On stability properties of type II solutions

Joachim Krieger (EPFL)

CIRM, July 2017

Bubbling off blow up solutions

- Consider a hamiltonian nonlinear wave equation of the form

$$\Box u = F(u, \nabla u)$$

admitting a scaling symmetry

$$u(t,x) \to u_\lambda(t,x) := \lambda^\alpha u(\lambda t, \lambda x),\ \alpha \geq 0$$

Bubbling off blow up solutions

- Consider a hamiltonian nonlinear wave equation of the form

$$\Box u = F(u, \nabla u)$$

admitting a scaling symmetry

$$u(t,x) \to u_\lambda(t,x) := \lambda^\alpha u(\lambda t, \lambda x),\ \alpha \geq 0$$

- We call a solution of the form

$$u(t,x) \sim W_{\lambda(t)}(x)$$

with $\lim_{t\to T} \lambda(t) = +\infty$ and $W(x)$ of finite energy a bubbling off singularity (blow up).

Bubbling off blow up solutions

- Consider a hamiltonian nonlinear wave equation of the form

$$\Box u = F(u, \nabla u)$$

admitting a scaling symmetry

$$u(t,x) \to u_\lambda(t,x) := \lambda^\alpha u(\lambda t, \lambda x),\ \alpha \geq 0$$

- We call a solution of the form

$$u(t,x) \sim W_{\lambda(t)}(x)$$

with $\lim_{t\to T} \lambda(t) = +\infty$ and $W(x)$ of finite energy a bubbling off singularity (blow up).

- Many models display this form of blow up.

Models with bubbling off blow ups I

- Experience has revealed that these types of blow ups occur for energy critical models, i. e. such that the conserved energy is invariant under the natural scaling.

Models with bubbling off blow ups I

- Experience has revealed that these types of blow ups occur for energy critical models, i. e. such that the conserved energy is invariant under the natural scaling.
- Wave Maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ (energy critical), given by

$$\Box u = -(-|u_t|^2 + |\nabla_x u|^2)u$$

If W denotes for example the ground state harmonic map $W : \mathbf{R}^2 \longrightarrow S^2$, then bubbling off solutions of the form

$$u(t,x) \sim W_{\lambda(t)}(x)$$

with infinitely many different scaling laws $\lambda(t)$ are known to exist.

Models with bubbling off blow ups II

- Yang Mills equations in $4+1$ dimensions (energy critical) under a symmetry reduction, leading to a radial problem in $2+1$-dimensions

$$\Box u = \frac{2}{r^2}(1-u^2),\ r=|x|$$

Here $W(x)=\frac{1-r^2}{1+r^2}$, and bubbling off solutions of the form

$$W_{\lambda(t)}(x) \sim \frac{1-\lambda^2(t)r^2}{1+\lambda^2(t)r^2}$$

Again if one works in energy space, infinitely many different $\lambda(t)$'s possible.

Models with bubbling off blow ups II

- The energy critical focussing NLW in $\mathbf{R}^{3+1}$:

$$\Box u = -u^5.$$

Here $W(x) = \frac{1}{(1+\frac{|x|^2}{3})^{\frac{1}{2}}}$ and there are infinitely many different type two blow ups of the form

$$u(t,x) \sim W_{\lambda(t)}(x) = \lambda^{\frac{1}{2}}(t)W(\lambda(t)x).$$

Models with bubbling off blow ups II

- The energy critical focussing NLW in $\mathbf{R}^{3+1}$:

$$\Box u = -u^5.$$

Here $W(x) = \frac{1}{(1+\frac{|x|^2}{3})^{\frac{1}{2}}}$ and there are infinitely many different type two blow ups of the form

$$u(t,x) \sim W_{\lambda(t)}(x) = \lambda^{\frac{1}{2}}(t) W(\lambda(t)x).$$

- All of these solutions were obtained by implementing **symmetry reduction** of the original problem. For example, for critical Wave Maps, one obtains an equation of the form

$$-u_{tt} + u_{rr} + \frac{1}{r} u_r = \frac{\sin(2u)}{2r^2}.$$

Significance of these solutions

- For energy critical models with a positive hamiltonian, such as Wave Maps or Yang Mills, these solutions as well as their 'superpositions' are expected to be the only possible blow up dynamics.

Significance of these solutions

- For energy critical models with a positive hamiltonian, such as Wave Maps or Yang Mills, these solutions as well as their 'superpositions' are expected to be the only possible blow up dynamics.
- More precisely, say for Wave Maps $\mathbf{R}^{2+1} \longrightarrow S^2$, one expects a type of soliton resolution result near a singularity of the form

$$u(t,x) = \sum_{i=1}^{N} \mathcal{L}_i[W_i(\lambda_i(t)(x-x_i))] + v(t,x), \ \lim_{t\to T} \lambda_i(t) = +\infty.$$

where W_i are harmonic maps and $\mathcal{L}_i$ are Lorentz transformations.

Significance of these solutions

- For energy critical models with a positive hamiltonian, such as Wave Maps or Yang Mills, these solutions as well as their 'superpositions' are expected to be the only possible blow up dynamics.
- More precisely, say for Wave Maps $\mathbf{R}^{2+1} \longrightarrow S^2$, one expects a type of soliton resolution result near a singularity of the form

$$u(t,x) = \sum_{i=1}^{N} \mathcal{L}_i[W_i(\lambda_i(t)(x-x_i))] + v(t,x), \ \lim_{t\to T} \lambda_i(t) = +\infty.$$

  where W_i are harmonic maps and $\mathcal{L}_i$ are Lorentz transformations.
- Such results along sequences of times have been proved by Struwe for the harmonic map heat flow in the late eighties.

The issue of stability of these solutions

- It appears that at this point in time, the **full stability properties** of any of these solutions (i. e. outside the symmetry reduction) are **unknown**.

The issue of stability of these solutions

- It appears that at this point in time, the **full stability properties** of any of these solutions (i. e. outside the symmetry reduction) are **unknown**.
- Even within the restricted symmetry class, establishing stability of these kinds of blow ups is a highly delicate issue.

The issue of stability of these solutions

- It appears that at this point in time, the **full stability properties** of any of these solutions (i. e. outside the symmetry reduction) are **unknown**.
- Even within the restricted symmetry class, establishing stability of these kinds of blow ups is a highly delicate issue.
- As far as the models from before are concerned, there have been developed **two approaches** to building such bubbling off blow ups : one approach by Merle-Raphael and another by K.-Schlag-Tataru.

