SINGULAR PERTURBATIONS OF DIRAC HAMILTONIANS: self-adjointness and spectrum.

Luis Vega

basque center for applied mathematics

eman ta zabal zazu

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HADE

The Operator

- $\partial_t \psi = iH\psi \quad ; \quad H = H_0 + \mathbb{V} \quad , \quad \psi = \psi(x,t) \quad , \quad \mathbb{V}(x)$ hermitian
- $H_0 = \frac{1}{i}\alpha \cdot \nabla + m\beta$
- $H_0^2 = -\Delta + m^2$

$$\alpha \cdot \alpha = \mathbb{1} \qquad \alpha = (\alpha_j)$$
$$\alpha\beta + \beta\alpha = 0$$
$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad j \neq k \quad ; \quad \alpha_j^2 = 1 \qquad j = 1,2,3$$

- If $x \in \mathbb{R}^3$ then $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, $\phi, \chi \in \mathbb{C}^2$ (spinors).
- $\mathbb{V}$: "critical" $\qquad \frac{1}{\lambda}\mathbb{V}\left(\frac{x}{\lambda}\right) \sim \mathbb{V}(x)$

Example: Coulomb $\qquad \mathbb{V} = \frac{-\lambda}{|x|}\mathbb{1}$

- $\alpha_j = \begin{pmatrix} 0 & \widehat{\sigma}_j \\ \widehat{\sigma}_j & 0 \end{pmatrix} \qquad j = 1,2,3 \quad , \quad \widehat{\sigma}_j$ Pauli matrices

- $\widehat{\sigma} \cdot A \quad \widehat{\sigma} \cdot B = A \cdot B + i\widehat{\sigma} \cdot A \wedge B$

- $\widehat{\sigma} \cdot \frac{x}{|x|} \quad \widehat{\sigma} \cdot \nabla = \frac{x}{|x|} \cdot \nabla + i\widehat{\sigma} \frac{x}{|x|} \wedge \nabla = \partial_r - \frac{1}{r}\widehat{\sigma} \cdot L$

- $(1 + \widehat{\sigma} \cdot L)^2 \geq 1$

$$H = -i\alpha \cdot \nabla + m\beta = \begin{pmatrix} m & 0 & -i\partial_3 & -\partial_2 - i\partial_1 \\ 0 & m & \partial_2 - i\partial_1 & i\partial_3 \\ -i\partial_3 & -\partial_2 - i\partial_1 & -m & 0 \\ \partial_2 - i\partial_1 & i\partial_3 & 0 & -m \end{pmatrix}$$

$$\begin{pmatrix} m & -\partial_z \\ \partial_{\overline{z}} & -m \end{pmatrix} \begin{pmatrix} m & -\partial_z \\ \partial_{\overline{z}} & -m \end{pmatrix} = \begin{pmatrix} m^2 - \Delta & 0 \\ 0 & -\Delta + m^2 \end{pmatrix}$$

General Questions

(a) Self–adjointness.

(b) Spectrum: Characterization of the ground state by the "right inequality".

Similar questions for a non linear $\mathbb{V}$ always assume some smallness condition on $\mathbb{V}$.

(c) What is a small/big perturbation of H_0?

Coulomb Potential

- $H_0 - \dfrac{\lambda}{|x|}$

(a) Self–adjointness: **Rellich '53**, **Schminke '72**, **Wust '75**, **Nenciu '76**, **Kato '80– '83** (Kato–Nenciu inequality)

Final answer: $|\lambda| < 1$.

(b) "Ground state" $(\lambda > 0)$ Minimization process (**Dolbeault**, **Esteban**, **Séré '00**):

- Variational inequality for $\phi \left(\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \right)$.
- Hardy–Kato–Nenciu type inequalities (**Dolbeault**, **Duoandikoetxea**, **Esteban**, **Loss**, **V. '00**).

