Global stability of Minkowski space for the Einstein equation with a massive scalar field

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(with A. Ionescu)

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Outline

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Nonlinear Optics: **NLS**...

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Two key ideas in our work are that

- There is a strong unity in all these problems despite key specificities ("In varietate unitas")
- The full problem is richer than the sum of its part and incorporating back each specific interaction leads to interesting new problems ("Stronger together")

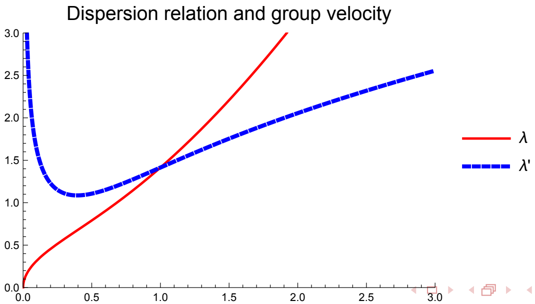
Water-waves dispersion relation

Linearize at equilibrium:

$$\left(\partial_t + i\sqrt{|\nabla|(g - \sigma\Delta)}\right)\mathbf{U} = 0.$$

Dispersion relation

$$\Lambda(\xi) = \lambda(|\xi|), \qquad \lambda(r) = \sqrt{g\,r + \sigma\,r^3}.$$



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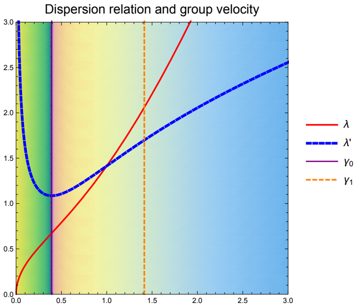

In water, $\gamma_0 \sim 58\text{m}^{-1}$, $2\pi/\gamma_0 \simeq 1.7\text{cm}$.

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These problems reduce, after appropriate manipulations to proving small data - global existence results.

Outline

The Einstein equation

Our motivation is to consider the stability of Minkowski space for the **Einstein equation** in the presence of matter fields:

$$E(\mathbf{g}) = \mathrm{Ric}_{\mathbf{g}} - \frac{1}{2}\mathrm{Scal}_{\mathbf{g}}\mathbf{g} = 8\pi G\mathcal{T}$$

where $\mathrm{Ric}_{\mathbf{g}}$ denotes the *Ricci* tensor and $\mathrm{Scal}_{\mathbf{g}}$ the *scalar curvature* of a Lorentzian metric $\mathbf{g}$ and $\mathcal{T}$ is an *energy-momentum* tensor representing the action of the matter on the space-time $(M, \mathbf{g})$. The unknowns are $\mathbf{g}$ and $\mathcal{T}$.

Dynamic equations

In an Harmonic (or "de Donder") gauge, the principal symbol of Ric_g is

$$(\mathrm{Ric}_{\mathbf{g}})_{\alpha\beta} = \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu\mathbf{g}_{\alpha\beta} + \text{l.o.t.}$$

and since $\mathbf{g}$ is *Lorentzian*, the metric equations are hyperbolic. The dynamic equations for the matter field then follows from the *Bianchi identities* which yield

$$\nabla^\alpha\mathcal{T}_{\alpha\beta} = 0.$$

This usually leads to hyperbolic equations and makes the system self-contained.

Previous results about the Einstein equation

Many previous results have focused on the stability of the Minkownski space for Einstein equations in vacuum [**Christodoulou-Klainerman**, **Bieri**, **Nicolo**, **Friedrich**], with an electromagnetic field [**Zipser**], with a massless scalar field [**Lindblad-Rodnianski**].
Also: many works around stability of more involved solutions, w/o cosmological constants (Schwartzshild, Kerr, FLRW, Oppenheimer-Snyder) [**Alexakis, Dafermos, Huneau, Rodnianski, Schlue, Slapenkov-Rothman, Speck, Tataru, Tohanehanu, Vasy**. . .]

The matter field

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Challenge: need robust methods not just taylored to one specific dispersion relation (here wave): fewer vector fields, fewer null terms...

Massive scalar field

Simple model: massive scalar field $\psi : M \to \mathbb{R}$ whose stress-energy tensor is given by

$$\mathcal{T}_{\alpha\beta} = \partial_\alpha \psi \cdot \partial_\beta \psi - \frac{1}{2}(\mathbf{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + \psi^2) \cdot \mathbf{g}_{\alpha\beta}$$

The *Bianchi identity* then gives that

$$0 = \mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\psi - 2\psi = -(\partial_t^2 - \Delta + 2)\psi + \text{nonlinear}$$

which is a (quasilinear) Klein-Gordon equation satisfying all the requirements above.