The issue of stability of these solutions

- It appears that at this point in time, the **full stability properties** of any of these solutions (i. e. outside the symmetry reduction) are **unknown**.
- Even within the restricted symmetry class, establishing stability of these kinds of blow ups is a highly delicate issue.
- As far as the models from before are concerned, there have been developed **two approaches** to building such bubbling off blow ups : one approach by Merle-Raphael and another by K.-Schlag-Tataru.
- The Merle-Raphael approach gives also stability of solutions obtained, while this information was missing for the solutions by K.-Schlag-Tataru. The latter do not have fixed blow up rate, but a continuum of rates, while the former tend to come with a fixed blow up rate. Also, the solutions differ in terms of their smoothness.

Very rough outline of Merle-Raphael construction

- Method consists of two steps : first, exhibit an approximate profile (Q_b) as an expansion in terms of a variable b, where eventually $b \sim \lambda_t/\lambda^2$. One may think of Q_b as an approximate profile around which one perturbs to find the exact solution.

$$Q_b = Q_\lambda + \sum_i b^i T_i$$

The functions T_i are found by solving suitable elliptic equations.

Very rough outline of Merle-Raphael construction

- Method consists of two steps : first, exhibit an approximate profile (Q_b) as an expansion in terms of a variable b, where eventually $b \sim \lambda_t/\lambda^2$. One may think of Q_b as an approximate profile around which one perturbs to find the exact solution.

$$Q_b = Q_\lambda + \sum_i b^i T_i$$

  The functions T_i are found by solving suitable elliptic equations.

- Next, one complements this to an exact solution

$$u = Q_b(\lambda(t)r) + \epsilon(t, r).$$

  To control ϵ, one foces a suitable vanishing condition on ϵ. The problem then becomes controlling both the evolution of b and that of ϵ.

Very rough outline of Merle-Raphael construction

- Method consists of two steps : first, exhibit an approximate profile (Q_b) as an expansion in terms of a variable b, where eventually $b \sim \lambda_t/\lambda^2$. One may think of Q_b as an approximate profile around which one perturbs to find the exact solution.

$$Q_b = Q_\lambda + \sum_i b^i T_i$$

  The functions T_i are found by solving suitable elliptic equations.
- Next, one complements this to an exact solution

$$u = Q_b(\lambda(t)r) + \epsilon(t, r).$$

  To control ϵ, one foces a suitable vanishing condition on ϵ. The problem then becomes controlling both the evolution of b and that of ϵ.
- In a nutshell, one uses virial type identifies obtained by differentiating in time certain energy type functionals.

Very rough outline of KST construction I

- The usefulness of these virial type identities depends on subtle positivity properties which hinge on the monotonicity of λ (sign of b).

Very rough outline of KST construction I

- The usefulness of these virial type identities depends on subtle positivity properties which hinge on the monotonicity of λ (sign of b).
- The evolution of the quantities b, ϵ is studied simultaneously.

Very rough outline of KST construction I

- The usefulness of these virial type identities depends on subtle positivity properties which hinge on the monotonicity of λ (sign of b).
- The evolution of the quantities b, ϵ is studied simultaneously.
- In KST method, one fixes $\lambda(t) = t^{-1-\nu}$ (for Wave Maps and critical NLW). Then one proceeds in two steps : construction of an approximate solution of the form

$$u_e(t,x) = W_{\lambda(t)}(x) + \sum_{i=1}^{N} v_i$$

This u_e is roughly analogous to the Q_b of the Merle-Raphael solution, but in fact here one may construct arbitrarily many (but finitely many !) corrections v_i, resulting in an error of size $O(t^N)$.

Very rough outline of KST construction II

- Completion of approximate solution to an exact solution

$$u(t,x) = u_e(t,x) + \epsilon(t,x),$$

where $\epsilon(t,x)$ is forced to vanish at blow up time. This can be solved via parametrix methods, by exploiting the very rapid decay of the source term (instead of a vanishing condition as in Merle-Raphael).

Very rough outline of KST construction II

- Completion of approximate solution to an exact solution

$$u(t,x) = u_e(t,x) + \epsilon(t,x),$$

  where $\epsilon(t,x)$ is forced to vanish at blow up time. This can be solved via parametrix methods, by exploiting the very rapid decay of the source term (instead of a vanishing condition as in Merle-Raphael).
- This last fact seemed to suggest that the KST-solutions are all very unstable compared to the ones obtained by the Merle-Raphael method.

Very rough outline of KST construction II

- Completion of approximate solution to an exact solution

$$u(t,x) = u_e(t,x) + \epsilon(t,x),$$

  where $\epsilon(t,x)$ is forced to vanish at blow up time. This can be solved via parametrix methods, by exploiting the very rapid decay of the source term (instead of a vanishing condition as in Merle-Raphael).
- This last fact seemed to suggest that the KST-solutions are all very unstable compared to the ones obtained by the Merle-Raphael method.
- The point of this talk is to disprove this last point, and to explain a novel way to think about the stability problem of special bubbling off blow ups. We do this in the context of the critical focussing NLW, which is somehow the simplest model to analyse. Method should have much broader applicability.

The focussing critical NLW on $\mathbf{R}^{3+1}$.

- Recall the equation

$$\Box u = -u^5,\ (t, x) \in \mathbf{R}^{3+1},\ \Box = -\partial_t^2 + \triangle.$$

The focussing critical NLW on $\mathbf{R}^{3+1}$.

- Recall the equation

$$\Box u = -u^5,\ (t, x) \in \mathbf{R}^{3+1},\ \Box = -\partial_t^2 + \triangle.$$

- This model admits the explicit self-similar blow up solutions

$$u(t, x) = \frac{c}{(T-t)^{\frac{1}{2}}}$$

  These are known to be stable in a very strong sense (in the radial class) by a recent result by R. Donninger (2015).

The focussing critical NLW on $\mathbf{R}^{3+1}$.

- Recall the equation

$$\Box u = -u^5,\ (t,x) \in \mathbf{R}^{3+1},\ \Box = -\partial_t^2 + \triangle.$$

- This model admits the explicit self-similar blow up solutions

$$u(t,x) = \frac{c}{(T-t)^{\frac{1}{2}}}$$

These are known to be stable in a very strong sense (in the radial class) by a recent result by R. Donninger (2015).

- The static solution $W(x) = \frac{1}{(1+\frac{|x|^2}{3})^{\frac{1}{2}}}$ may be perturbed to result in bubbling off blow up solutions of the form

$$u(t,x) = W_{\lambda(t)}(x) + \epsilon(t,x),\ \lambda(t) = t^{-1-\nu}, \nu > 0.$$

By contrast to s.s. blow up, this one has bounded H^1-norm (of type II) (KST '09, KS '12).