Electrostatic Shell Interactions:

$\Omega \subset \mathbb{R}^3$ bounded smooth domain

$\sigma =$ surface measure on $\partial\Omega$

$N =$ outward unit normal vector field on $\partial\Omega$

Electrostatic shell potential $V_\lambda = \lambda\delta_{\partial\Omega}$:

$$\lambda \in \mathbb{R}, \qquad V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)$$

$\varphi_\pm =$ non–tangential boundary values of $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$ when approaching from Ω or $\mathbb{R}^3 \setminus \overline{\Omega}$

Electrostatic shell interaction for H: $H + V_\lambda$

(a) Self-Adjointess

If $\lambda \neq \pm 2 \implies H + V_\lambda$ is self-adjoint on $\mathcal{D}(H + V_\lambda)$.

(**[Arrizabalaga, Mas, V., 2014]**,
more general **[Posilicano, 2008]**
Ω ball $\longrightarrow$ **[Dittrich, Exner, Seba, 1989]**)

$a \in (-m, m)$

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}\,|x|}}{4\pi|x|}\left[a + m\beta + \left(1 - \sqrt{m^2 - a^2}\,|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right]$$

$$= \text{ fundamental solution of } H - a$$

$$\mathcal{D}(H + V_\lambda) = \Big\{\varphi : \quad \varphi = \phi^0 * (G dx + g d\sigma),\ G \in L^2((R)^3)^4 \ \ g \in L^2(\partial\Omega)^4,$$

$$\lambda\left(\phi^0 * (G dx)\right)\big|_{\partial\Omega} = -\left(1 + \lambda C^0_{\partial\Omega}\right) g)\Big\}$$

where $C^a_{\partial\Omega}(g)(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} \phi^a(x - y) g(y) d\sigma(y)\ ,\ x \in \partial\Omega.$

(b) Point Spectrum on $(-m, m)$ for $H + V_\lambda$

Birman–Schwinger principle: $a \in (-m, m), \quad \lambda \in \mathbb{R} \setminus \{0\},$

$$(*) \ \ker(H + V_\lambda - a) \neq 0 \iff \ker\left(\frac{1}{\lambda} + C^a_{\partial\Omega}\right) \neq 0$$

(problem in $\mathbb{R}^3$) **(problem in $\partial\Omega$)**

Properties of $C^a_{\partial\Omega}, \quad a \in [-m, m]$:

(a) $C^a_{\partial\Omega}$ bounded self–adjoint operator in $L^2(\partial\Omega)^4$.

(b) $\left[C^a_{\partial\Omega}(\alpha \cdot N)\right]^2 = -\frac{1}{4} I_d. \quad \left(\alpha \cdot N = \sum_{j=1}^{3} \alpha_j N_j \text{ multiplication operator}\right)$

$$\ker\left(\frac{1}{\lambda} + C^a_{\partial\Omega}\right) \neq 0 \quad \begin{cases} \overset{(a)}{\Longrightarrow} & |\lambda| \geq \lambda_l(\partial\Omega) > 0 \quad \text{and} \quad \lambda_l(\partial\Omega) \leq 2 \\ \overset{(b)}{\Longrightarrow} & |\lambda| \leq \lambda_u(\partial\Omega) < +\infty \quad \text{and} \quad \lambda_u(\partial\Omega) \geq 2 \end{cases}$$

Therefore, $\ker(H + V_\lambda - a) \neq 0 \implies |\lambda| \in [\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$

Main result:

Question: How small can $\left[\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)\right]$ be?

(Isoperimetric–type statement w.r.t. Ω)

(Find optimizers)

Examples: $\Omega \subset \mathbb{R}^3$ bounded smooth domain

- Isoperimetric inequality: $\mathrm{Vol}\,(\Omega)^2 \leq \frac{1}{36}\,\mathrm{Area}\,(\partial\Omega)^3$.
- Pólya–Szegö inequality:

$$\mathrm{Cap}\,(\overline{\Omega}) = \left(\inf_{\nu} \iint \frac{d\nu(x)d\nu(y)}{4\pi|x-y|}\right)^{-1}$$

ν probability Borel measure $\mathrm{supp}\,\nu \subset \overline{\Omega}$

$$\mathrm{Cap}\,(\overline{\Omega}) \geq 2\big(6\pi^2 \mathrm{Vol}\,(\Omega)\big)^{1/3}. \quad \longleftarrow \quad \text{[Pólya, Szegö, 1951]}$$

In both cases, $=$ holds $\iff$ Ω is a ball.

