On the transport property of Gaussian measures under Hamiltonian PDE dynamics

Tadahiro Oh

The University of Edinburgh

July 03, 2017

with

Nikolay Tzvetkov (Cergy-Pontoise) and Philippe Sosoe (Harvard)

Analyse asymptotique des équations d'évolution
Centre International de Rencontres Mathématiques

Nonlinear dispersive Hamiltonian PDEs

Nonlinear Schrödinger equations (NLS):

$$i\partial_t u - \Delta u \pm |u|^{p-1}u = 0, \qquad x \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$

- Hamiltonian: $H(u) = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2 dx \pm \frac{1}{p+1}\int_{\mathbb{T}^d} |u|^{p+1} dx$

Nonlinear wave equations (NLW):

$$\partial_t^2 u + mu - \Delta u \pm |u|^{p-1}u = 0, \qquad m \geq 0, \quad x \in \mathbb{T}^d$$

Goal:

Study transport properties of (weighted) Gaussian measures on $\mathcal{D}'(\mathbb{T}^d)$ under Hamiltonian PDE dynamics

Gaussian measures on periodic functions on $\mathbb{T}^d$

Gaussian measures on "$H^s(\mathbb{T}^d)$":

$$\text{“}d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n\in\mathbb{Z}^d} e^{-\frac{1}{2}\langle n\rangle^{2s}|\widehat{u}_n|^2} d\widehat{u}_n\text{”}$$

where $\langle n\rangle = (1+|n|^2)^{\frac{1}{2}}$

- μ_s is *not* a measure on $H^s(\mathbb{T}^d)$
  $\Longrightarrow$ We need to *enlarge the space* in order to make sense of μ_s
- μ_s is a Gaussian probability measure on $H^\sigma(\mathbb{T}^d)$ for $\sigma < s - \frac{d}{2}$
- Under μ_s, a random function u is represented by the random Fourier series:

$$u(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{2\pi i n\cdot x} \in H^\sigma(\mathbb{T}^d) \setminus H^{s-\frac{d}{2}}(\mathbb{T}^d), \text{ almost surely}$$

  where $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ = independent standard $\mathbb{C}$-valued Gaussian r.v.'s
- The triplet (H^s, H^σ, μ_s) forms an abstract Wiener space
  - Also, $(H^s, W^{\sigma,p}, \mu_s)$ for any $p \le \infty$
  - When $s = 1$, μ_1 is basically the periodic *Wiener measure* (strictly speaking, corresponding to the OU process)

Gaussian measures on periodic functions on $\mathbb{T}^d$

Gaussian measures on "$H^s(\mathbb{T}^d)$":

$$\text{“}d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^d} e^{-\frac{1}{2}\langle n \rangle^{2s} |\widehat{u}_n|^2} d\widehat{u}_n\text{”}$$

where $\langle n \rangle = (1 + |n|^2)^{\frac{1}{2}}$

- μ_s is *not* a measure on $H^s(\mathbb{T}^d)$
  $\Longrightarrow$ We need to *enlarge the space* in order to make sense of μ_s
- μ_s is a Gaussian probability measure on $H^\sigma(\mathbb{T}^d)$ for $\sigma < s - \frac{d}{2}$
- Under μ_s, a random function u is represented by the random Fourier series:

  $$u(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^s} e^{2\pi i n \cdot x} \in H^\sigma(\mathbb{T}^d) \setminus H^{s - \frac{d}{2}}(\mathbb{T}^d), \text{ almost surely}$$

  where $\{g_n(\omega)\}_{n \in \mathbb{Z}}$ = independent standard $\mathbb{C}$-valued Gaussian r.v.'s

- The triplet (H^s, H^σ, μ_s) forms an abstract Wiener space
  - Also, $(H^s, W^{\sigma,p}, \mu_s)$ for any $p \leq \infty$
  - When $s = 1$, μ_1 is basically the periodic *Wiener measure* (strictly speaking, corresponding to the OU process)

(H, B, μ), abstract Wiener space, H = Cameron-Martin space

Cameron-Martin Theorem: Consider the following translation map:

$$T_h : u \mapsto u + h \qquad \text{for some } h \in B$$

Q: What is the relation between the original Gaussian measure μ on B and the translated measure $\mu^h(\,\cdot\,) = (T_h)_* \mu(\,\cdot\,) = \mu(\,\cdot - h)$?

Theorem: Cameron-Martin '44

- If $h \in H$, μ and μ^h are equivalent (= mutually absolutely continuous).
  Namely, μ is quasi-invariant under T_h
- Otherwise, they are mutually singular

- This allows us to take a derivative of μ in the direction of $h \in H$ (= H-derivative)
  $\Longrightarrow$ starting point of Malliavin calculus
- For μ_s on $\mathcal{D}'(\mathbb{T}^d)$, μ_s and μ_s^h are equivalent if and only if $h \in H^s(\mathbb{T}^d)$.
  Namely, h is $(\frac{d}{2} + \varepsilon)$-*smoother* than typical elements $u \in H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$

(H, B, μ), abstract Wiener space, H = Cameron-Martin space

Cameron-Martin Theorem: Consider the following translation map:

$$T_h : u \mapsto u + h \qquad \text{for some } h \in B$$

Q: What is the relation between the original Gaussian measure μ on B and the translated measure $\mu^h(\,\cdot\,) = (T_h)_*\mu(\,\cdot\,) = \mu(\,\cdot - h)$?