Remarks on blow up solutions by KST.

- The solution corresponding to scaling parameter $\lambda(t) = t^{-1-\nu}$ has regularity $H^{1+\frac{\nu}{2}-}$ and no better. This comes from the approximate solution u_e and more precisely the second correction v_2.

Remarks on blow up solutions by KST.

- The solution corresponding to scaling parameter $\lambda(t) = t^{-1-\nu}$ has regularity $H^{1+\frac{\nu}{2}-}$ and no better. This comes from the approximate solution u_e and more precisely the second correction v_2.
- The method of Merle-Raphael has apparently not yet been implemented for this model. However, there is a result due to Hillairet-Raphael('10) which produces solutions of the form

$$u(t,x) \sim W_{\lambda(t)}(x),\ \lambda(t) = t^{-1}e^{\sqrt{|\log t|}}$$

for the $4+1$-dimensional analogue, with C^{∞}-data. They assert that these are stable for data perturbations along a **co-dimension one** Lipschitz manifold. This is natural as we will explain shortly.

General Facts about stability of type II solutions

- As noted before, all bubbling off solutions are type II. In general, a solution of

$$\Box u = -u^5$$

on $\mathbf{R}^{3+1}$ is called **type II**, provided we have

$$\sup_{t\in I} \left\|\nabla_{t,x} u(t,\cdot)\right\|_{L^2_x} < \infty.$$

General Facts about stability of type II solutions

- As noted before, all bubbling off solutions are type II. In general, a solution of

$$\Box u = -u^5$$

on $\mathbf{R}^{3+1}$ is called **type II**, provided we have

$$\sup_{t\in I} \left\|\nabla_{t,x} u(t,\cdot)\right\|_{L^2_x} < \infty.$$

- There is a **complete classification** of **radial type II** solutions by Duyckaerts-Kenig-Merle('09) :

$$u(t,x) = \sum_{i=1}^{N} \kappa_i W_{\lambda_i(t)}(x) + v(t,x).$$

General Facts about stability of type II solutions

- As noted before, all bubbling off solutions are type II. In general, a solution of
$$\Box u = -u^5$$
on $\mathbf{R}^{3+1}$ is called **type II**, provided we have
$$\sup_{t\in I} \left\|\nabla_{t,x} u(t,\cdot)\right\|_{L^2_x} < \infty.$$

- There is a **complete classification** of **radial type II** solutions by Duyckaerts-Kenig-Merle('09) :
$$u(t,x) = \sum_{i=1}^{N} \kappa_i W_{\lambda_i(t)}(x) + v(t,x).$$

- There is also a partial result on **stability** of type II solutions, provided they have only one profile :

General Facts about stability of type II solutions

- (K.-Nakanishi-Schlag '13) Let
$$u(t,x) = W_{\lambda(t)}(x) + v(t,x)$$
with $\sup_{t\in I} \|\nabla_{t,x} v\|_{L^2_x} < \delta_0 \ll 1$. Then there is a **co-dimension one** Lipschitz manifold Σ of data perturbations resulting in type II solutions. Data 'below' Σ scatter to zero in forward time, while data 'above' blow up in finite time (but unknown if of type I or type II).

General Facts about stability of type II solutions

- (K.-Nakanishi-Schlag '13) Let
$$u(t,x) = W_{\lambda(t)}(x) + v(t,x)$$
with $\sup_{t\in I} \|\nabla_{t,x} v\|_{L^2_x} < \delta_0 \ll 1$. Then there is a **co-dimension one** Lipschitz manifold Σ of data perturbations resulting in type II solutions. Data 'below' Σ scatter to zero in forward time, while data 'above' blow up in finite time (but unknown if of type I or type II).
- This result makes no assertion of the character of the solution with perturbed data along Σ, other than that it's type II. This is necessarily so, as the energy topology is too crude to distinguish between finite time blow ups or globally existing solutions.

General Facts about stability of type II solutions

- (K.-Nakanishi-Schlag '13) Let
$$u(t,x) = W_{\lambda(t)}(x) + v(t,x)$$
with $\sup_{t\in I} \|\nabla_{t,x} v\|_{L^2_x} < \delta_0 \ll 1$. Then there is a **co-dimension one** Lipschitz manifold Σ of data perturbations resulting in type II solutions. Data 'below' Σ scatter to zero in forward time, while data 'above' blow up in finite time (but unknown if of type I or type II).
- This result makes no assertion of the character of the solution with perturbed data along Σ, other than that it's type II. This is necessarily so, as the energy topology is too crude to distinguish between finite time blow ups or globally existing solutions.
- The co-dimension one results from an unstable mode for the linear operator $\mathcal{L} = -\triangle - 5W^4$ arising upon linearising around W.

Stability of the KST blow up solutions

- Our goal was to develop methods which allow us to analyse the stability properties of the KST blow up solutions constructed for

$$\Box u = -u^5,\ (t,x) \in \mathbf{R}^{3+1}.$$

More precisely, rather than trying to follow the method by Merle-Raphael, we tried to build on the tools introduced in the work by KST.

Stability of the KST blow up solutions

- Our goal was to develop methods which allow us to analyse the stability properties of the KST blow up solutions constructed for

$$\Box u = -u^5,\ (t,x) \in \mathbf{R}^{3+1}.$$

More precisely, rather than trying to follow the method by Merle-Raphael, we tried to build on the tools introduced in the work by KST.

- **Theorem**(Burzio-K. '17) Let $\nu > 0$ be sufficiently small. Then the KST blow up solutions

$$u(t,x) = W_{\lambda(t)}(x) + \epsilon(t,x),\ \lambda(t) = t^{-1-\nu}$$

are stable along a co-dimension one Lipschitz hyper surface of radial data perturbations in a suitable topology.

Remarks on result

- This result is optimal in light of K.-Nakanishi-Schlag theorem('13), and analogous to the assertion by Hillairet-Raphael on stability obtained by Merle-Raphael method.

Remarks on result

- This result is optimal in light of K.-Nakanishi-Schlag theorem('13), and analogous to the assertion by Hillairet-Raphael on stability obtained by Merle-Raphael method.
- The same method can be used to obtain conditional stability results for larger ν, which then involve higher co-dimension.