Theorem [AMV2016].– $\Omega \subset \mathbb{R}^3$ bounded smooth domain. If

$$m \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\overline{\Omega})} > \frac{1}{4\sqrt{2}},$$

then

$$\begin{aligned} &\sup\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \ \text{ for some } \ a \in (-m, m)\} \\ &\geq 4\left(m \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\overline{\Omega})} + \sqrt{m^2 \frac{\mathrm{Area}(\partial\Omega)^2}{\mathrm{Cap}(\overline{\Omega})^2} + \frac{1}{4}}\right) \end{aligned}$$

and

$$\begin{aligned} &\inf\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \ \text{ for some } \ a \in (-m, m)\} \\ &\leq 4\left(-m \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\overline{\Omega})} + \sqrt{m^2 \frac{\mathrm{Area}(\partial\Omega)^2}{\mathrm{Cap}(\overline{\Omega})^2} + \frac{1}{4}}\right) \end{aligned}$$

In both cases, $=$ holds $\iff$ Ω is a ball.

Ingredients of the proof:

(1) The monotonicity of $\lambda(a)$ in $\ker\left(\dfrac{1}{\lambda(a)} + C^a_{\partial\Omega}\right)$ reduces the study of (∗) to $a = \pm m$.

(2) The quadratic form inequality relates $\sup\{|\lambda| : \ker(H + V_\lambda - a)) \neq 0 \ \text{ for some } \ a \in (-m, m)\}$ in (∗) with the optimal constant of an inequality involving the single layer potential K and a SIO. (Here appears the $1/4\sqrt{2}$)

(3) Isoperimetric type statement for K in terms of $\text{Area}\,(\partial\Omega) \setminus \text{Cap}\,(\overline{\Omega})$.

Proof:

(1) $$\ker\left(\frac{1}{\lambda(a)} + C^a_{\partial\Omega}\right) \neq 0 \quad \Longrightarrow \quad C^a_{\partial\Omega} g_a = \frac{1}{\lambda(a)} g_a \ , \ \ \|g_a\| = 1$$

$$\Longrightarrow \quad \frac{1}{\lambda(a)} = \frac{1}{\lambda(a)}\langle g_a, g_a\rangle = \langle C^a_{\partial\Omega} g_a, g_a\rangle$$

$$C^a_{\partial\Omega} \hookrightarrow (H-a)^{-1} \quad \Longrightarrow \quad \frac{d}{da} C^a_{\partial\Omega} \hookrightarrow (H-a)^{-2}$$

$$\Longrightarrow \quad \frac{d}{da}\left(\frac{1}{\lambda(a)}\right) \sim \langle (H-a)^{-2} g_a, g_a\rangle = \|(H-a)^{-1} g_a\|^2 \geq 0$$

(assume g_a independent of a)

(2)

$$\left.\begin{array}{l} Kf(x) = \dfrac{1}{4\pi}\displaystyle\int \frac{f(y)}{|x-y|} d\sigma y \qquad \begin{pmatrix}\text{compact}\\ \text{positive}\end{pmatrix} \\ Wf(x) = \dfrac{1}{4\pi}\lim_{\epsilon\to 0}\displaystyle\int_{|x-y|>\epsilon} i\cdot\widehat{\sigma}\cdot\frac{x-y}{|x-y|^3} f(y)\, d\sigma(y) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(SIO)} \end{array}\right\} C^{a}_{\partial\Omega} = \begin{pmatrix} 2mK & W \\ W & 0 \end{pmatrix}$$

$$\left(\widehat{\sigma} = (\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right)$$