Previous work on Einstein-Klein-Gordon

The stability of Minkowski space for restricted initial data has been studied by Q. **Wang** and **LeFloch-Ma** adapting the *hyperbolic foliation method* of **Klainerman** to the quasilinear case.

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Our goal is to provide a new proof which allows to consider general initial data.

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Method

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1. Reduction to a (linearly-)diagonal dispersive system

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3. Dispersive analysis

Reduction to a diagonal dispersive system

Goal: find **U** which controls all the relevant quantities and satisfies a diagonal dispersive system:

$$\left(\partial_t + iT_{\Sigma_j}\right)\mathbf{U}_j = \mathcal{N}(\mathbf{U}) = O(\mathbf{U}^2 + h.o.t.)$$

where Σ_j are real, dispersive symbols:

$$\det \nabla^2_{\zeta\zeta}\Sigma_j(x,\zeta) \neq 0.$$

Energy estimates

- Developed by **John, Klainerman, Shatah** (see also also [**Alinhac, Wu, Germain-Masmoudi, Alazard-Delort, Hunter-Ifrim-Tataru, I.-Pusateri, Deng-I.-P.**])

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- **Goal**: Control of L^2-norms based on vector fields that commute with the linear operators:

$$\mathcal{E}(\mathbf{U}) = \|\mathbf{U}\|_{L^2}^2 + \|\mathcal{V}\mathbf{U}\|_{L^2}^2 + \|\mathcal{V}_1\mathcal{V}_2\mathbf{U}\|_{L^2} \ldots$$

Related to symmetries/invariances of the problem.

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  Typical $\mathcal{V}$: flat derivatives: ∂_x, ∂_t, rotations: $\Omega_{jk} = x_j\partial_k - x_k\partial_j$, scaling: $\mathcal{S} = x \cdot \nabla + ct\partial_t$, Lorentz boosts $\Gamma_j = x_j\partial_t + t\partial_j$
- **Implementation:**

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- **Implementation:**
  - Relatively straightforward if fast convergence to equilibrium (integrable decay of solutions $\|\mathbf{U}(t)\|_{L^\infty} \lesssim 1/t$)

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Related to symmetries/invariances of the problem.
- **Implementation:**
  - Relatively straightforward if fast convergence to equilibrium (integrable decay of solutions $\|\mathbf{U}(t)\|_{L^\infty} \lesssim 1/t$)
  - Otherwise more delicate, need paralinearization, information about resonances (normal forms), restricted nondegeneracy condition...

Dispersive analysis

Building on ideas by **Delort-Fang-Xue**, **Gustafson-Nakanishi-Tsai**, **Germain-Masmoudi-Shatah** and developed by **Deng, Guo, Hani, Ifrim, Ionescu, Kato, P., Pusateri, Tataru, Tzvetkov, Wang, Stingo**. . .

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Key: devise a good norm.

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One can start by postulating some form of (modified) *scattering* back to the equilibrium and hope that a good asymptotic model for the evolution is

$$(\partial_t + i\Lambda)\mathbf{U} = O(\text{“perturbative”})$$

Express the solution as superposition of linear solutions:

$$\mathbf{U}(t) = e^{-it\Lambda}\mathbf{V}, \qquad \int_{\mathbb{R}} \|\partial_t \mathbf{V}(t)\| dt < \infty$$

Designer norm

$\mathbf{V}(t)$ given by the Duhamel formula

$$\mathbf{V}(t) = \mathbf{V}(0) - i\int_{s=0}^{t} e^{is\Lambda}\left\{e^{-is\Lambda}\mathbf{V}(s)\cdot e^{-is\Lambda}\mathbf{V}(s)\right\}ds$$

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First approx.: Schwartz inputs, constant in time. Integral becomes

$$\mathcal{F}^{-1} \int_{\mathbb{R}^3} \frac{e^{it\Phi} - 1}{\Phi} \widehat{\mathbf{V}}(\xi - \eta)\widehat{\mathbf{V}}(\eta) d\eta, \quad \Phi = \Lambda(\xi) - \Lambda(\xi - \eta) - \Lambda(\eta)$$

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Main contributions from resonances: $\{\Phi = 0\}$ and from coherence (stationary points: $\{\nabla_\eta \Phi = 0\}$).
In case the coherence are nondegenerate, $\det \nabla^2_{\eta\eta}\Phi \neq 0$, one can use stationary phase analysis to obtain precise control **[I.-P.]**.

