Universität Bielefeld

SFB 701

Scattering for the Cubic Dirac equation

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Asymptotic Analysis of Evolution Equations

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Cubic Dirac equation

$$\left.\begin{array}{c} -i\gamma^\mu \partial_\mu \psi + M\psi = (\overline{\psi}\psi)\psi \\ \psi(0) = f \end{array}\right\} \text{ on } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

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- $M \geqslant 0$, and $\psi(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}^N$ with $N = \begin{cases} 2 & n = 1,2 \\ 4 & n = 3. \end{cases}$

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- Repeated Greek indices are summed over $\mu = 0, ..., n$, and $\partial_0 = \partial_t$, $\partial_j = \partial_{x_j}$ $(j \geqslant 1)$.

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- Repeated Greek indices are summed over $\mu = 0, ..., n$, and $\partial_0 = \partial_t$, $\partial_j = \partial_{x_j}$ $(j \geqslant 1)$.
- γ^μ are (constant) $N \times N$ complex matrices such that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I_{N\times N}, \qquad (\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^j)^\dagger = -\gamma^j$$

  and $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In particular,

$$(-i\gamma^\mu\partial_\mu + M)^\dagger(-i\gamma^\mu\partial_\mu + M) = \partial_t^2 - \Delta + M^2 = \Box + M^2.$$

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- If $n = 3$, one choice is

$$\gamma^0 = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

where the Pauli matrices σ^j are defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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- If $n = 1, 2$, we take

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^2, \qquad \gamma^2 = -i\sigma^1.$$

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- Dirac adjoint $\overline{\psi} = \psi^\dagger \gamma^0$ (implies $\overline{\psi}\psi \in \mathbb{R}$).

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- Two main models

$(\overline{\psi}\psi)\psi$	Soler Model [Soler'70]
$(\overline{\psi}\gamma_\mu\psi)\gamma^\mu\psi$	Thirring Model [Thirring'58]

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- Squaring the Dirac equation leads to an equation of the form

$$\Box\psi + M^2\psi = M\psi^3 + \psi^2\partial\psi.$$

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- Basic conserved quantities are the Charge

$$Q[\psi] = \|\psi\|_{L^2_x(\mathbb{R}^n)}$$

and the Energy

$$E[\psi] = \int_{\mathbb{R}^n} \frac{i}{2}\big(\overline{\psi}\gamma^0\partial_t\psi - \overline{\partial_t\psi}\gamma^0\psi\big) + \frac{1}{2}\big(\overline{\psi}\psi\big)^2 dx.$$

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Can decompose $\psi = \psi_+ + \psi_-$

$$E[\psi] = \int_{\mathbb{R}^n} \big|\langle\nabla\rangle^{\frac{1}{2}}\psi_+\big|^2 - \big|\langle\nabla\rangle^{\frac{1}{2}}\psi_-\big|^2 - \frac{3}{2}(\overline{\psi}\psi)^2 dx.$$

Scaling

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- Thus the Cubic Dirac equation is critical in $H^{\frac{n-1}{2}}$, in particular,

$n=1$ problem is Charge critical, scale invariant space is L^2.

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- Basic Questions:
  1. (LWP) Given data $f \in H^s$ can we find a time $T > 0$ and a unique solution $\psi \in C([0,T], H^s)$ which depends continuously on the data?
  2. (GWP and asymptotic behaviour) Can we extend local solution to a global solution $\psi \in C(\mathbb{R}, H^s)$? What happens as $t \to \infty$?

Case $n = 1$: large data GWP

Focus on Thirring model

$$-i\gamma^\mu \partial_\mu \psi + M\psi = (\overline{\psi}\gamma^\mu \psi)\gamma_\mu \psi.$$

Theorem (C.'12)

Let $n = 1$ and $M \geqslant 0$. Then the Thirring model is globally well-posed from large data in $L^2_x(\mathbb{R})$.

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- Previous results: gwp for regular large data [Delgado'78], large data global existence $s > \frac{1}{2}$ [Selberg-Tesfahun'10].

Case $n = 1$: modified scattering

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Theorem (C.-Lindblad'16)

Let $n = 1$ *and* $M = 1$. *If the data satisfies* $\| \langle x \rangle^4 f \|_{H^5} \ll 1$ *then for the solution* $\psi^T = (\psi_1, \psi_2)$ *we have the pointwise asymptotics as* $\rho = \sqrt{t^2 - x^2} \to \infty$

$$\begin{aligned}
&\sqrt{t-x}(\psi_1 + \psi_2) \\
&\qquad = e^{i\rho + 2i|f_+(\frac{x}{t})|^2 \ln(\rho)} f_+(\tfrac{x}{t}) + e^{-i\rho + 2i|f_-(\frac{x}{t})|^2 \ln(\rho)} f_-(\tfrac{x}{t}) + \mathcal{O}\big(\rho^{-\frac{1}{2}}\big) \\
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- For linear Dirac log correction vanishes.
- In the massless case $M = 0$, can explicitly write down solution in terms of data.

Sketch of proof

Argument follows original approach to modified scattering for Klein-Gordon equation [Delort'01], [Lindblad-Soffer'05].