Then,

$$[C^{m}_{\partial\Omega}(\alpha\cdot N)]^2 = -\frac{1}{4} \quad\Longrightarrow\quad \begin{cases} \{(\widehat{\sigma}\cdot N)K, (\widehat{\sigma}\cdot N)W\} = 0 \\ [(\widehat{\sigma}\cdot N)W]^2 = -\dfrac{1}{4} \end{cases} \qquad (**)$$

$$\ker\left(\frac{1}{\lambda}+C^m_{\partial\Omega}\right)\neq 0 \quad\Longrightarrow\quad C^m_{\partial\Omega}g=\frac{1}{\lambda}g \qquad g=\begin{pmatrix}\mu\\ h\end{pmatrix}$$

$$\Longrightarrow\quad \begin{cases} 2mK\mu+Wh &= -\dfrac{1}{\lambda}\mu \\[2ex] W\mu &= -\dfrac{1}{\lambda}h \end{cases}$$

$$\overset{(**)}{\Longrightarrow}\quad \exists f\in L^2(\partial\Omega)^2,\ f\neq 0 \ \text{such that}\ \left(-\frac{8m}{\lambda}K+1-\frac{16}{\lambda^2}W^2\right)f=0$$

Multiply by $\overline{f}$ and integrate on $\partial\Omega$:

$$\begin{pmatrix}\text{decreasing}\\ \text{on }\lambda>0\end{pmatrix}\qquad \left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\underbrace{\int_{\partial\Omega}Kf\cdot\overline{f}}_{\geq 0}=\int_{\partial\Omega}|f|^2$$

Quadratic form inequality:

$$\lambda_\Omega=\inf\left\{\lambda>0:\left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\int_{\partial\Omega}Kf\cdot\overline{f}\leq\int_{\partial\Omega}|f|^2\qquad\forall f\in L^2(\partial\Omega)^2\right\}$$

[Esteban,Séré, 1997]

(3) Ω ball $\quad\Longrightarrow\quad \|W\|^2_{\partial\Omega} = \frac{1}{4}$

$$\left(\text{“} \Longleftarrow \text{”} \quad \begin{matrix} \text{[Khavinson, Putinar, Shapiro, 2007]} \\ \text{[Hofmann, Marmolejo–Olea, Mitrea, Pérez–Esteva, Taylor, 2009]} \end{matrix} \right)$$

$$\Longrightarrow \quad \lambda_\Omega = 4\left(m\|K\|_{\partial\Omega} + \sqrt{m^2\|K\|^2_{\partial\Omega} + \|W\|^2_{\partial\Omega}} \right)$$

Ω general,

$$\|K\|_{\partial\Omega} = \sup_{f\neq 0} \frac{1}{\|f\|^2_{\partial\Omega}} \int_{\partial\Omega} Kf \cdot \overline{f} \geq \iint \frac{d\sigma(y)}{4\pi|x-y|} \frac{d\sigma(x)}{\sigma(\partial\Omega)} \geq \frac{\text{Area}\,(\partial\Omega)}{\text{Cap}\,(\overline{\Omega})}$$

$(\mathbf{f} = \mathbf{1})$

(“ = ” $\quad\Longleftrightarrow\quad$ Ω is a ball: **Gruber’s conjecture** [Reichel, 1996, 1997])

Coulomb Potential again

Recall Birman–Schwinger principle:

$$\frac{d}{da}\left(\frac{1}{\lambda(a)}\right) \sim \langle (H-a)^{-2} g_a, g_a \rangle = \|(H-a)^{-1} g_a\|^2 \geq 0$$

(assume g_a independent of a)

This suggests another way of obtaining the ground state for the Coulomb potential $V(x) = -\frac{\lambda}{|x|}$:

$$\frac{m^2 - a^2}{m^2} \int \frac{|\psi|^2}{|x|} \leq \int \left| \left(\frac{1}{i} \alpha \cdot \nabla + m\beta + a \right) \psi \right|^2 |x|$$

(**Arrizabalaga, Duoandikoetxea, V. '13**; **Cassano, Pizzichilo, V. '17**)

The inequality is optimal and it is achieved for $a > 0$ by the ground state of $V_a(x) = -\frac{m^2 - a^2}{m^2} \frac{1}{|x|}$.

The proof is a consequence of the "uncertainty principle".

- $2\mathrm{Re}\,\langle S\psi, A\psi\rangle = \langle (SA - AS)\psi, \psi\rangle$ if $S^* = S$ and $A^* = -A$.

- $2\mathrm{Re}\,\langle A_1\psi, A_2\psi\rangle = -\langle (A_1A_2 + A_2A_1)\psi, \psi\rangle$ if $A_1^* = -A_1$ and $A_2^* = -A_2$.