Remarks on result

- This result is optimal in light of K.-Nakanishi-Schlag theorem('13), and analogous to the assertion by Hillairet-Raphael on stability obtained by Merle-Raphael method.
- The same method can be used to obtain conditional stability results for larger ν, which then involve higher co-dimension.
- Conclusion is that optimal stability properties (i. e. co-dimension one stability) do not imply a fixed blow up rate. However, it is reasonably to conjecture that **imposing C^{∞}-smooth data** does force a quantised set of possible blow up speeds.

Remarks on result

- This result is optimal in light of K.-Nakanishi-Schlag theorem('13), and analogous to the assertion by Hillairet-Raphael on stability obtained by Merle-Raphael method.
- The same method can be used to obtain conditional stability results for larger ν, which then involve higher co-dimension.
- Conclusion is that optimal stability properties (i. e. co-dimension one stability) do not imply a fixed blow up rate. However, it is reasonably to conjecture that **imposing C^∞-smooth data** does force a quantised set of possible blow up speeds.
- This is in stark contrast to **parabolic problems**, such as heat flow for harmonic maps, where it seems that there is only one stable blow up rate. Shows that the finite time blow ups by KST are a truly hyperbolic phenomenon.

General strategy for Burzio-K. stability result

- Start with a blow up solution

$$u_{specific}(t,x) = W_{\lambda(t)} + v$$

Keep the $\lambda(t)$ fixed and try to analyse the stability of this under as general perturbations as possible :

$$u_{specific} \longrightarrow u_{specific} + \epsilon.$$

Do this by working with the linear operator

$$-\partial_t^2 + \triangle + 5W_{\lambda(t)}^4$$

and developing a parametrix for it. This will use the monotonicity of λ.

General strategy for Burzio-K. stability result

- Start with a blow up solution

$$u_{specific}(t,x) = W_{\lambda(t)} + v$$

Keep the $\lambda(t)$ fixed and try to analyse the stability of this under as general perturbations as possible :

$$u_{specific} \longrightarrow u_{specific} + \epsilon.$$

Do this by working with the linear operator

$$-\partial_t^2 + \triangle + 5W_{\lambda(t)}^4$$

and developing a parametrix for it. This will use the monotonicity of λ.

- Solve the nonlinear perturbation problem. This is where proximity of $\lambda(t)$ to t^{-1}, i. e. smallness of ν, comes in.

General strategy for Burzio-K. stability result

- Start with a blow up solution

$$u_{specific}(t,x) = W_{\lambda(t)} + v$$

  Keep the $\lambda(t)$ fixed and try to analyse the stability of this under as general perturbations as possible :

$$u_{specific} \longrightarrow u_{specific} + \epsilon.$$

  Do this by working with the linear operator

$$-\partial_t^2 + \triangle + 5W_{\lambda(t)}^4$$

  and developing a parametrix for it. This will use the monotonicity of λ.
- Solve the nonlinear perturbation problem. This is where proximity of $\lambda(t)$ to t^{-1}, i. e. smallness of ν, comes in.
- The preceding will result in a certain vanishing condition for data. Then use the freedom in 'modulating' $\lambda(t)$ to satisfy the vanishing condition.

Key issue

- There appears no general method for proving estimates let along constructing a global parametrix for operators of the form

$$-\partial_t^2 + \triangle + V(t,x)$$

Key issue

- There appears no general method for proving estimates let along constructing a global parametrix for operators of the form
$$-\partial_t^2 + \triangle + V(t,x)$$
- The closest general technique for proving Strichartz estimates for the corresponding wave equation was announced by Beceanu in '14. However, does not seem to allow resonances or eigenvalue at zero.

Key issue

- There appears no general method for proving estimates let along constructing a global parametrix for operators of the form

$$-\partial_t^2 + \triangle + V(t,x)$$

- The closest general technique for proving Strichartz estimates for the corresponding wave equation was announced by Beceanu in '14. However, does not seem to allow resonances or eigenvalue at zero.
- We develop a parametrix construction in case $V = \lambda^2(t)W(\lambda(t)x)$. Method only requires good spectral representation as well as monotonicity properties of $\lambda(t)$.

Main technical ingredient : spectral representation

- Linearization $\mathcal{L} = -\partial_R^2 - 5W^4$ admits one unstable eigenmode ϕ_d
$$\mathcal{L}\phi_d = -k_d^2\phi_d$$
as well as a resonance at zero $\phi(R,0) = R\frac{1-\frac{R^2}{3}}{(1+\frac{R^2}{3})^{\frac{3}{2}}}$, satisfying
$$\mathcal{L}\phi(R,0) = 0.$$

Main technical ingredient : spectral representation

- Linearization $\mathcal{L} = -\partial_R^2 - 5W^4$ admits one unstable eigenmode ϕ_d
$$\mathcal{L}\phi_d = -k_d^2\phi_d$$
as well as a resonance at zero $\phi(R,0) = R\frac{1-\frac{R^2}{3}}{(1+\frac{R^2}{3})^{\frac{3}{2}}}$, satisfying
$$\mathcal{L}\phi(R,0) = 0.$$

- For general f we have the representation
$$f(R) = x_d\phi_d(R) + \int_0^\infty x(\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$
with $x_d = \langle f, \phi_d\rangle_{L^2(\mathbf{R})}$, $x(\xi) = \langle \phi(R,\xi), f\rangle_{L^2(\mathbf{R})}$.

Main technical ingredient : spectral representation

- Linearization $\mathcal{L} = -\partial_R^2 - 5W^4$ admits one unstable eigenmode ϕ_d

$$\mathcal{L}\phi_d = -k_d^2\phi_d$$

as well as a resonance at zero $\phi(R,0) = R\frac{1-\frac{R^2}{3}}{(1+\frac{R^2}{3})^{\frac{3}{2}}}$, satisfying

$$\mathcal{L}\phi(R,0) = 0.$$

- For general f we have the representation

$$f(R) = x_d\phi_d(R) + \int_0^\infty x(\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

with $x_d = \langle f, \phi_d\rangle_{L^2(\mathbf{R})}$, $x(\xi) = \langle \phi(R,\xi), f\rangle_{L^2(\mathbf{R})}$.

- $\phi(R,\xi) = \phi(R,0) + O(R^2\xi)$.