Harmonic gauge

The Einstein equations are *covariant* and, in any set of coordinates, have a large degeneracy corresponding to change of variables. This is fixed by choosing a gauge. A commonly used one is the *Harmonic* gauge:

$$\forall \nu, \quad \Box_{\mathbf{g}} x^{\nu} = 0 \quad \Leftrightarrow \quad \partial_\alpha \mathbf{g}^{\alpha\nu} + \frac{1}{2}\mathbf{g}^{\mu\nu}\mathbf{g}^{\alpha\beta}\partial_\mu \mathbf{g}_{\alpha\beta} = 0$$

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To understand them better, we introduce a *Hodge decomposition* adapted to the Minkowski decomposition $\mathbb{R}_t \oplus \mathbb{R}^3_x$:

$$\mathbf{g} = \begin{pmatrix} a & \vec{v} \\ \vec{v}^T & M_{3\times 3}, \end{pmatrix}, \qquad \vec{v} = \nabla b + \nabla \times \omega,$$

$$M_{3\times 3} = \nabla^2 c + \left[\nabla(\nabla \times \Omega) + \nabla(\nabla \times \Omega)^T\right] + C$$

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The decomposition of the matrix M is associated to the maps

$$\Gamma(\mathbb{S}_{3\times 3}) \xrightarrow{\mathrm{Div}} \Gamma(\mathbb{R}^3) \xrightarrow{\mathrm{div}} \Gamma(\mathbb{R})$$

Einstein equations in Hodge decomposition

Hodge decomposition *parameterizes* the set of Lorentzian metric fields by $F = \frac{1}{2}(a + c)$, $\underline{F} = \frac{1}{2}(a - c)$, $b \in \mathbb{R}$, ω, $\Omega \in \mathbb{R}^3$ and $C \in \mathbb{S}^0_{3\times 3}$.

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Harmonic gauge equations become

$$\begin{aligned} \partial_t \underline{F} - b + \partial_t Tr(C) &= Nonlinear, \\ \partial_t b - \Delta \underline{F} + \Delta Tr(C) &= Nonlinear, \\ \partial_t \omega + \Delta \Omega &= Nonlinear. \end{aligned}$$

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This shows that Ω, b and $Tr(C)$ are slaved (to linear order) to the other unknowns. Thus Einstein has 6 dynamics unknowns:

$$F, \quad \underline{F}, \quad \omega, \quad \vartheta = C - \frac{1}{3} Tr(C)\delta_{ij}$$

Outline

The Einstein-massive scalar field

The Einstein equations then read (neglecting null forms)

$$\begin{aligned} \Box\vartheta &= 0, \qquad \Box\omega = 0, \\ \Box \mathbf{F} &= -\left[(\partial_t\psi)^2 + R_pR_r(\partial_p\psi\partial_r\psi) - \psi^2\right], \\ \Box \underline{\mathbf{F}} &= -(\partial_t\psi)^2 + R_jR_k(\partial_j\psi\partial_k\psi) + Q[\vartheta,\vartheta], \\ (\Box - 1)\psi &= -(\mathbf{g}^{\alpha\beta} - m^{\alpha\beta})\partial_{\alpha\partial_\beta}\psi \end{aligned}$$

Note that the quasilinear terms for the metric are null and have been omitted, but this requires additional care.

The Wave-Klein-Gordon system

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This led to the following system:

$$\begin{aligned}(\partial_t^2 - \Delta)\mathbf{u} &= A\mathbf{v}\cdot\mathbf{v} + B^{ij}\partial_i\mathbf{v}\cdot\partial_j\mathbf{v},\\ (\partial_t^2 - \Delta + 1)\mathbf{v} &= \mathbf{u}\cdot H^{ij}\partial_i\partial_j\mathbf{v}\end{aligned}$$

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This is a quasilinear hyperbolic system of coupled wave and Klein-Gordon equations.
We first propose to study this system.

