The Benjamin-Ono equation

Mihaela Ifrim

University of California, Berkeley

July 6, 2017

This is joint work with Daniel Tataru

Introduction: Origins

The Benjamin Ono equation (BO)

$$(\partial_t + H\partial_x^2)\phi = \frac{1}{2}\partial_x(\phi^2), \qquad \phi(0) = \phi_0,$$

where ϕ is a real valued function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

- It was first observed by Benjamin (1967) and Ono (1975), and it represents a good, physically relevant model for water waves; more examples: KdV, NLS, Ostrovsky-Hunter, etc.

- BO is a model for the propagation of one dimensional internal waves in deep water: wave propagation at the interface of layers of fluids with different densities

About BO

- It is known to be completely integrable: it has an associated Lax pair, an inverse scattering transform, an infinite hierancy of conservation laws

$$E_0 = \int \phi^2 \, dx$$

momentum

$$E_1 = \int \phi H \phi_x - \frac{1}{3}\phi^3 \, dx$$

as well as energy

$$E_2 = \int \phi_x^2 - \frac{3}{4}\phi^2 H\phi_x + \frac{1}{8}\phi^4 \, dx$$

- It has a Hamiltonian structure; uses the symplectic form

$$\omega(\psi_1, \psi_2) = \int \psi_1 \partial_x \psi_2 \, dx$$

with momentum as the Hamiltonian.

About BO

- It is a dispersive equation

$$\boxed{\omega(\xi) = -\xi|\xi|}$$

Restricting ourselves to positive/negative frequencies leads to a linear Schrödinger equation with a choice of signs.

- Group velocity:

$$\boxed{v = 2|\xi|}$$

suggests that waves propagate to the right and solitons to the left.

- It is more of a quasilinear nature, than semilinear: the ∂_x in the nonlinearity is strong so that $\partial_x(\phi^2)$ is not perturbative.

What is known about the well-posedness for BO

The BO equation: $$\begin{cases} (\partial_t + H\partial_x^2)\phi = \dfrac{1}{2}\partial_x(\phi^2), \\ \phi(0) = \phi_0 \end{cases}$$

- Solutions are invariant under the scaling

$$\phi(t, x) \Rightarrow \lambda\phi(\lambda^2 t, \lambda x)$$

The scaling invariant Sobolev space is $\dot{H}^{-\frac{1}{2}}$. This is a natural threshold for the well-posedness.

- LWP & GWP is well understood in H^s, $s \geq 0$. . Well-posedness in the range $-1/2 \leq s < 0$ appears to be an open question.

Goals:

1. Revisit the global well posedness result
2. What can we say about the long time behaviour of the solutions?

Previous works

- Many developments in the theory of local and global theory for the BO equations: Iorio (1986), Saut (1979), Ponce (1991), Molinet, Saut, and Tzvetkov (2001), Tao (2001), Koch and Tzvetkov (2003), Kenig and Koenig (2003), Ionescu and Kenig (2005), Burq and Planchon (2005), Kenig and Martel (2009), Molinet and Pilot (2012)

- Relevant to our work are the papers of Saut (1979) and Tao (2001). The ideas we use to reach the L^2 threshold for the well-possedness result are the normal form method (Shatah), and the renormalization idea introduced by Tao (2001) in the wave maps context.

1. We will discuss the first goal of this talk: the well-posedness result.

Local and Global Well-Posedness for BO

Theorem

The Benjamin-Ono equation is globally well-posed in L^2.

By well-posedness we mean the following:

(i) Existence of regular solutions: For each initial data $\phi_0 \in H^3$ there exists a unique global solution $\phi \in C(\mathbb{R}; H^3)$.

(ii) Existence and uniqueness of rough solutions: For each initial data $\phi_0 \in L^2$ there exists a solution $\phi \in C(\mathbb{R}; L^2)$, which is the unique limit of regular solutions.

(iii) Continuous dependence: The data to solution map $\phi_0 \to \phi$ is continuous from L^2 into $C(L^2)$, locally in time.

(iv) Higher regularity: The data to solution map $\phi_0 \to \phi$ is continuous from H^s into $C(H^s)$, locally in time, for each $s > 0$.