Translation of problem to distorted Fourier side

- Start by replacing perturbation ϵ by $\tilde{\epsilon} = R\epsilon$. Work with new variables

$$R = \lambda(t)r,\ \tau = \int_t^\infty \lambda(s)\,ds,\ \lambda(t) = t^{-1-\nu}.$$

Translation of problem to distorted Fourier side

- Start by replacing perturbation ϵ by $\tilde{\epsilon} = R\epsilon$. Work with new variables

$$R = \lambda(t)r,\ \tau = \int_t^\infty \lambda(s)\,ds,\ \lambda(t) = t^{-1-\nu}.$$

- Equation in terms of new coordinates

$$\begin{aligned}&(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)^2\tilde{\epsilon} - \beta_\nu(\tau)(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)\tilde{\epsilon} + \mathcal{L}\tilde{\epsilon}\\ &= \lambda^{-2}(\tau)RN(\epsilon) + \partial_\tau(\dot{\lambda}\lambda^{-1})\tilde{\epsilon};\ \beta_\nu(\tau) = \dot{\lambda}(\tau)\lambda^{-1}(\tau),\end{aligned} \tag{1}$$

Translation of problem to distorted Fourier side

- Start by replacing perturbation ϵ by $\tilde{\epsilon} = R\epsilon$. Work with new variables

$$R = \lambda(t)r,\ \tau = \int_t^\infty \lambda(s)\,ds,\ \lambda(t) = t^{-1-\nu}.$$

- Equation in terms of new coordinates

$$\begin{aligned}&(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)^2\tilde{\epsilon} - \beta_\nu(\tau)(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)\tilde{\epsilon} + \mathcal{L}\tilde{\epsilon}\\ &= \lambda^{-2}(\tau)RN(\epsilon) + \partial_\tau(\dot{\lambda}\lambda^{-1})\tilde{\epsilon};\ \beta_\nu(\tau) = \dot{\lambda}(\tau)\lambda^{-1}(\tau),\end{aligned} \tag{1}$$

- This is translated to Fourier side. Complication since

$$R\partial_R\phi(R,\xi) \neq \xi\partial_\xi\phi(R,\xi).$$

This leads to certain nonlocal linear source terms for equation on Fourier side which are somewat complicated to deal with.

Translation of problem to distorted Fourier side

- In terms of $x(\tau, \xi) = \mathcal{F}(\tilde{\epsilon}(\tau, \cdot))(\xi)$, get the equation

$$(\mathcal{D}_\tau^2 + \beta_\nu(\tau)\mathcal{D}_\tau + \underline{\xi})\underline{x}(\tau, \xi) = \mathcal{R}(\tau, \underline{x}) + \underline{f}(\tau, \xi)$$

where

$$\mathcal{D}_\tau = \partial_\tau - \beta_\nu(\tau)(2\xi\partial_\xi + \frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)})$$

Translation of problem to distorted Fourier side

- In terms of $x(\tau, \xi) = \mathcal{F}(\tilde{\epsilon}(\tau, \cdot))(\xi)$, get the equation

$$(\mathcal{D}_\tau^2 + \beta_\nu(\tau)\mathcal{D}_\tau + \underline{\xi})\underline{x}(\tau, \xi) = \mathcal{R}(\tau, \underline{x}) + \underline{f}(\tau, \xi)$$

  where

$$\mathcal{D}_\tau = \partial_\tau - \beta_\nu(\tau)(2\xi\partial_\xi + \frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)})$$

- The term $\mathcal{R}(\tau, \underline{x})$ represents certain non-local operators of the form

$$\beta_\nu(\tau)\int_0^\infty \frac{F(\xi, \eta)\rho(\eta)}{\xi - \eta}\mathcal{D}_\tau x(\tau, \eta)\, d\eta$$
$$\beta_\nu^2(\tau)\int_0^\infty \frac{F(\xi, \eta)\rho(\eta)}{\xi - \eta}x(\tau, \eta)\, d\eta$$

  where the kernel satisfies suitable vanishing properties.

Translation of problem to distorted Fourier side

- The terms $\mathcal{R}(\tau, \underline{x})$ are in some term borderline, in that one does not gain smallness for them. Iterating them away requires a kind of Volterra iteration, which takes decisive advantage of the monotonicity properties of λ.

Translation of problem to distorted Fourier side

- The terms $\mathcal{R}(\tau, \underline{x})$ are in some term borderline, in that one does not gain smallness for them. Iterating them away requires a kind of Volterra iteration, which takes decisive advantage of the monotonicity properties of λ.
- The other source term $\underline{f}(\tau, \xi)$ stands for the Fourier transform of all the nonlinear interaction terms. Dealing with it will require $\lambda(t)$ to be close enough to t^{-1}, i. e. ν to be small enough.

Translation of problem to distorted Fourier side

- The terms $\mathcal{R}(\tau, \underline{x})$ are in some term borderline, in that one does not gain smallness for them. Iterating them away requires a kind of Volterra iteration, which takes decisive advantage of the monotonicity properties of λ.
- The other source term $\underline{f}(\tau, \xi)$ stands for the Fourier transform of all the nonlinear interaction terms. Dealing with it will require $\lambda(t)$ to be close enough to t^{-1}, i. e. ν to be small enough.
- The idea is to solve the x-equation via a suitable iteration scheme, obtaining convergence by imposing suitable vanishing conditions on the initial data

$$\big(\underline{x}(\tau_0), \mathcal{D}_\tau \underline{x}(\tau_0)\big).$$

The iteration scheme ; zeroth iterate

- To begin with, we start with the zeroth iterate, which solves the free transport equation

$$(\mathcal{D}_\tau^2 + \beta_\nu(\tau)\mathcal{D}_\tau + \underline{\xi})\underline{x}(\tau, \xi) = 0.$$

This equation can be solved completely explicitly for the continuous spectral part :

$$\frac{\lambda^{\frac{5}{2}}(\tau)}{\lambda^{\frac{5}{2}}(\tau_0)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \cos\left[\lambda(\tau)\xi^{\frac{1}{2}} \int_{\tau_0}^{\tau} \lambda^{-1}(u)\, du\right] x_0\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)$$
$$+ \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\tau_0)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \frac{\sin\left[\lambda(\tau)\xi^{\frac{1}{2}} \int_{\tau_0}^{\tau} \lambda^{-1}(u)\, du\right]}{\xi^{\frac{1}{2}}} x_1\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)$$

Here $\lambda(\tau) \sim \tau^{1+\nu^{-1}}$, and $\tau \to \infty$ as $t \to 0$.

Analysis of the zeroth iterate

- The issue becomes how the function

$$\tilde{\epsilon}(\tau, R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\, d\xi$$

grows as $\tau \to +\infty$ when substituting the preceding parametrix.