In our case the right choice is:

$$2\mathrm{Re}\,\langle \underbrace{(\alpha\cdot\nabla + i(m\beta + a))}_{A_1}\psi, (1+\sigma\cdot L)\mathbb{1}\,\alpha\cdot\frac{x}{|x|}\Big(\underbrace{\frac{a}{m}\beta}_{S} + \underbrace{1}_{A_2}\Big)\rangle.$$

$\lambda^2 = 4$

Joint work with T. Ourmieres-Bonafos.

Recent work by

- Benguria, Fournais, Stockmeyer, Van den Bosch.
- Behrndt, Exner, Holzmann, Lotoreichik. $(\lambda^2 = 4c^2 \quad ; \quad c \to \infty)$
- Behrndt, Holzmann.
- Mas, Pizzichilo. $(\lambda^2 \ \textbf{small})$

For $\lambda \in \mathbb{R}$, we introduce the matrix valued function:

$$\mathcal{P}_\lambda = \frac{\lambda}{2} + i(\alpha \cdot \mathbf{n}).$$

For $(u_+, u_-) \in H^1(\Omega_+)^4 \times H^1(\Omega_-)^4$ we define the following transmission condition in $H^{1/2}(\partial\Omega)^4$

$$(*) \qquad \mathcal{P}_\lambda t_{\partial\Omega} u_+ + \mathcal{P}_\lambda^* t_{\partial\Omega} u_- = 0, \quad \text{on } \partial\Omega.$$

Alternativaley, as $\mathcal{P}_\lambda$ is invertible, we can see the transmission condition as

$$t_{\partial\Omega} u_+ = \mathcal{R}_\lambda t_{\partial\Omega} u_-, \text{ with } \mathcal{R}_\lambda := \frac{1}{\lambda^2/4 + 1}\big(1 - \frac{\lambda^2}{4} + \lambda(i\alpha \cdot \mathbf{n})\big).$$

Definition.– Let $\lambda \in \mathbb{R}$ and $m \in \mathbb{R}$. The Dirac operator coupled with an electrostatic δ–shell interaction of strength λ is the operator $\Big(\mathcal{H}_\lambda(m), \mathrm{dom}\,(\mathcal{H}_\lambda(m))\Big)$, acting on $L^2(\mathbb{R}^3)^4$ and defined on the domain

$$\mathrm{dom}\,\Big(\mathcal{H}_\lambda(m)\Big) = \Big\{(u_+, u_-) \in H^1(\Omega_+)^4 \times H^1(\Omega_-)^4 \,:\, (u_+, u_-) \text{ satisfies } (*)\Big\}$$

It acts in the sense of distributions as $\mathcal{H}_\lambda(m)u = \Big(\mathcal{H}(m)u_+, \mathcal{H}(m)u_-\Big)$ where we identify an element of $L^2(\Omega_+)^4 \times L^2(\Omega_-)^4$ with an element of $L^2(\mathbb{R}^3)^4$.

Theorem.– Let $m \in \mathbb{R}$. The following holds:

(i) If $\lambda \neq \pm 2$, the operator $\Big(\mathcal{H}_\lambda(m), \mathrm{dom}\,\big(\mathcal{H}_\lambda(m)\big)\Big)$ is self-adjoint.

(ii) If $\lambda = \pm 2$, the operator $\Big(\mathcal{H}_\lambda(m), \mathrm{dom}\,\big(\mathcal{H}_\lambda(m)\big)\Big)$ is essentially self-adjoint and we have

$$\mathrm{dom}\,(\mathcal{H}_\lambda(m)) \subsetneq \mathrm{dom}\,(\overline{\mathcal{H}}_\lambda(m)) = \Big\{(u_+, u_-) \in H(\alpha, \Omega_+) \times H(\alpha, \Omega_-) : (u_+, u_-) \text{ satisfies } (*)\Big\},$$

where the transmission condition holds in $H^{-1/2}(\partial\Omega)^4$.