Theorem: Cameron-Martin '44

- If $h \in H$, μ and μ^h are equivalent (= mutually absolutely continuous). Namely, μ is quasi-invariant under T_h
- Otherwise, they are mutually singular

- This allows us to take a derivative of μ in the direction of $h \in H$ (= H-derivative) $\Longrightarrow$ starting point of Malliavin calculus
- For μ_s on $\mathcal{D}'(\mathbb{T}^d)$, μ_s and μ_s^h are equivalent if and only if $h \in H^s(\mathbb{T}^d)$. Namely, h is $(\frac{d}{2} + \varepsilon)$-*smoother* than typical elements $u \in H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$

Ramer's generalization of Cameron-Martin Theorem:

$$T : u \mapsto u + F(u)$$

- We now allow the shift $F(u)$ to depend on a random element $u \in B$

Theorem: Ramer '74 (rough formulation)

μ is quasi-invariant under T if the H-derivative of F at u:

$$DF(u) : H \to H$$

is a Hilbert-Schmidt map for every $u \in B$

- For μ_s on $\mathcal{D}'(\mathbb{T}^d)$, (roughly speaking)

  μ_s is quasi-invariant under T if $F(u)$ is *$(d+\varepsilon)$-smoother* than $u \in H^\sigma(\mathbb{T}^d)$

  (More smoothing than Cameron-Martin Theorem, now that the shift is random)

There are also works on quasi-invariance of μ under flows generated by vector fields: Cruzeiro '83, Peters '95, Bogachev and Mayer-Wolf '99, Ambrosio-Figalli '09, etc.

Duhamel formulation (for NLS):

$$u(t) = \Phi(t)u_0 = S(t)u_0 + \int_0^t S(t-t')|u|^{p-1}u(t')dt'$$
$$= S(t)\left\{u_0 + \underbrace{\int_0^t S(-t')|u|^{p-1}u(t')dt'}_{=F(u_0)}\right\}$$

- Gaussian measure μ_s is *invariant* under the linear solution map $S(t) = e^{-it\Delta}$ (by the rotational invariance of $\mathbb{C}$-valued Gaussian r.v.'s)

  $\Longrightarrow$ The solution map $\Phi(t)$ is of the form "$u_0 + F(u_0)$"

Q: Can we study transport properties (such as invariance, quasi-invariance, singularity) of μ_s under nonlinear dispersive Hamiltonian PDEs?

Part 1

Invariant Gibbs measures for Hamiltonian PDEs

Invariant Gibbs measures

Finite dimensional Hamiltonian dynamics on $\mathbb{R}^{2n}$:

$$\dot{p}_j = \frac{\partial H}{\partial q_j}, \qquad \dot{q}_j = -\frac{\partial H}{\partial p_j}$$

with Hamiltonian $H(p,q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$

- By *Liouville's theorem*, Lebesgue measure $dpdq = \prod_{j=1}^n dp_j dq_j$ is invariant
- Hamiltonian $H(p(t), q(t))$ is invariant under the flow

$\Longrightarrow$ **Gibbs measure**: $d\rho = Z^{-1} e^{-H(p,q)} dpdq$ is **invariant**

Namely,

$$\rho(\Phi(-t)A) = \rho(A) \quad \text{for all } t \in \mathbb{R}$$

- Moreover, if $F(p,q)$ is a "nice" conserved quantity, then

$$d\mu_F = Z^{-1} \exp(-F(p,q)) \prod_{j=1}^n dp_j dq_j$$

  is also invariant

NLS on $\mathbb{T}$: $i\partial_t u - \partial_x^2 u \pm |u|^{p-1}u = 0, \qquad x \in \mathbb{T}$

- NLS is a Hamiltonian PDE:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |u_x|^2 dx \pm \frac{1}{p+1}\int_{\mathbb{T}} |u|^{p+1} dx, \quad M(u) = \int_{\mathbb{T}} |u|^2 dx,$$

- $H(u)$ is conserved under the NLS dynamics

Gibbs measure: "$d\rho = Z^{-1}e^{-H(u)}du$" is "expected" to be ***invariant***

- We actually consider

$$d\rho = Z^{-1}e^{\mp\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}dx} e^{-\frac{1}{2}\int_{\mathbb{T}}|u_x|^2dx - \frac{1}{2}\int_{\mathbb{T}}|u|^2dx} du$$
$$= Z^{-1}e^{\mp\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}dx} \underbrace{e^{-\frac{1}{2}\|u\|^2_{H^1}} du}_{=d\mu_1}$$

$\Longrightarrow$ ρ is a probability measure on $H^\sigma(\mathbb{T})$, $\sigma < \frac{1}{2}$:

- defocusing case ($-$ sign) : all $p > 1$
- focusing case ($+$ sign):
  Lebowitz-Rose-Speer '88: $p \leq 5$ (with L^2-cutoff)
  - related to existence of finite time blowup solutions when $p \geq 5$

NLS on $\mathbb{T}$: $\quad i\partial_t u - \partial_x^2 u \pm |u|^{p-1}u = 0, \qquad x \in \mathbb{T}$

- NLS is a Hamiltonian PDE:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |u_x|^2 dx \pm \frac{1}{p+1}\int_{\mathbb{T}} |u|^{p+1} dx, \quad M(u) = \int_{\mathbb{T}} |u|^2 dx,$$

- $H(u)$ is conserved under the NLS dynamics

Gibbs measure: "$d\rho = Z^{-1}e^{-H(u)}du$" is "expected" to be ***invariant***

- We actually consider

$$\begin{aligned} d\rho &= Z^{-1}e^{\mp\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}dx} e^{-\frac{1}{2}\int_{\mathbb{T}}|u_x|^2 dx - \frac{1}{2}\int_{\mathbb{T}}|u|^2 dx} du \\ &= Z^{-1}e^{\mp\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}dx} \underbrace{e^{-\frac{1}{2}\|u\|_{H^1}^2} du}_{= d\mu_1} \end{aligned}$$

$\Longrightarrow$ ρ is a probability measure on $H^\sigma(\mathbb{T})$, $\sigma < \frac{1}{2}$:

- defocusing case ($-$ sign) : all $p > 1$
- focusing case ($+$ sign):
  Lebowitz-Rose-Speer '88: $p \leq 5$ (with L^2-cutoff)
  - related to existence of finite time blowup solutions when $p \geq 5$

Difficulty: Construction of global-in-time dynamics on $\mathrm{supp}(\rho)$

- Bourgain '94: use "formal invariance of ρ as a replacement of a conservation law"

  $\Longrightarrow$ invariance of Gibbs measure ρ for NLS, KdV, mKdV, etc.

- Also, Friedlander '85 (NLW), McKean mid '90s, mid '00s$\sim$: Tzvetkov, Burq, Thomann, Oh, etc.