Analysis of the zeroth iterate

- The issue becomes how the function

$$\tilde{\epsilon}(\tau, R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

grows as $\tau \to +\infty$ when substituting the preceding parametrix.

- The key is the relation

$$\phi(R,\xi) = \phi(R,0) + O(R^2\xi).$$

It allows one to infer a relation

$$\tilde{\epsilon}(\tau, R) = c(\tau)\phi(R.0) + \tilde{\epsilon}_{good}(\tau, R)$$

Analysis of the zeroth iterate

- The issue becomes how the function

$$\tilde{\epsilon}(\tau, R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau, \xi)\phi(R, \xi)\rho(\xi)\, d\xi$$

  grows as $\tau \to +\infty$ when substituting the preceding parametrix.

- The key is the relation

$$\phi(R, \xi) = \phi(R, 0) + O(R^2\xi).$$

  It allows one to infer a relation

$$\tilde{\epsilon}(\tau, R) = c(\tau)\phi(R.0) + \tilde{\epsilon}_{good}(\tau, R)$$

- It turns out that imposing simple vanishing relations on x_0, x_1, one can achieve only linear growth on $c(\tau)$.

Analysis of the zeroth iterate

- Precisely, the following result obtains :
  Lemma Assume that we have

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_0(\xi)}{\xi^{\frac{1}{4}}} \cos[\nu\tau_0\xi^{\frac{1}{2}}]\, d\xi = 0$$

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_1(\xi)}{\xi^{\frac{3}{4}}} \sin[\nu\tau_0\xi^{\frac{1}{2}}]\, d\xi = 0.$$

  Assume a further co-dimension one condition to prevent exponential growth from unstable mode ϕ_d. Then we have

$$\|\frac{\tilde{\epsilon}(\tau,\cdot)}{R}\|_{L^\infty_{dR}} \lesssim \tau.$$

Analysis of the zeroth iterate

- Precisely, the following result obtains :
  Lemma Assume that we have

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi = 0$$
$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_1(\xi)}{\xi^{\frac{3}{4}}}\sin[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi = 0.$$

  Assume a further co-dimension one condition to prevent exponential growth from unstable mode ϕ_d. Then we have

$$\|\frac{\tilde{\epsilon}(\tau,\cdot)}{R}\|_{L^\infty_{dR}} \lesssim \tau.$$

- This growth turns out to be manageable in the nonlinear terms, provided $\nu > 0$ is sufficiently small.

Analysis of the first iterate

- This comes from the fact that in the τ, R-variables, the nonlinear term

$$\lambda^{-2}(\tau)RN(\epsilon) = \lambda^{-2}(\tau)RN(\frac{\tilde{\epsilon}}{R})$$

comes with the weight $\lambda^{-2}(\tau) \sim \tau^{-2(1+\nu^{-1})}$.

Analysis of the first iterate

- This comes from the fact that in the τ, R-variables, the nonlinear term
$$\lambda^{-2}(\tau)RN(\epsilon) = \lambda^{-2}(\tau)RN(\frac{\tilde{\epsilon}}{R})$$
comes with the weight $\lambda^{-2}(\tau) \sim \tau^{-2(1+\nu^{-1})}$.
- The first iterate then solves
$$(\mathcal{D}_\tau^2 + \beta_\nu(\tau)\mathcal{D}_\tau + \underline{\xi})\underline{x}^{(1)}(\tau,\xi) = \mathcal{R}(\tau, \underline{x}^{(0)}) + \underline{f}^{(0)}(\tau,\xi)$$

Analysis of the first iterate

- This comes from the fact that in the τ, R-variables, the nonlinear term

$$\lambda^{-2}(\tau)RN(\epsilon) = \lambda^{-2}(\tau)RN(\frac{\tilde{\epsilon}}{R})$$

comes with the weight $\lambda^{-2}(\tau) \sim \tau^{-2(1+\nu^{-1})}$.

- The first iterate then solves

$$(\mathcal{D}_\tau^2 + \beta_\nu(\tau)\mathcal{D}_\tau + \underline{\xi})\underline{x}^{(1)}(\tau, \xi) = \mathcal{R}(\tau, \underline{x}^{(0)}) + \underline{f}^{(0)}(\tau, \xi)$$

- Leads to the Duhamel term

$$\int_{\tau_0}^{\tau} \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \frac{\sin[\lambda(\tau)\xi^{\frac{1}{2}} \int_\tau^\sigma \lambda^{-1}(u)\, du]}{\xi^{\frac{1}{2}}} \mathcal{R}(\sigma, \underline{x}^{(0)})(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\, d\sigma$$

Analysis of the first iterate

- This is 'well-behaved' for $\xi > 1$, but one again runs into difficulties for $\xi < 1$.

Analysis of the first iterate

- This is 'well-behaved' for $\xi > 1$, but one again runs into difficulties for $\xi < 1$.
- Use the decomposition

$$\int_{\tau_0}^{\tau} \ldots = \int_{\tau_0}^{\infty} \ldots - \int_{\tau}^{\infty} \ldots$$

Then the first integral on right is solution of free transport equation, and second integral is well-behaved since

$$\frac{\lambda(\tau)}{\lambda(\sigma)} \leq 1, \; \sigma \geq \tau.$$

Analysis of the first iterate

- This is 'well-behaved' for $\xi > 1$, but one again runs into difficulties for $\xi < 1$.
- Use the decomposition

$$\int_{\tau_0}^{\tau} \ldots = \int_{\tau_0}^{\infty} \ldots - \int_{\tau}^{\infty} \ldots$$

Then the first integral on right is solution of free transport equation, and second integral is well-behaved since

$$\frac{\lambda(\tau)}{\lambda(\sigma)} \leq 1, \, \sigma \geq \tau.$$

- The free term

$$\int_{\tau_0}^{\infty} \ldots$$

no longer satisfies the vanishing conditions like for zeroth iterate.

Analysis of the first iterate

- Thus we are forced to add a small correction to achieve the required vanishing conditions. This means replacing the pure Duhamel term

$$\int_{\tau_0}^{\tau} \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \frac{\sin[\lambda(\tau)\xi^{\frac{1}{2}} \int_{\tau}^{\sigma} \lambda^{-1}(u)\, du]}{\xi^{\frac{1}{2}}} \mathcal{R}(\sigma, \underline{x}^{(0)})(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\, d$$

by a modified one :

$$\int_{\tau_0}^{\tau} \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \frac{\sin[\lambda(\tau)\xi^{\frac{1}{2}} \int_{\tau}^{\sigma} \lambda^{-1}(u)\, du]}{\xi^{\frac{1}{2}}} \mathcal{R}(\sigma, \underline{x}^{(0)})(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\, d$$
$$+ S(\tau)(\triangle\tilde{\tilde{x}}_0^{(1)}, \triangle\tilde{\tilde{x}}_1^{(1)})$$

Here the data $\triangle\tilde{\tilde{x}}_0^{(1)}, \triangle\tilde{\tilde{x}}_1^{(1)}$ are the Fourier transforms of suitable multiples of truncated resonance.