(v) Weak Lipschitz dependence*: The flow map for L^2 solutions is locally Lipschitz in the $H^{-\frac{1}{2}}$ topology.

Understanding well-posedness in low regularity setting

Energy estimates for the linearization is as follows

$$\frac{d}{dt}\|v\|_{L^2}^2 = \int_{\mathbb{R}} v\partial_x(\phi v)\,dx = \frac{1}{2}\int_{\mathbb{R}} v^2\partial_x\phi\,dx \lesssim \|\phi_x\|_{L^\infty}\|v\|_{L^2}^2$$

1. Thus, in order to have these energy estimates, one needs to assume $\|\phi_x\|_{L^\infty}$ bound, so Gronwall applies- this is what Saut did.

2. Alternatevely, one can ask that ϕ_x is only uniformly bounded in time, i.e., $\phi_x \in L_t^1 L_x^\infty$, so again Gronwall applies. But in terms of lowering the regularity of the initial data, it is not doing anything.

3. Tao observed that if one can prove $\phi_x \in L_t^4 L_x^\infty$, then, this will lower the reg.: eg the H^3 reg. required for the initial data is H^1 now. But this leads to Stricharz est. $\rightarrow$ a semilinear Schrödinger eq with perturbative nonlinearity $\rightarrow$ renormalization

Tools needed for H^1 well-posedness (Tao)

- Renormalization akin to a Cole-Hopf transformation

$$\phi \to w = P^+_{high} e^{-i\Phi}, \text{ where } \Phi_x := \frac{1}{2}\phi$$

applied to the ϕ, which removes the worst part of the quadratic nonlinearity

$$(\partial_t - i\partial_x^2)w = \text{ better semilinear terms}$$

- Bootrstrap argument for the Stricharz estimates.
- Frequency envelopes: We say that a sequence $c_k \in l^2$ is an L^2 frequency envelope for $\phi \in L^2$ if
  i) $\sum_{k=0}^{\infty} c_k^2 \lesssim 1$ (extra $c_0 \sim 1$);
  ii) it is slowly varying, $c_j/c_k \leq 2^{-\delta|j-k|}$, with δ a very small universal constant;
  iii) it bounds the dyadic norms of ϕ, namely $\|P_k\phi\|_{L^2} \leq c_k$.

Given a frequency envelope c_k we denote $c_{\leq k} = (\sum_{j\leq k} c_j^2)^{\frac{1}{2}}$.

Tools used for L^2 well-posedness
(Ionescu & Kenig and Molinet & Pilod)

- Renormalization, which transform the problem into a perturbative problem

$$w^+ = e^{-i\Phi_{low}} P^+_{high}\phi \quad (\approx \partial_x w)$$

- Modified $X^{s,b}$ spaces for perturbative estimates in w^+ equation, inspired from spaces introduced by Tataru and Tao for wave maps.

- Bilinear estimates in $X^{s,b}$ spaces and Strichartz estimates.

Ingredients for our proof of the L^2 well-posedness

Tools we use:

• A partial normal form transformation which is bounded and removes some of the quadratic nonlinearity.

• A conjugation via a suitable exponential (also called gauge transform Tao'01), which removes in a bounded way the remaining part of the quadratic nonlinearity.

• Strichartz and bilinear Strichartz estimates.

• Bootstrap argument for both Strichartz and bilinear Strichartz bounds in a frequency envelopes fashion.

New result:

We also prove $H^{-\frac{1}{2}}$ well-posedness for the linearized equation.

- Strichartz estimates:

Assume that ψ solves the linear Schrödinger equation in $\mathbb{R} \times \mathbb{R}$. Then the following estimate holds.

$$\|\phi\|_S \lesssim \|\phi_0\|_{L^2} + \|f\|_{S'},$$

where

$$S = L_t^\infty L_x^2 \cap L_t^4 L_x^\infty, \quad S' = L_t^1 L_x^2 + L_t^{\frac{4}{3}} L_x^1$$

- Bilinear estimates in L^2:

Let ψ^1, ψ^2 be two solutions to the inhomogeneous Schrödinger equation with data ψ_0^1, ψ_0^2 and inhomogeneous terms f^1 and f^2. Assume that the sets

$$E_i = \{|\xi|, \xi \in \text{supp } \hat{\psi}^i\}$$

are disjoint. Then we have

$$\|\psi^1\psi^2\|_{L^2} \lesssim \frac{1}{\text{dist}(E_1, E_2)} (\|\psi_0^1\|_{L^2} + \|f^1\|_{S'})(\|\psi_0^2\|_{L^2} + \|f^2\|_{S'})$$

Normal form transformations for BO

Definition: Normal forms are nonlinear transformations that remove *nonresonant* quadratic nonlinearities (Shatah' 83)

$$\tilde{\phi} \to \phi + B(\phi, \phi).$$

For BO we formally obtain

$$\tilde{\phi} = \phi - \frac{1}{4} H\phi \cdot \partial_x^{-1}\phi - \frac{1}{4} H\left[\phi \cdot \partial_x^{-1}\phi\right].$$

Difficulty: unbounded when ∂_x^{-1} falls on the low frequency

Solution: applying the normal form transformation in two steps, which are both bounded, and where the second step is a renormalization.

Start with the frequency localized equation

$$\left(i\partial_t + \partial_x^2\right)\phi_k^+ = iP_k^+(\phi \cdot \phi_x)$$

→ The issue: is at the level of $P_k^+(\phi \cdot \phi_x)$ when ϕ_x is at high frequency and ϕ is at low frequency.

→ The solution: move the principal part $\phi_{<k} \cdot \partial_x \phi_k^+$ of $P_k^+(\phi \cdot \phi_x)$ into the linear operator. Denoting

$$A_{BO}^{k,+} = \left(i\partial_t + \partial_x^2 - i\phi_{<k} \cdot \partial_x\right) + \frac{1}{2}(H+i)\,\partial_x \phi_{<k}$$

our equation becomes

$$A_{BO}^{k,+}\,\phi_k^+ = iP_k^+(\phi_{\geq k} \cdot \phi_x) + i\left[P_k^+, \phi_{<k}\right]\phi_x + \frac{1}{2}(H+i)\,\partial_x\phi_{<k} \cdot \phi_k^+$$

Eliminate the harmless nonlinear quadratic part via a bounded nft

$$\tilde{\phi}_k^+ := \phi_k^+ + B_k(\phi, \phi)$$

Appplying the nft: $\phi_k^+ \to \tilde{\phi}_k^+$ gives an equation of the form

$$A_{BO}^{k,+}\,\tilde{\phi}_k^+ = Q_k^3(\phi,\phi,\phi)$$

Renormalization idea: second bounded normal form transformation that will remove the paradifferential terms in the LHS

$$\psi_k^+ := \tilde{\phi}_k^+ \cdot e^{-i\Phi_{<k}}$$

$$\downarrow$$

$$(i\partial_t + \partial_x^2)\,\psi_k^+ = [\tilde{Q_k}^3(\phi,\phi,\phi) + \tilde{Q_k}^4(\phi,\phi,\phi,\phi)]e^{-i\Phi_{<k}}$$

! Validity of the steps above: bdd of the transf. seq. $\phi_k \to \tilde{\phi}_k \to \psi_k$

! $L_t^1 L_x^2$ bounds for $\tilde{Q_k}^3$ and $\tilde{Q_k}^4$; bounds obtained using frequency envelopes and Strichartz and bilinear Strichatrz estimates

! Bootstrap arg. is the last tool finishes the proof of the apriori bdds.

2. The second goal of this talk: optimal time-scale up to which the solutions have linear dispersive decay.

Linear dispersive decay

Theorem

If the initial data ϕ_0 for linear Benjamin-Ono equation satisfies

$$\|\phi_0\|_{L^2} + \|x\phi_0\|_{L^2} \leq \epsilon,$$

then the solution ϕ satisfies the globally in time dispersive decay bounds

$$|\phi(t,x)| + |H\phi(t,x)| \lesssim \langle t\rangle^{-\frac{1}{2}} \langle x_{-}t^{-\frac{1}{2}}\rangle^{-\frac{1}{2}}$$

→ Corresponding linear operator

$$L\phi = x\phi - 2tH\phi_x$$

which commutes with the linear BO operator.