- As a consequence of invariance of (finite dimensional) Gibbs measure:

  Logarithmic growth bound: Let $\sigma < \frac{1}{2}$. Then, we have

  $$\|u(t)\|_{H^\sigma} \lesssim C(u_0^\omega)\big\{\log(1+|t|)\big\}^{\frac{1}{2}} \text{ for any } t \in \mathbb{R}$$

  almost surely

  Remark: Such a logarithmic growth bound is *beyond* the usual deterministic polynomial growth bounds

Gibbs measures on $\mathbb{T}^2$

Consider **defocusing NLS on** $\mathbb{T}^2$**:**

$$i\partial_t u - \Delta u + |u|^{p-1}u = 0, \qquad p \in 2\mathbb{N}+1,$$

with the associated Gibbs measure: $d\rho = Z^{-1} e^{-\frac{1}{p+1}\int_{\mathbb{T}^2}|u|^{p+1}} d\mu_1$

Difficulty: Wiener measure μ_1 on $\mathbb{T}^2$ is support on $H^\sigma(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $\sigma < 0$.
They are not even functions!! In particular, $\int_{\mathbb{T}^2}|u|^{p+1} = \infty$, a.s.

Two problems:

- Construction of the Gibbs measure: **renormalization** (*Wick ordering*)
  - Euclidean quantum field theory (70's $\sim$): Nelson, Simon, Glimm-Jaffe, ...
  - **No** Gibbs measure in the focusing case: Brydges-Slade '96
- Well-posedness for **defocusing Wick ordered NLS on** $\mathbb{T}^2$**:**

$$i\partial_t u - \Delta u + \underbrace{:|u|^{p-1}u:}_{\text{Wick ordered nonlinearity}} = 0$$

  - Gibbs measure on $H^\sigma(\mathbb{T}^2)$, $\sigma < 0$
  - ill-posed for $\sigma < s_{\text{crit}} = 1 - \frac{2}{p-1}$: $s_{\text{crit}} = 0$ if $p = 3$, $s_{\text{crit}} = \frac{1}{2}$ if $p = 5$, ...

Gibbs measures on $\mathbb{T}^2$

Consider **defocusing NLS on $\mathbb{T}^2$:**

$$i\partial_t u - \Delta u + |u|^{p-1}u = 0, \qquad p \in 2\mathbb{N}+1,$$

with the associated Gibbs measure: $d\rho = Z^{-1} e^{-\frac{1}{p+1}\int_{\mathbb{T}^2}|u|^{p+1}} d\mu_1$

Difficulty: Wiener measure μ_1 on $\mathbb{T}^2$ is support on $H^\sigma(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $\sigma < 0$.
They are not even functions!! In particular, $\int_{\mathbb{T}^2}|u|^{p+1} = \infty$, a.s.

Two problems:

- Construction of the Gibbs measure: **renormalization** (*Wick ordering*)
  - Euclidean quantum field theory (70's $\sim$): Nelson, Simon, Glimm-Jaffe, ...
  - **No** Gibbs measure in the focusing case: Brydges-Slade '96
- Well-posedness for **defocusing Wick ordered NLS on $\mathbb{T}^2$:**

$$i\partial_t u - \Delta u + \underbrace{:|u|^{p-1}u:}_{\text{Wick ordered nonlinearity}} = 0$$

  - Gibbs measure on $H^\sigma(\mathbb{T}^2)$, $\sigma < 0$
  - ill-posed for $\sigma < s_{\text{crit}} = 1 - \frac{2}{p-1}$: $s_{\text{crit}} = 0$ if $p = 3$, $s_{\text{crit}} = \frac{1}{2}$ if $p = 5$, ...

Main difficulty: local well-posedness on $\text{supp}(\rho)$

(a) *Probabilistic* local well-posedness:

McKean '95, Bourgain '96, Burq-Tzvetkov '08, '14, Oh '11, Bourgain-Bulut '14

- Construct (local) solutions a.s. with respect to $u_0^\omega = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle} e^{2\pi i n \cdot x}$
- *gain of integrability* of linear solution under randomization:

(b) "compactness" argument (via invariance of finite dim'l Gibbs measures)

$\Longrightarrow$ a.s. global existence (without uniqueness): "energy solutions" and "invariance" of Gibbs measure

Burq-Thomann-Tzvetkov '14

Defocusing Wick ordered NLS on $\mathbb{T}^2$: $i\partial_t u - \Delta u + :|u|^{p-1}u: = 0, \quad p \in 2\mathbb{N}+1$

- $p = 3$: ε-gap between $\sigma < 0$ and $s_{\text{crit}} = 0$

  Bourgain '96: probabilistic local well-posedness

  $\Longrightarrow$ almost sure global well-posedness and invariance of Gibbs measure

- $p \geq 5$: regularity gap $= s_{\text{crit}} + \varepsilon > \frac{1}{2}$ is too large

  Oh-Thomann '15: (non-unique) global-in-time "energy solutions"

  $\Longrightarrow$ "invariance" of Gibbs measure

Defocusing Wick ordered NLW on $\mathbb{T}^2$: $\partial_t^2 u + mu - \Delta u + :u^p: = 0, \quad p \in 2\mathbb{N}+1$

- Oh-Thomann '17: probabilistic local well-posedness

  $\Longrightarrow$ a.s. GWP and invariance of Gibbs measure

Weak universality:

- WNLW appears as a *scaling limit* of NLW on a dilated torus $(\varepsilon^{-1}\mathbb{T})^2$:

$$\partial_t^2 v_\varepsilon - \Delta v_\varepsilon + m_\varepsilon v_\varepsilon + f(v_\varepsilon) = 0$$

  and scaling v_ε back to the standard torus $\mathbb{T}^2$

On $\mathbb{T}^3$?

- Gibbs measure on $\mathbb{T}^3$ is renormalizable only for $p = 3$ & defocusing
  - Wick ordering is not enough (need second order correction)
  - very rough $\sim H^{-\frac{1}{2}-}(\mathbb{T}^3)$
- Stochastic quantization equation:

$$\partial_t u = \Delta u - u^3 + \infty \cdot u + \underbrace{\xi}_{\text{space-time white noise}}$$

  - formally preserves the Gibbs measure
  - "local well-posedness": Hairer '14 (regularity structure), Kupiainen '16 (RG method), Catellier-Chouk '16 (paracontrolled distribution introduced by Gubinelli-Imkeller-Perkowski '15)
  - invariance of Gibbs measure: Hairer-Matetski '15
  - global well-posedness: Mourrat-Weber '16
- (renormalized) defocusing cubic NLS/NLW on $\mathbb{T}^3$?