Analysis of the first and higher iterates

- What makes things work is that the special structure of $\mathcal{R}$ implies that **with respect to a suitable norm**, the corrections

$$\triangle\tilde{\tilde{x}}_0^{(1)}, \triangle\tilde{\tilde{x}}_1^{(1)}$$

are smaller than original perturbation x_0, x_1 by a factor τ_0^{-1+}.

Analysis of the first and higher iterates

- What makes things work is that the special structure of $\mathcal{R}$ implies that **with respect to a suitable norm**, the corrections

$$\triangle\tilde{\tilde{x}}_0^{(1)}, \triangle\tilde{\tilde{x}}_1^{(1)}$$

are smaller than original perturbation x_0, x_1 by a factor τ_0^{-1+}.

- One then proceeds similarly for higher iterates : one shows that for $\xi < 1$, one can always split

$$x^{(i)}(\tau,\xi) = x^{(i)}(\tau,\xi)_{good} + S(\tau)(\triangle\tilde{\tilde{x}}_0^{(i)}, \triangle\tilde{\tilde{x}}_1^{(i)})$$

where $x^{(i)}(\tau,\xi)_{good}$ satisfies good bounds, while the free term has data satisfying the required vanishing conditions.

Analysis of the first and higher iterates

- What makes things work is that the special structure of $\mathcal{R}$ implies that **with respect to a suitable norm**, the corrections

$$\triangle\tilde{\tilde{x}}_0^{(1)}, \triangle\tilde{\tilde{x}}_1^{(1)}$$

are smaller than original perturbation x_0, x_1 by a factor τ_0^{-1+}.

- One then proceeds similarly for higher iterates : one shows that for $\xi < 1$, one can always split

$$x^{(i)}(\tau,\xi) = x^{(i)}(\tau,\xi)_{good} + S(\tau)(\triangle\tilde{\tilde{x}}_0^{(i)}, \triangle\tilde{\tilde{x}}_1^{(i)})$$

where $x^{(i)}(\tau,\xi)_{good}$ satisfies good bounds, while the free term has data satisfying the required vanishing conditions.

- The corrections $(\triangle\tilde{\tilde{x}}_0^{(i)}, \triangle\tilde{\tilde{x}}_1^{(i)}$ converge exponentially fast to an overall correction much smaller than the original perturbation.

Some technical remarks

- To control the $x^{(i)}$, one works with norms which penalise both large and small frequencies, implying a regularity of $H^{\frac{3}{2}+}$ on physical side.

$$\sup_{\tau\geq\tau_0}\Big(\frac{\tau}{\tau_0}\Big)^{-\kappa}\|\xi^{-0-}\langle\xi\rangle^{1+}x^{(i)}(\tau,\xi)\|_{L^2_{d\xi}}$$

This needs to be complemented with a square-sum norm over dyadic scales for the time derivatives

$$\Big(\sum_{\substack{\tau\sim N\gtrsim\tau_0\\ N\,\text{dyadic}}}\Big(\frac{\tau}{\tau_0}\Big)^{\kappa}\|\xi^{-0-}\langle\xi\rangle^{\frac{1}{2}+}\mathcal{D}_\tau x^{(i)}(\tau,\xi)\|^2_{L^2_{d\xi}}\Big)^{\frac{1}{2}}.$$

Some technical remarks

- To control the $x^{(i)}$, one works with norms which penalise both large and small frequencies, implying a regularity of $H^{\frac{3}{2}+}$ on physical side.

$$\sup_{\tau\geq\tau_0}\left(\frac{\tau}{\tau_0}\right)^{-\kappa}\|\xi^{-0-}\langle\xi\rangle^{1+}x^{(i)}(\tau,\xi)\|_{L^2_{d\xi}}$$

This needs to be complemented with a square-sum norm over dyadic scales for the time derivatives

$$\Big(\sum_{\substack{\tau\sim N\gtrsim\tau_0\\ N\,\text{dyadic}}}\left(\frac{\tau}{\tau_0}\right)^{\kappa}\|\xi^{-0-}\langle\xi\rangle^{\frac{1}{2}+}\mathcal{D}_\tau x^{(i)}(\tau,\xi)\|^2_{L^2_{d\xi}}\Big)^{\frac{1}{2}}.$$

- The fact that this iteration scheme actually converges relies on a somewhat delicate re-iteration procedure, exploiting eventual gains over many repetitions of the iterative step.

Intermediate stability result

- The preceding scheme can be shown to converge in suitable norm topology, leading to
  Intermediate theorem : there is a co-dimension two Lipschitz manifold of initial data perturbations (ϵ_0, ϵ_1) such that the perturbed data

$$u_{specific}[t_0] + (\epsilon_0, \epsilon_1)$$

lead to a solution blowing up in $(0, 0)$ of the form

$$u_{specific}(t, x) + \epsilon(t, x),$$

where we recall

$$u_{specific}(t, x) = W_{\lambda(t)}(x) + v(t, x)$$

is the original type II blow up as in KST '09, KS'12 that we perturb around.

From intermediate to optimal stability

- Up until now the method did not use much of the specific structure of the KST blow up solutions. In the next step, the detailed structure of these solutions will become important.

From intermediate to optimal stability

- Up until now the method did not use much of the specific structure of the KST blow up solutions. In the next step, the detailed structure of these solutions will become important.
- The issue now becomes how to get rid of the vanishing condition on the data perturbation which prevents the growth of resonance. Once this is achieved, the desired optimal co-dimension one stability result ensues.

From intermediate to optimal stability

- Up until now the method did not use much of the specific structure of the KST blow up solutions. In the next step, the detailed structure of these solutions will become important.
- The issue now becomes how to get rid of the vanishing condition on the data perturbation which prevents the growth of resonance. Once this is achieved, the desired optimal co-dimension one stability result ensues.
- Typically, this involves 'modulation', i. .e exploitation of the symmetries of the equation. Since we are working radially, only scaling and time translation are possible candidates.