Quadratic phase

The nonlinearity contains essentially only one interaction: $KG \times KG \times Wa$. This corresponds to the quadratic phase

$$\Phi(\xi_1, \xi_2) = |\xi_1 + \xi_2| \pm \Lambda(\xi_1) \pm \Lambda(\xi_2), \qquad \Lambda(x) = \sqrt{1 + x^2}$$

$$|\Phi(\xi_1, \xi_2)| \gtrsim (1 + |\xi_1|^2 + |\xi_2|^2)^{-1} |\xi_1 + \xi_2|.$$

Thus many quadratic interactions are *nonresonant*, but the case when $t|\xi_1 + \xi_2| \simeq 1$ requires a particular care.

Nonlinear asymptotic behavior

Simple to see that wave solution will behave differently from linear: for

$$-\Box \mathbf{u} = \mathbf{v}^2 + |\nabla \mathbf{v}|^2, \qquad \mathbf{v}^2 + |\nabla \mathbf{v}|^2 \gtrsim \frac{1}{\langle t \rangle^3} \mathbb{1}_{\{|x| \le t/2\}}$$

and radial data, one can easily see that

$$\mathbf{u}(r,t) \gtrsim \frac{1}{t} \mathbb{1}_{\{|x| \le t/4\}}$$

and u has significant (nonintegrable) presence *inside* the light-cone. This in turn makes the quasilinear term

$$\left(\partial_t^2 - (1 + \mathbf{u})\Delta + 1\right) \mathbf{v} = 0$$

play a significant role over large time.

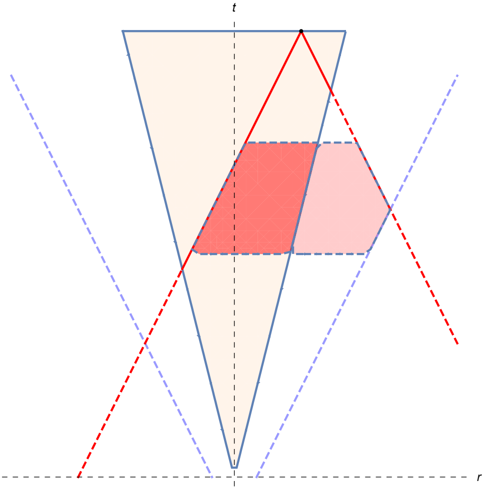

Figure: The wave solution has significant presence *inside* the light cone.

Modified scattering for the Klein-Gordon unknown

After a paradifferential analysis, one can show that the asymptotic behavior follows the equation

$$\left(\partial_t^2 - \Theta^2\right)\mathbf{v}_\infty = 0,$$
$$\widehat{\Theta^2}(\xi, t) = |\xi|^2 + 1 + \mathbf{u}(t\nabla\Lambda(\xi), t)H^{ij}\xi_i\xi_j, \qquad \Lambda(\xi) = \sqrt{1 + |\xi|^2}$$

that is, the dispersion relation has to be modified by the value of u *on the linear KG-characterisitics*.

Derivative imbalance

Another fundamental difficulty comes from the derivative imbalance of the nonlinearity. Indeed, the (linear) energy allows to control

$$\mathcal{E}(t) \simeq \|\nabla_{x,t}\mathbf{U}(t)\|_{L^2}^2 + \|\mathbf{V}(t)\|_{H^1}^2, \qquad \mathbf{U} = \mathcal{V}\mathbf{u} \qquad \mathbf{V} = \mathcal{V}\mathbf{v}$$

where $\mathcal{V}$ denotes one of the commuting vector fields. Thus after commuting with a vector field, we obtain terms of the form

$$\Box\mathcal{V}\mathbf{u} = \mathbf{v}\cdot\mathcal{V}\mathbf{v},$$
$$(\Box - 1)\mathcal{V}\mathbf{v} = \mathcal{V}\mathbf{u}\cdot\partial^2\mathbf{v} + \mathbf{u}\cdot\partial^2(\mathcal{V}\mathbf{v})$$