Completely open

Remarks & comments

This recent development also lead to

1. **Probabilistic well-posedness** beyond deterministic analysis:
2. **Singular stochastic dispersive PDEs**: *space-time white noise forcing*
    - stochastic KdV on $\mathbb{T}$: LWP (Oh '09), global dynamics (Oh-Quastel-Sosoe '17)
    - stochastic NLW on $\mathbb{T}^2$: LWP (Gubinelli-Koch-Oh '17)
      GWP (Gubinelli-Koch-Oh-Tolomeo '17)
      time-dependent renormalization
    - stochastic cubic NLS on $\mathbb{T}$: *completely open*
      important in fiber optics

Dynamical properties?

1. Recurrence property: Poincaré, Furstenberg '77
2. Ergodicity and 'asymptotic stability' of ρ?
    - Completely open
    - These questions have been answered for some stochastic PDEs. This is mainly due to *uniqueness* of invariant measures. However, for Hamiltonian PDEs, there are more than one (formally) invariant measures and such questions are out of reach at this point...

Part 2

Quasi-invariant measures for Hamiltonian PDEs

Goal: Investigate transport properties of Gaussian measure μ_s:

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du \quad \text{on } H^\sigma(\mathbb{T}^d), \ \sigma < s - \frac{d}{2}$$

- $s = 0$: **White noise** on $\mathbb{T}$: very rough

$$u(x) = \sum_{n\in\mathbb{Z}} g_n(\omega) e^{inx} \in H^\sigma(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}), \quad \sigma < -\frac{1}{2}$$

Invariance of white noise

- KdV: Quastel-Valkó '08, Oh '09, Oh-Quastel-Valkó '12
- (renormalized) cubic fourth order NLS (4NLS): Oh-Tzvetkov-Wang '17

Q: Is white noise μ_0 invariant under (renormalized) cubic NLS on $\mathbb{T}$?

- **Very** difficult
- μ_0 is a limit of invariant measures for cubic NLS (Oh-Quastel-Valkó '12) but no well-posedness...

Q: Can we study transport properties of μ_s for general (non-small) s?

- When s is large, this question is *not* about rough solutions

Goal: Investigate transport properties of Gaussian measure μ_s:

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du \quad \text{on } H^\sigma(\mathbb{T}^d), \ \sigma < s - \tfrac{d}{2}$$

- $s = 0$: **White noise** on $\mathbb{T}$: very rough

$$u(x) = \sum_{n \in \mathbb{Z}} g_n(\omega) e^{inx} \in H^\sigma(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}), \quad \sigma < -\tfrac{1}{2}$$

Invariance of white noise

- KdV: Quastel-Valkó '08, Oh '09, Oh-Quastel-Valkó '12
- (renormalized) cubic fourth order NLS (4NLS): Oh-Tzvetkov-Wang '17

Q: Is white noise μ_0 invariant under (renormalized) cubic NLS on $\mathbb{T}$?

- **Very** difficult
- μ_0 is a limit of invariant measures for cubic NLS (Oh-Quastel-Valkó '12) but no well-posedness...

Q: Can we study transport properties of μ_s for general (non-small) s?

- When s is large, this question is *not* about rough solutions

Goal: Investigate transport properties of Gaussian measure μ_s:

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du \quad \text{on } H^\sigma(\mathbb{T}^d), \ \sigma < s - \tfrac{d}{2}$$

- $s = 0$: **White noise** on $\mathbb{T}$: very rough

$$u(x) = \sum_{n\in\mathbb{Z}} g_n(\omega) e^{inx} \in H^\sigma(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}), \quad \sigma < -\tfrac{1}{2}$$

Invariance of white noise

- KdV: Quastel-Valkó '08, Oh '09, Oh-Quastel-Valkó '12
- (renormalized) cubic fourth order NLS (4NLS): Oh-Tzvetkov-Wang '17

Q: Is white noise μ_0 invariant under (renormalized) cubic NLS on $\mathbb{T}$?

- **Very** difficult
- μ_0 is a limit of invariant measures for cubic NLS (Oh-Quastel-Valkó '12) but no well-posedness...

Q: Can we study transport properties of μ_s for general (non-small) s?

- When s is large, this question is *not* about rough solutions

Quasi-invariance of Gaussian measures μ_s

- $(s = 0)$ white noise μ_0 on $\mathbb{T}$: KdV and (Wick ordered) 4NLS
  invariance $\Longrightarrow$ quasi-invariance
- $(s = 1)$ invariant Gibbs measure (= Gaussian measure μ_1 with weight)
  $\Longrightarrow$ μ_1 is quasi-invariant
- completely integrable PDEs with infinitely many conservation laws
  $\Longrightarrow$ invariant measures ρ_k (= μ_k with weight) for every integer $k \geq 2$
  - cubic NLS on $\mathbb{T}$, KdV on $\mathbb{T}$, Benjamin-Ono equation on $\mathbb{T}$
    (Zhidkov '01, Tzvetkov-Visciglia '14-15, Deng-Tz-V '15)
  - derivative NLS on $\mathbb{T}$: open (only construction)

Q: Gel'fand '96: Can we directly prove quasi-invariance of μ_s (at least for s large) for (non-integrable) PDEs?

Remark:

- Gibbs measure problem: study of rough solutions
- When s is large, this question is *not* about rough solutions

Quasi-invariance of Gaussian measures μ_s

- $(s = 0)$ white noise μ_0 on $\mathbb{T}$: KdV and (Wick ordered) 4NLS
  invariance $\implies$ quasi-invariance
- $(s = 1)$ invariant Gibbs measure (= Gaussian measure μ_1 with weight)
  $\implies \mu_1$ is quasi-invariant
- completely integrable PDEs with infinitely many conservation laws
  $\implies$ invariant measures ρ_k (= μ_k with weight) for every integer $k \geq 2$
  - cubic NLS on $\mathbb{T}$, KdV on $\mathbb{T}$, Benjamin-Ono equation on $\mathbb{T}$
    (Zhidkov '01, Tzvetkov-Visciglia '14-15, Deng-Tz-V '15)
  - derivative NLS on $\mathbb{T}$: open (only construction)

Q: Gel'fand '96: Can we directly prove quasi-invariance of μ_s (at least for s large) for (non-integrable) PDEs?