From intermediate to optimal stability

- Up until now the method did not use much of the specific structure of the KST blow up solutions. In the next step, the detailed structure of these solutions will become important.
- The issue now becomes how to get rid of the vanishing condition on the data perturbation which prevents the growth of resonance. Once this is achieved, the desired optimal co-dimension one stability result ensues.
- Typically, this involves 'modulation', i. .e exploitation of the symmetries of the equation. Since we are working radially, only scaling and time translation are possible candidates.
- However, this does not work here, due to the precise structure of the KST solutions, and more precisely, their 'shock behaviour' across the light cone.

Detailed structure of the KST solutions

- As mentioned earlier, the KST solutions are obtained by first constructing an approximate solution u_e of the form

$$u_e(t,x) = W_{\lambda(t)}(x) + \sum_{i=1}^{N} v_i$$

and then adding a final correction via solving a suitable wave equation. The corrections v_i are solving suitable elliptic problems.

Detailed structure of the KST solutions

- As mentioned earlier, the KST solutions are obtained by first constructing an approximate solution u_e of the form

$$u_e(t,x) = W_{\lambda(t)}(x) + \sum_{i=1}^{N} v_i$$

  and then adding a final correction via solving a suitable wave equation. The corrections v_i are solving suitable elliptic problems.

- Specifically, the v_{2k} solve certain ODEs in the self-similar variable $a = \frac{r}{t}$. This reduction hinges on the precise structure of $\lambda(t) = t^{-1-\nu}$, namely being an exact power law, as well as making a suitable ansatz for v_{2k}.

Detailed structure of the KST solutions

- As mentioned earlier, the KST solutions are obtained by first constructing an approximate solution u_e of the form

$$u_e(t,x) = W_{\lambda(t)}(x) + \sum_{i=1}^{N} v_i$$

and then adding a final correction via solving a suitable wave equation. The corrections v_i are solving suitable elliptic problems.

- Specifically, the v_{2k} solve certain ODEs in the self-similar variable $a = \frac{r}{t}$. This reduction hinges on the precise structure of $\lambda(t) = t^{-1-\nu}$, namely being an exact power law, as well as making a suitable ansatz for v_{2k}.
- Roughly speaking we have $v_{2k}(t,r) = \frac{\lambda^{\frac{1}{2}}R}{(\lambda t)^{2k}}(1-a)^{(k-\frac{1}{2})\nu+\frac{1}{2}}$

Detailed structure of the KST solutions

- Rescaling or time-translating u_e then changes it by an amount which is infinite with respect to the norm of admissible data perturbations. These need to be of regularity $H^{\frac{3}{2}+}$.

Detailed structure of the KST solutions

- Rescaling or time-translating u_e then changes it by an amount which is infinite with respect to the norm of admissible data perturbations. These need to be of regularity $H^{\frac{3}{2}+}$.
- A more subtle way to modulate is required : replace $\lambda(t) = t^{-1-\nu}$ by (for $N \gg 1$).

$$\lambda_{\gamma_1,\gamma_2}(t) = (1 + \gamma_1 \frac{t^N}{\langle t^N \rangle} + \gamma_2 \log t \frac{t^N}{\langle t^N \rangle}) t^{-1-\nu}, \nu > 0.$$

Detailed structure of the KST solutions

- Rescaling or time-translating u_e then changes it by an amount which is infinite with respect to the norm of admissible data perturbations. These need to be of regularity $H^{\frac{3}{2}+}$.
- A more subtle way to modulate is required : replace $\lambda(t) = t^{-1-\nu}$ by (for $N \gg 1$).

$$\lambda_{\gamma_1,\gamma_2}(t) = (1 + \gamma_1 \frac{t^N}{\langle t^N \rangle} + \gamma_2 \log t \frac{t^N}{\langle t^N \rangle})t^{-1-\nu}, \nu > 0.$$

- It turns out that the exact same procedure that gave rise to the KST '09 solutions can be applied to build solutions of the form

$$u_{\gamma_1,\gamma_2}(t,x) = W_{\lambda_{\gamma_1,\gamma_2}(t)}(x) + v(t,x).$$

Detailed structure of the KST solutions

- Rescaling or time-translating u_e then changes it by an amount which is infinite with respect to the norm of admissible data perturbations. These need to be of regularity $H^{\frac{3}{2}+}$.
- A more subtle way to modulate is required : replace $\lambda(t) = t^{-1-\nu}$ by (for $N \gg 1$).

$$\lambda_{\gamma_1,\gamma_2}(t) = (1 + \gamma_1 \frac{t^N}{\langle t^N \rangle} + \gamma_2 \log t \frac{t^N}{\langle t^N \rangle})t^{-1-\nu}, \nu > 0.$$

- It turns out that the exact same procedure that gave rise to the KST '09 solutions can be applied to build solutions of the form

$$u_{\gamma_1,\gamma_2}(t,x) = W_{\lambda_{\gamma_1,\gamma_2}(t)}(x) + v(t,x).$$

- The $u_{0,0}(t,x)$ coincides with the original KST '09, KS' 12 solutions.

Detailed structure of the KST solutions

- Modulation step :
  Lemma (Burzio -K.) Given a pair of data perturbations (ϵ_1, ϵ_2) small enough in a suitable norm, there exist unique γ_1, γ_2, such that we have

$$u_{0,0}[t_0] + (\epsilon_1, \epsilon_2) = u_{\gamma_1,\gamma_2}[t_0] + (\bar{\epsilon}_1, \bar{\epsilon}_2),$$

  where $(\bar{\epsilon}_1, \bar{\epsilon}_2)$ satisfy the vanishing conditions required to bound resonance growth with respect to the new scaling parameter $\lambda_{\gamma_1,\gamma_2}$.

Detailed structure of the KST solutions

- Modulation step :
  Lemma (Burzio -K.) Given a pair of data perturbations (ϵ_1, ϵ_2) small enough in a suitable norm, there exist unique γ_1, γ_2, such that we have

$$u_{0,0}[t_0] + (\epsilon_1, \epsilon_2) = u_{\gamma_1,\gamma_2}[t_0] + (\bar{\epsilon}_1, \bar{\epsilon}_2),$$

  where $(\bar{\epsilon}_1, \bar{\epsilon}_2)$ satisfy the vanishing conditions required to bound resonance growth with respect to the new scaling parameter $\lambda_{\gamma_1,\gamma_2}$.
- At this point, one can re-iterate the whole process leading to the intermediate stability result, replacing $\lambda(t)$ by $\lambda_{\gamma_1,\gamma_2}(t)$, to infer the optimal co-dimension one stability result.