→ Linear pointwise bounds:

$$\boxed{\|\phi(t)\|^2_{L^\infty} + \|H\phi(t)\|^2_{L^\infty} \lesssim t^{-1}\|\phi\|_{L^2}\|L\phi\|_{L^2}}$$

Solitons

Benjamin-Ono's solitons are given by

$$\phi(t,c) = cQ(c(x+ct)), \qquad \text{where } Q(x) = \frac{4}{1+x^2}.$$

Question: can solitons emerge from small and localized data?

Inverse scattering: solitons correspond to negative eigenvalues $-\lambda$ of the Lax operator

$$\mathcal{L} := H\partial_x + \phi$$

Key property: if ϕ is of size ϵ then

$$\lambda \leq e^{-\frac{\alpha}{\epsilon}}, \text{ where } \alpha = const.$$

Conclusion: for small data solitions can emerge from the self similar region only at almost-global time $T \approx e^{-\frac{\alpha}{\epsilon}}$

Nonlinear dispersive decay

Theorem

Assume that the initial data ϕ_0 for the Benjamin-Ono equation satisfies

$$\|\phi_0\|_{L^2} + \|x\phi_0\|_{L^2} \leq \epsilon.$$

Then the solution ϕ satisfies the dispersive decay bounds

$$|\phi(t,x)| + |H\phi(t,x)| \lesssim \langle t\rangle^{-\frac{1}{2}} \langle x_{-} t^{-\frac{1}{2}}\rangle^{-\frac{1}{2}}$$

up to time

$$|t| \lesssim T_\epsilon := e^{\frac{c}{\epsilon}}.$$

Conjecture: For any sufficiently small and localized data the BO solution is either dispersive or splits into a dispersive part and a soliton.

A new conserved energy

$$\boxed{G(\phi) = \int x^2\phi^2 - 4xt\left(\phi H\phi_x - \frac{1}{3}\phi^3\right) + 4t^2\left(\phi_x^2 - \frac{3}{4}\phi^2 H\phi_x + \frac{1}{8}\phi^4\right) dx}$$

$\rightarrow$ It is positive definite,

$$G(\phi) = \|L^{NL}\phi\|_{L^2}^2$$

where

$$L^{NL}\phi = x\phi - 2t(H\phi_x - \frac{1}{8}(3\phi^2 - (H\phi)^2)$$

A bootstrap argument

From the conservation laws we have

$$\begin{aligned} \|\phi(t)\|_{L^2} &\leq \epsilon, \\ \|L^{NL}\phi(t)\|_{L^2} &\leq \epsilon, \\ \int_{-\infty}^{\infty} \phi \, dx = c, \qquad |c| &\leq \epsilon. \end{aligned}$$

We further make the bootstrap assumption

$$\|\phi(t)\|_{L^\infty} + \|H\phi(t)\|_{L^\infty} \leq 2C\epsilon\langle t\rangle^{-\frac{1}{2}}, \qquad |t| \leq T \leq e^{\frac{c}{\epsilon}}$$

Then we prove

$$\|\phi(t)\|_{L^\infty}^2 + \|H\phi(t)\|_{L^\infty}^2 \lesssim \epsilon^2 t^{-1}(1 + C + C^3\epsilon \log t + C^4\epsilon^2 \log^2 t),$$

which suffices up to $T \leq e^{\frac{c}{\epsilon}}$.

Schetch of the proof

Recall

$$L^{NL}\phi = x\phi - 2t(H\phi_x - \frac{1}{8}(3\phi^2 - (H\phi)^2)$$

- First attempt: use the previous estimate for L and treat the quadratic terms in L^{NL} perturbatively. Does not work!
- Second attempt: treat these terms nonpertubatively:
    - pointwise bound for $\partial^{-1}\phi$
    - write an ODE for

$$\frac{d}{dx}\left(|\phi|^2 + |H\phi|^2\right) = F$$

- Evaluate F taking advantage of bootstrap assumptions as well as a certain commutator structure for the nonlinear contributions.

Thank you for your attention.