Remark:

- Gibbs measure problem: study of rough solutions
- When s is large, this question is *not* about rough solutions

Benjamin-Bona-Mahony equation (BBM) on $\mathbb{T}$: small amplitude long surface waves

$$\partial_t u + \partial_x u - \partial_t \partial_x^2 u + \partial_x (u^2) = 0$$

$$\Longrightarrow \quad \partial_t u + (1-\partial_x^2)^{-1}\partial_x u + (1-\partial_x^2)^{-1}\partial_x (u^2) = 0$$

- **Ramer's result:** μ_s on $\mathcal{D}'(\mathbb{T})$ is quasi-invariant under the map

$$T : u_0 \mapsto u_0 + F(u_0)$$

if $F(u_0)$ is $(d+\varepsilon)$-*smoothing* $\Longrightarrow$ not sufficient for BBM

Tzvetkov '15: For $s \in \mathbb{N}$, μ_s is quasi-invariant under BBM

- A similar result holds for generalized BBM model with less smoothing
- introduced a new method to establish quasi-invariance of μ_s beyond Ramer
- uses the explicit smoothing in the nonlinearity but *not* dispersive effect

Q: Can we find a good model to prove quasi-invariance via *dispersive effect*?

Benjamin-Bona-Mahony equation (BBM) on $\mathbb{T}$: small amplitude long surface waves

$$\partial_t u + \partial_x u - \partial_t \partial_x^2 u + \partial_x (u^2) = 0$$
$$\Longrightarrow \quad \partial_t u + (1-\partial_x^2)^{-1}\partial_x u + (1-\partial_x^2)^{-1}\partial_x (u^2) = 0$$

- **Ramer's result:** μ_s on $\mathcal{D}'(\mathbb{T})$ is quasi-invariant under the map

$$T : u_0 \mapsto u_0 + F(u_0)$$

  if $F(u_0)$ is $(d+\varepsilon)$-*smoothing* $\Longrightarrow$ not sufficient for BBM

Tzvetkov '15: For $s \in \mathbb{N}$, μ_s is quasi-invariant under BBM

- A similar result holds for generalized BBM model with less smoothing
- introduced a new method to establish quasi-invariance of μ_s beyond Ramer
- uses the explicit smoothing in the nonlinearity but *not* dispersive effect

Q: Can we find a good model to prove quasi-invariance via *dispersive effect*?

Cubic fourth order NLS (4NLS) on $\mathbb{T}$:

$$i\partial_t u - \partial_x^4 u = |u|^2 u$$

- Globally well-posed in $H^\sigma(\mathbb{T})$, $\sigma \geq 0$
- Strongly ill-posed for $\sigma < 0$ (Oh-Wang '17: non-existence in negative Sobolev spaces)

Theorem: Oh-Tzvetkov '16, Oh-Sosoe-Tzvetkov '17

Let $s > \frac{1}{2}$. Then, the Gaussian measure μ_s is quasi-invariant under 4NLS

- This theorem is *optimal*: μ_s is supported on $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$
- Unlike BBM, there is *no* apparent smoothing in 4NLS. We exhibit smoothing effects *via dispersion* after using some *gauge transform* and *normal form reductions*
- The proof consists of *local & global analysis* (in the phase space $H^\sigma(\mathbb{T})$)
  - local PDE analysis (normal form reductions, energy estimates)
  - global phase space analysis (gauge transform, a change-of-variable formula)

Key role of dispersion

Q: Is *dispersion* essential for quasi-invariance of μ_s?

Yes. Consider the dispersionless model on $\mathbb{T}$:

$$i\partial_t u = |u|^2 u$$

- Explicit solution formula $u(t,x) = e^{-it|u(0,x)|^2} u(0,x)$
- Globally well-posed in $H^\sigma(\mathbb{T}), \sigma > \frac{1}{2}$

  Note: our random data u is a.s. continous for $s > \frac{1}{2} \quad \Longrightarrow \quad \sigma > 0$

Theorem: Oh-Sosoe-Tzvetkov '17

Let $s > \frac{1}{2}$. Then, μ_s is *not* quasi-invariant under the dispersionless model

- The proof uses law of iterated logarithms, a fine criterion to measure the regularity of a typical function w.r.t. μ_s (= fractional Brownian loop). This property regularity property is destroyed by the flow of the dispersionless model

Key role of dispersion

Q: Is *dispersion* essential for quasi-invariance of μ_s?

Yes. Consider the dispersionless model on $\mathbb{T}$:

$$i\partial_t u = |u|^2 u$$

- Explicit solution formula $u(t,x) = e^{-it|u(0,x)|^2} u(0,x)$
- Globally well-posed in $H^\sigma(\mathbb{T}), \sigma > \frac{1}{2}$

  Note: our random data u is a.s. continous for $s > \frac{1}{2} \quad \Longrightarrow \quad \sigma > 0$

Theorem: Oh-Sosoe-Tzvetkov '17

Let $s > \frac{1}{2}$. Then, μ_s is *not* quasi-invariant under the dispersionless model

- The proof uses law of iterated logarithms, a fine criterion to measure the regularity of a typical function w.r.t. μ_s (= fractional Brownian loop). This property regularity property is destroyed by the flow of the dispersionless model

Rough idea

Goal : Compute $\frac{d}{dt}\mu_s(\Phi(t)(A))$

1. **Energy estimate** (local analysis): $\frac{d}{dt}\|\Phi(t)(u)\|^2_{H^s} \leq \underbrace{C(\|u\|_{L^2})}_{\text{conserved}} \underbrace{\|u\|^{2-\theta}_{H^{s-\frac{1}{2}-\varepsilon}}}_{\text{supp}(\mu_s)}$

2. **A change-of-variable formula** (global analysis):

$$\mu_s(\Phi(t)(A)) = Z_s^{-1}\int_{\Phi(t)A} e^{-\frac{1}{2}\|u\|^2_{H^s}}\,du \text{ “}=\text{” } Z_s^{-1}\int_A e^{-\frac{1}{2}\|\Phi(t)(u)\|^2_{H^s}}\,du$$

$\Longrightarrow$ (Yudovich) Given $t \in \mathbb{R}$ and $\delta > 0$, there exists $C = C(t,\delta) > 0$ such that

$$\mu_s(\Phi(t)(A)) \leq C(t,\delta)\{\mu_s(A)\}^{1-\delta}$$

$\Longrightarrow$ quasi-invariance!!