Here:

- $$H(\alpha,\Omega) := \left\{u \in L^2(\Omega)^4 : \mathcal{H}u \in L^2(\Omega)^4\right\} = \left\{u \in L^2(\Omega)^4 : (\alpha\cdot\mathbf{D})u \in L^2(\Omega)^4\right\},$$
- $\alpha\cdot\mathbf{D} = \dfrac{1}{i}\alpha\cdot\nabla.$

Let $\varepsilon = \pm 1$ and $\lambda = 2\varepsilon$. Let $u = (u_+, u_-) \in \mathrm{dom}\left(\mathcal{H}_\lambda(m)\right)$, $u_\pm$ can be rewritten $u_\pm = (u_\pm^{[1]}, u_\pm^{[2]})$ and, for $x \in \partial\Omega$, the transmission condition reads

$$\begin{pmatrix} u_+^{[1]}(x) \\ u_+^{[2]}(x) \end{pmatrix} = \begin{pmatrix} 0 & -i\varepsilon\sigma\cdot\mathbf{n}(x) \\ -i\varepsilon\sigma\cdot\mathbf{n}(x) & 0 \end{pmatrix}\begin{pmatrix} u_-^{[1]}(x) \\ u_-^{[2]}(x) \end{pmatrix} = \begin{pmatrix} -i\varepsilon\sigma\cdot\mathbf{n}u_-^{[2]}(x) \\ -i\varepsilon\sigma\cdot\mathbf{n}u_-^{[1]}(x) \end{pmatrix}.$$

The Calderón projectors are the bounded linear operators from $H^{-1/2}(\partial\Omega)^4$ onto itself defined as:

$$\mathcal{C}_{\pm} = \pm i C_{\pm}(\alpha \cdot \mathbf{n}).$$

As $\partial\Omega$ is $\mathcal{C}^2$, the multiplication by $\alpha \cdot \mathbf{n}$ is a bounded linear operator from $H^{-1/2}(\partial\Omega)^4$ onto itself. Thus the definition makes sense.

Their formal adjoints are:

$$\mathcal{C}_{\pm}^* = \mp i(\alpha \cdot \mathbf{n}) C_{\mp}.$$

By definition, $\mathcal{C}_{\pm}^*$ is a linear bounded operator from $H^{-1/2}(\partial\Omega)^4$ onto itself.

Note that the Calderón projectors satisfy:

$$\mathcal{C}_{\pm} - \mathcal{C}_{\pm}^{*} = \pm i\mathcal{A},$$

where $\mathcal{A}$ does not depend on the sign $\pm$. Roughly speaking, $\mathcal{A}$ measures the defect of self-adjointness of the Calderón projectors.

Proposition.– The operator $\mathcal{A}$ extends into a bounded operator from $H^{-1/2}(\partial\Omega)^4$ to $H^{1/2}(\partial\Omega)^4$ and it is compact.

This system rewrites as:

$$\begin{pmatrix} \frac{\lambda}{2} & -i\alpha\cdot\mathbf{n} \\ i\alpha\cdot\mathbf{n} & \frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} \mathcal{C}_+(t_{\partial\Omega}u_+) \\ \mathcal{C}_-(t_{\partial\Omega}u_-) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\lambda}{2} & -i\alpha\cdot\mathbf{n} \\ i\alpha\cdot\mathbf{n} & -\frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} \mathcal{C}_+(t_{\partial\Omega}u_-) \\ \mathcal{C}_-(t_{\partial\Omega}u_+) \end{pmatrix} + \begin{pmatrix} (\alpha\cdot\mathbf{n})\mathcal{A}(t_{\partial\Omega}u_+ - t_{\partial\Omega}u_-) \\ -(\alpha\cdot\mathbf{n})\mathcal{A}(t_{\partial\Omega}u_+ - t_{\partial\Omega}u_-) \end{pmatrix}.$$

The right-hand side is in $H^{1/2}(\partial\omega)^8$ and the matrix in the left-hand side is invertible in $H^{1/2}(\partial\Omega)^8$ as long as $\lambda \neq \pm 2$. Thus $t_{\partial\Omega}u_\pm \in H^{1/2}(\partial\Omega)^4$ and $\mathrm{dom}\left(\mathcal{H}_\lambda(m)^*\right) \subset \mathrm{dom}\left(\mathcal{H}_\lambda(m)\right)$. The reciprocal inclusion is similar.