partial_t + b\partial_{\alpha'})^2 \bar{Z}_t + i\frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\bar{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i)$$

behaves like

$$(\partial_t + b\partial_{\alpha'})^2 \bar{Z}_t + i\mathcal{D}_u \bar{Z}_t = l.o.t.$$

Thanks you very much for your attention.