- In Step 1, we need to apply two transformations on the phase space. Then, perform (an infinite iteration of) normal form reductions to prove the energy estimate on a *modified* energy $E = \|u\|^2_{H^s} + R$
- In Step 2, we need to insert the frequency truncation $\mathbf{P}_{\leq N}$. Moreover, we need to consider a *modified* measure associated to the modified energy

Rough idea

Goal : Compute $\frac{d}{dt}\mu_s(\Phi(t)(A))$

1. **Energy estimate** (local analysis): $\frac{d}{dt}\|\Phi(t)(u)\|_{H^s}^2 \le \underbrace{C(\|u\|_{L^2})}_{\text{conserved}}\underbrace{\|u\|_{H^{s-\frac{1}{2}-\varepsilon}}^{2-\theta}}_{\text{supp}(\mu_s)}$

2. **A change-of-variable formula** (global analysis):

$$\mu_s(\Phi(t)(A)) = Z_s^{-1}\int_{\Phi(t)A} e^{-\frac{1}{2}\|u\|_{H^s}^2}du \text{ “ = ” } Z_s^{-1}\int_A e^{-\frac{1}{2}\|\Phi(t)(u)\|_{H^s}^2}du$$

$\Longrightarrow$ (Yudovich) Given $t \in \mathbb{R}$ and $\delta > 0$, there exists $C = C(t,\delta) > 0$ such that

$$\mu_s(\Phi(t)(A)) \le C(t,\delta)\{\mu_s(A)\}^{1-\delta}$$

$\Longrightarrow$ quasi-invariance!!

- In Step 1, we need to apply two transformations on the phase space. Then, perform (an infinite iteration of) normal form reductions to prove the energy estimate on a *modified* energy $E = \|u\|_{H^s}^2 + R$
- In Step 2, we need to insert the frequency truncation $\mathbf{P}_{\le N}$. Moreover, we need to consider a *modified* measure associated to the modified energy

Global analysis, Part 1

Decomposition of solution map:

$$\Phi(t) = \mathcal{G}_{-t} \circ S(t) \circ \Psi(t)$$

1. **Gauge transform:** Given $t \in \mathbb{R}$, define $\mathcal{G}_t$ on $L^2(\mathbb{T})$ by setting
$$\mathcal{G}_t[f] := e^{it \int_{\mathbb{T}} |f|^2} f$$
2. **Interaction representation:** $v(t) = S(-t)\mathcal{G}_t[u(t)]$, where $S(t) = e^{-it\partial_x^4}$

- $\Phi(t)$ = solution map of the original 4NLS
- $\Psi(t)$ = solution map of $v(0) \mapsto v(t) = S(-t)\mathcal{G}_t[u(t)]$

Proposition

Let $s > \frac{1}{2}$. For every $t \in \mathbb{R}$, the Gaussian measure μ_s is invariant under $S(-t)$ and $\mathcal{G}_t$

$\Longrightarrow$ Suffices to prove quasi-invariance of μ_s under $\Psi(t)$

Local analysis: modified energy and energy estimate

- $v = S(-t) \circ \mathcal{G}_t[u(t)]$ satisfies

$$\partial_t \widehat{v}_n = -i \sum_{\{\phi(\bar{n}) \neq 0\}} e^{-i\phi(\bar{n})t} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} + i|\widehat{v}_n|^2 \widehat{v}_n$$

  On $\Gamma(n) \stackrel{\text{def}}{=} \{\phi(\bar{n}) \neq 0\}$, we have $|\phi(\bar{n})| \gtrsim n_{\max}^2 \quad \Longleftarrow \quad$ rapid oscillation

- **Modified energy:** $E(v) = \|v\|_{H^s}^2 + R(v) \longleftarrow$ correction term

Proposition: energy estimate with smoothing

Let $s > \frac{3}{4}$. Then, for any small $\varepsilon > 0$, there exist $\theta > 0$ and $C > 0$ such that

$$\left| \frac{d}{dt} E(\mathbf{P}_{\leq N} v) \right| \leq \underbrace{C(\|v\|_{L^2})}_{\text{conserved}} \underbrace{\|v\|_{H^{s-\frac{1}{2}-\varepsilon}}^{2-\theta}}_{\text{supp}(\mu_s)}$$

- (Infinite iteration of) **normal form reductions** $\longrightarrow$ correction term R
- Standard (deterministic) PDE analysis
- The proof relies on elementary number theory (divisor counting argument)

By normal form reduction (IBP in time), we have

$$\begin{aligned}
\frac{d}{dt}\|v(t)\|_{H^s}^2 &= -2\operatorname{Re} i \sum_{n\in\mathbb{Z}}\sum_{\Gamma(n)} e^{-i\phi(\bar n)t}\langle n\rangle^{2s} v_{n_1}\overline{v_{n_2}}v_{n_3}\overline{v_n} \\
&= -2i\operatorname{Re}\frac{d}{dt}\underbrace{\bigg[\sum_{n\in\mathbb{Z}}\sum_{\Gamma(n)}\frac{e^{-i\phi(\bar n)t}}{\phi(\bar n)}\langle n\rangle^{2s} v_{n_1}\overline{v_{n_2}}v_{n_3}\overline{v_n}\bigg]}_{=:-R(v)} \\
&\qquad + 2i\operatorname{Re}\sum_{n\in\mathbb{Z}}\sum_{\Gamma(n)}\frac{e^{-i\phi(\bar n)t}}{\phi(\bar n)}\langle n\rangle^{2s}\underbrace{\partial_t(v_{n_1}\overline{v_{n_2}}v_{n_3}\overline{v_n})}_{=6\text{-linear}}
\end{aligned}$$