Negative derivative

This derivative imbalance is important since u can develop significant low-frequency ($\sim 1/t$) presence. To overcome this, we gain twice a $1/2$ derivative by exploiting the *fast* decay of the Klein-Gordon equation: $|\partial^j \mathbf{v}| \lesssim 1/t^{\frac{3}{2}}$:

$$\begin{aligned}
\frac{d}{dt}\|\nabla_{x,t}|\nabla|^{-\frac{1}{2}}\mathcal{V}\mathbf{u}\|_{L^2} &\lesssim \||\nabla|^{-\frac{1}{2}}\left\{\mathbf{v}\cdot\mathcal{V}\mathbf{v}\right\}\|_{L^2} \\
&\lesssim \langle t\rangle^{\frac{1}{2}}\|\mathbf{v}\|_{L^\infty}\|\mathcal{V}\mathbf{v}\|_{L^2}, \\
\frac{d}{dt}\|\langle\nabla\rangle\mathcal{V}\mathbf{v}\|_{L^2} &\lesssim \|\mathcal{V}\mathbf{u}\|_{L^2}\|\partial^2\mathbf{v}\|_{L^\infty} \\
&\lesssim \langle t\rangle^{\frac{1}{2}}\|\nabla_{x,t}|\nabla|^{-\frac{1}{2}}\mathcal{V}\mathbf{u}\|_{L^2}\|\partial^2\mathbf{v}\|_{L^\infty}
\end{aligned}$$

Main result

In a joint work with A. Ionescu, we obtain

Global existence for the Wave-Klein-Gordon system, [I. P.]

Given I.D. $(u(t=0), \partial_t u(t=0), v(t=0), \partial_t v(t=0))$ satisfying

$$\| \nabla |\nabla|^{-\frac{1}{2}} \mathcal{V}\vec{\mathbf{u}}(t=0)\|_{L^2} + \|\mathcal{V}\vec{\mathbf{v}}(t=0)\|_{H^1} \leq \varepsilon_0 \leq \overline{\varepsilon}$$

there exists a global solution to the Wave-Klein-Gordon system.
Energy grows slowly and modified scattering behavior for $t \gg 1$:

$$\Box \mathbf{u}_\infty = 0,$$

$$(\partial_t^2 - \Theta^2)\mathbf{v}_\infty = 0, \qquad \widehat{\Theta^2}(\xi, t) = |\xi|^2 + 1 + \mathbf{u}(t\nabla\Lambda(\xi), t)H^{ij}\xi_i\xi_j.$$

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- Our I.D. not necessarily in L^2. Important for application to Einstein equations.
- Analysis becomes significantly easier if quasilinear term is replaced by $\partial \mathbf{u} \cdot \partial^2 \mathbf{v}$. Then, recovers linear scattering [**Georgiev**, **Katayama**].

Outline

The positive mass theorem

Definition of I.V.P. for Einstein-equation involved, but positive mass theorem implies that, for suitable perturbations of Minkowski space,

$$\mathbf{g}_{\alpha\beta} = m_{\alpha\beta} + \frac{M}{r}\mathbf{g}^{(1)}_{\alpha\beta} + \frac{1}{r^2}\mathbf{g}^{(2)}_{\alpha\beta} + \dots$$

and $M = 0$ if and only if $\mathbf{g}_{\alpha\beta} = m_{\alpha\beta}$. Therefore it is important to consider initial data with slow decay (with $\mathbf{g} - m$ not in L^2!).

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Definition of I.V.P. for Einstein-equation involved, but positive mass theorem implies that, for suitable perturbations of Minkowski space,

$$\mathbf{g}_{\alpha\beta} = m_{\alpha\beta} + \frac{M}{r}\mathbf{g}^{(1)}_{\alpha\beta} + \frac{1}{r^2}\mathbf{g}^{(2)}_{\alpha\beta} + \dots$$

and $M = 0$ if and only if $\mathbf{g}_{\alpha\beta} = m_{\alpha\beta}$. Therefore it is important to consider initial data with slow decay (with $\mathbf{g} - m$ not in L^2!).
In addition, now expect same long range effect for wave equation:

$$(\partial_t^2 - \mathcal{D}^2)\mathbf{u}_\infty = 0, \quad \widehat{\mathcal{D}^2}(\xi, t) \simeq |\xi|^2 + \mathbf{g}^{\alpha\beta}(t\frac{\xi}{|\xi|}, t)\mathfrak{n}_\alpha\mathfrak{n}_\beta, \ \mathfrak{n} = (-|\xi|, \xi_j)$$

The asymptotic system

Another source of difficulty: logarithmic growth of some components of the metric. Fortunately, this is only present in some terms. In fact, nilpotent structure ([**L.R.**]):

$$\Box\phi_1 = 0, \qquad \Box\phi_2 = 0,$$
$$\Box\phi_3 = \phi_1 \cdot \partial^2\phi_3 + (\partial\phi_2)^2$$

which forces

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In turn, nonintegrable decay of metric: use paradifferential formulation of equations for energy estimates.