When $s\in(\frac{1}{2},\frac{3}{4}]$, iterate this process ***infinitely many times***:

$$\frac{d}{dt}\|v(t)\|_{H^s}^2 = \frac{d}{dt}\underbrace{\bigg[\sum_{j=2}^{\infty}\mathcal{N}_0^{(j)}(v)\bigg]}_{=:-R(v)} + \sum_{j=2}^{\infty}\mathcal{N}_1^{(j)}(v) + \sum_{j=2}^{\infty}\mathcal{R}^{(j)}(v)$$

$$\Longrightarrow \quad \bigg|\frac{d}{dt}E(\mathbf{P}_{\le N}v)\bigg| \le C(\|v\|_{L^2}), \qquad \text{where } E(v) = \|v\|_{H^s}^2 + R(v)$$

- Guo-Kwon-Oh '13: infinite iteration of NF reductions for cubic NLS on $\mathbb{T}$ (i.e. on the equation) in the context of low regularity uniqueness problem

Global analysis, Part2

1. **Weighted Gaussian measures:** $E(v) = \|v\|_{H^s}^2 + R(v)$
   - Construct a weighted Gaussian measure $\rho_{s,N,r}$ of the form:

$$\begin{aligned} d\rho_{s,N,r} &= Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2} E(\mathbf{P}_{\le N} v)} dv \\ &= Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2} R(\mathbf{P}_{\le N} v)} \underbrace{e^{-\frac{1}{2}\|v\|_{H^s}^2} dv}_{d\mu_s} \end{aligned}$$

2. **A change-of-variable formula:**

$$\rho_{s,N,r}(\Psi_N(t)(A)) = \hat{Z}_{s,N,r}^{-1} \int_A \mathbf{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2} E(\mathbf{P}_{\le N} \Psi_N(t)(v))} d(\mathbf{P}_{\le N} v) \otimes d\mu_{s,N}^{\perp}$$

3. Study **measure evolution** & take limits ($N \to \infty$, then $r \to \infty$)
   - compute time derivative (**energy estimate**)

   $\Longrightarrow$ quasi-invariance of $\rho_{s,N,r}$ under $\Psi_N(t)$

   $\overset{N\to\infty}{\Longrightarrow}$ quasi-invariance of $\rho_{s,r}$ (and $\mu_{s,r}$) under $\Psi(t)$

   $\overset{r\to\infty}{\Longrightarrow}$ quasi-invariance of μ_s under $\Psi(t)$!!

Nonlinear wave equation: Duhamel part enjoys 1-smoothing:

$$u(t) = S(t)(u_0, u_1) + \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} |u|^{p-1} u(t') dt'$$

Gaussian measure on $(u, \partial_t u)$: $\vec{\mu}_{s+1}(u, \partial_t u) = \mu_{s+1} \otimes \mu_s(u, \partial_t u)$

- $d = 1$: Tzvetkov '15 (implicit in a remark)

Theorem: Oh-Tzvetkov '17 ($d = 2$, defocusing cubic NLW)

Let $s \geq 2$ be an even integer. Then, $\vec{\mu}_{s+1}$ is quasi-invariant under the defocusing cubic NLW on $\mathbb{T}^2$

- A typical element (u, v) under $\vec{\mu}_{s+1}$ lives in $\mathcal{H}^\sigma = H^\sigma \times H^{\sigma-1}$, $\sigma < s$.
  Given a fixed $(h_1, h_2) \in \mathcal{H}^{\sigma+1}$, consider $T_h : (u, v) \mapsto (u, v) + (h_1, h_2)$
  Cameron-Martin $\Longrightarrow$ $\vec{\mu}_{s+1}$ and its transported measure are singular
- Given $(u_0, u_1) \in \mathcal{H}^\sigma$, we only have the nonlinear part for NLW in $\mathcal{H}^{\sigma+1}$

Main difficulty: energy estimate: $\partial_t \|(u, \partial_t u)\|^2_{\mathcal{H}^{s+1}}$

- *renormalized* energy (but ***no*** renormalization for the equation)

$\Longleftarrow$ We achieve this by introducing a *simultaneous* renormalization on both the $\mathcal{H}^{s+1}$-energy functional and its time derivative

- We establish a renormalized energy estimate in the *probabilistic* setting

In the following, we consider defocusing NLKG (for simplicity):

$$\partial_t^2 u + (1 - \Delta) u = -u^3$$

with Hamiltonian $E(u) = \frac{1}{2}\int (\partial_t u)^2 + \frac{1}{2}\int (Ju)^2 + \frac{1}{4}\int u^4, \quad J = \sqrt{1-\Delta}$

Goal: Define a renormalized energy $E_{s,\infty} \sim \|(u, \partial_t u)\|^2_{\mathcal{H}^{s+1}}$ with a good ∂_t-estimate

Ans: $E_{s,\infty} = \frac{1}{2}\int (J^s \partial_t u)^2 + \frac{1}{2}\int (J^{s+1} u)^2 + \underbrace{\frac{3}{2}\int (J^s u)^2 u^2}_{=\infty,\ \text{a.s.}} - \frac{3}{2}\infty \int u^2$

$\Longleftarrow$ Both $E_{s,\infty}$ and $\partial_t E_{s,\infty}$ behave "well"

Define σ_N by

$$\sigma_N = \mathbb{E}_{\vec{\mu}_{s+1}}\left[\int (J^s \mathbf{P}_{\leq N} u)^2\right] = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{1}{1+|n|^2} \sim \log N \longrightarrow \infty$$

$\Longrightarrow$ For each $p < \infty$, we have

$$X_N(\omega) := \underbrace{\int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N}_{\to \text{ ``}\infty - \infty\text{''}} = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{|g_n|^2 - 1}{1+|n|^2} \in L^p(\Omega)$$

with uniform bounds in $N \in \mathbb{N}$.

$\Longrightarrow$ X_N converges to X_∞ in $L^p(\Omega)$ for any $p < \infty$, allowing us to define

$$X_\infty(\omega) = \int (J^s u)^2 - \sigma_\infty := \lim_{N \to \infty} \left\{ \int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N \right\}$$

Goal: Define a renormalized energy $E_{s,\infty} \sim \|(u, \partial_t u)\|^2_{\mathcal{H}^{s+1}}$ with a good ∂_t-estimate

Ans: $E_{s,\infty} = \frac{1}{2}\int (J^s \partial_t u)^2 + \frac{1}{2}\int (J^{s+1} u)^2 + \underbrace{\frac{3}{2}\int (J^s u)^2 u^2}_{=\infty,\ \text{a.s.}} - \frac{3}{2}\infty \int u^2$

$\Longleftarrow$ Both $E_{s,\infty}$ and $\partial_t E_{s,\infty}$ behave "well"

Define σ_N by

$$\sigma_N = \mathbb{E}_{\vec{\mu}_{s+1}}\left[\int (J^s \mathbf{P}_{\leq N} u)^2\right] = \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\leq N}} \frac{1}{1+|n|^2} \sim \log N \longrightarrow \infty$$

$\Longrightarrow$ For each $p < \infty$, we have

$$X_N(\omega) := \underbrace{\int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N}_{\to\ \text{"}\infty - \infty\text{"}} = \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\leq N}} \frac{|g_n|^2 - 1}{1+|n|^2} \in L^p(\Omega)$$

with uniform bounds in $N \in \mathbb{N}$.

$\Longrightarrow$ X_N converges to X_∞ in $L^p(\Omega)$ for any $p < \infty$, allowing us to define

$$X_\infty(\omega) = \int (J^s u)^2 - \sigma_\infty := \lim_{N\to\infty}\left\{\int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N\right\}$$

$$\frac{1}{2}\partial_t\|(u,\partial_t u)\|^2_{\mathcal{H}^{s+1}} = -3\int(\partial_t J^s u)J^s u\cdot u^2 + \text{l.o.t.}$$

$$\overset{\text{IBP}}{=} -\frac{3}{2}\partial_t\left[\int (J^s u)^2 u^2\right] + 3\int (J^s u)^2\partial_t u\cdot u + \text{l.o.t.}$$

$$= -\frac{3}{2}\partial_t\left[\int \mathbf{P}_{\neq 0}[(J^s u)^2]\cdot\mathbf{P}_{\neq 0}[u^2]\right] + 3\int \mathbf{P}_{\neq 0}[(J^s u)^2]\cdot\mathbf{P}_{\neq 0}[\partial_t u\cdot u] + \text{l.o.t.}$$

$$\underbrace{-\frac{3}{2}\partial_t\left[\int (J^s u)^2\int u^2\right]}_{=\infty} + \underbrace{3\int (J^s u)^2\int \partial_t u\cdot u}_{=\infty}$$

With σ_N, we have

$$-\frac{3}{2}\partial_t\left[\int (J^s u)^2\int u^2\right] + 3\int (J^s u)^2\int \partial_t u\cdot u$$

$$= -\frac{3}{2}\partial_t\left[\Bigg(\underbrace{\int (J^s u)^2 - \sigma_N}_{=X_N}\Bigg)\int u^2\right] + 3\Bigg(\underbrace{\int (J^s u)^2 - \sigma_N}_{=X_N}\Bigg)\int \partial_t u\cdot u.$$

Define the renormalized energy $E_{s,N}(u, \partial_t u)$ by

$$E_{s,N}(u, \partial_t u) = \frac{1}{2}\int (J^s \partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int (J^s u)^2 u^2 - \frac{3}{2}\sigma_N \int u^2$$
$$= \frac{1}{2}\int (J^s \partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int \mathbf{P}_{\neq 0}[(J^s u)^2] \cdot \mathbf{P}_{\neq 0}[u^2]$$
$$+ \frac{3}{2}\bigg(\int (J^s u)^2 - \sigma_N\bigg)\int u^2$$

$$\Longrightarrow \partial_t E_{s,N}(u) = 3\int \mathbf{P}_{\neq 0}[(J^s u)^2] \cdot \mathbf{P}_{\neq 0}[\partial_t u \cdot u] + 3\bigg(\int (J^s u)^2 - \sigma_N\bigg)\int \partial_t u \cdot u + \text{l.o.t.}$$

Probabilistic renormalized energy estimate:

$$\bigg\{\int_{\{E(\mathbf{P}_{\leq N}u, \mathbf{P}_{\leq N}v) \leq r\}} \Big|\partial_t E_{s,N}(\pi_N \Phi_N(t)(u,v))|_{t=0}\Big|^p d\mu_s(u,v)\bigg\}^{\frac{1}{p}} \lesssim p$$

$\Longrightarrow E_{s,N} \to E_{s,\infty} = \frac{1}{2}\int (J^s \partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int (J^s u)^2 u^2 - \frac{3}{2}\infty \int u^2$, a.s.
and $E_{s,\infty}$ satisfies the same ∂_t-bound

Remarks

- We showed mutual absolute continuity of the transported measure $\Phi(t)_*\mu_s$ and the original Gaussian measure μ_s. Our argument, however, does not tell us much about the *time-dependent* Radon-Nikodym derivative (in $L^1(\mu_s)$) of $\Phi(t)_*\mu_s$ with respect to μ_s. It would be interesting to study more about the resulting Radon-Nikodym derivatives
  - higher integrability in $L^p(\mu_s)$, $p > 1$?
  - compactness in time? property of its time average?
- By an argument analogous to that for invariant measure, we can obtain

$$\|u(t)\|_{H^\sigma} \lesssim C(u_0^\omega)(1+|t|)^{\alpha(s)} \text{ for any } t \in \mathbb{R},$$

  where $\alpha(s) \to \infty$, as $s \to \infty$. It is very far from the logarithmic bound for invariant measures and may be obtained by deterministic techniques.

  Q: Can we establish *quantitative* versions of quasi-invariance and prove new growth bounds on higher Sobolev norms of solutions in a probabilistic manner?
- Our current understanding of the corresponding question for the (more complicated) NLS is very poor (except for 1-d cubic NLS)...