- We consider separately the exterior region $1 \leqslant t \leqslant \langle x \rangle$, and the interior region $t \geqslant \langle x \rangle$.

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- We consider separately the exterior region $1 \leqslant t \leqslant \langle x \rangle$, and the interior region $t \geqslant \langle x \rangle$.
- Exterior region Klein-Gordon equation has fast decay. Can exploit this by rewriting problem as a cubic Klein-Gordon equation which is schematically of form

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$$\Box \psi + \psi = \psi^3 + \psi^2 \partial \psi$$
together with weighted energy estimates as in [KLAINERMAN'93].
- This gives the decay bound
$$|\psi(t,x)| \lesssim \langle x \rangle^{-1} \| \langle x \rangle^4 \psi_0 \|_{H^5}$$
easily enough decay to close bootstrap argument.

Sketch of proof

- Remains to construct solution in the more interesting interior region $t \geqslant \langle x \rangle$. Introduce hyperbolic coordinates

$$t = \rho \cosh(y), \qquad x = \rho \sinh(y)$$

so $\rho = \sqrt{t^2 - x^2}$. After extracting linear decay/oscillations reduce to system of form

$$\partial_\rho \phi_\pm + e^{\mp 2i\rho} \frac{1}{\rho} \partial_y \phi_\pm = i|\phi_\pm|^2 \phi_\pm + \partial_\rho S_\pm + R_\pm$$

where $R_\pm = \mathcal{O}(\rho^{-2})$, and $\partial_\rho S_\pm \sim \rho^{-1}$.

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- Sharper methods can clearly be used to weaken assumptions [STINGO'15], [IFRIM-TATARU'15]...

Case $n = 1$: further results

- Thiring model is completely integrable, and in particular, explicit soliton solutions are known for the Thirring model, for instance for $|\omega| < M$

$$\psi = e^{it\omega} \begin{pmatrix} a_\omega(x) + a^\dagger_\omega(x) \\ a_\omega(x) - a^\dagger_\omega(x) \end{pmatrix}$$

with

$$U_\omega(x) = \frac{\sqrt{M - \omega^2}}{\sqrt{M + \omega} \cosh(\sqrt{M - \omega^2}\, x) + i\sqrt{M - \omega} \sinh(\sqrt{M - \omega^2}\, x)}.$$

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- Orbital stability in L^2_x of these solitons was recently obtained [Contreras-Pelinovsky-Shimabukuro'16] via inverse scattering methods.

Case $n = 2, 3$: small data gwp and scattering

Theorem (Bejenaru-Herr'15, '16, Bournaveas-C.'15)

Let $n = 2, 3$ and $M \geqslant 0$. There exists $\epsilon > 0$ such that if $\|f\|_{H^{\frac{n-1}{2}}} < \epsilon$ then there exists a global solution $\psi \in C(\mathbb{R}, H^{\frac{n-1}{2}})$ which is unique in a certain subspace, and depends continuously on the data. Moreover ψ scatters to a linear solution as $t \to \pm\infty$, thus there exists $\psi_{\pm\infty}$ with $(-i\gamma^\mu \partial_\mu + M)\psi_{\pm\infty} = 0$ such that

$$\lim_{t \to \pm\infty} \|\psi(t) - \psi_{\pm\infty}(t)\|_{H^{\frac{n-1}{2}}} = 0.$$

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- Result also holds in the case of the Thirring Model

$$-i\gamma^\mu \partial_\mu \psi + M\psi = (\overline{\psi}\gamma^\mu \psi)\gamma_\mu \psi.$$

When $n = 3$ can also add combinations of

$$(\overline{\psi}\gamma^5\psi)\psi, \qquad (\overline{\psi}\psi)\gamma^5\psi, \qquad (\overline{\psi}\gamma^5\psi)\gamma^5\psi$$

where $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ (essentially any Lorentz covariant nonlinearity).

Large data solutions $n = 3$

Theorem (C.-Herr'17)

Let $z \in \mathbb{C}$, $|z| = 1$, and $M, A \geqslant 0$. There exists $\epsilon = \epsilon(A) > 0$ such that if $\|\psi(0)\|_{H^1} \leqslant A$ and

$$\|\psi(0) + z\gamma^2\psi^*(0)\|_{H^1} \leqslant \epsilon$$

solution is globally well-posed and scatters to a free solution.

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- First observed by [CHADAM-GLASEY'74] with $\epsilon = 0$, [BACHELOT'89] for smooth data, [D'ANCONA-OKAMOTO'17] angular regularity plus potential.

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- Key point is that structural assumption implies that product $\overline{\psi}\psi$ is small, so can run perturbative argument as in the small data case.

Previous Results for Cubic Dirac:

- Local well-posedness for $s > 1$ (subcritical range) due to [ESCOBEDO-VEGA '97].

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- Local well-posedness for $s > 1$ (subcritical range) due to [Escobedo-Vega '97].
- Global well-posedness and scattering when $s > 1$ and $M > 0$, or $s = 1$ and some additional angular regularity due to [Machihara-Nakanishi-Ozawa '03, Machihara-Nakamura-Ozawa'04, Machihara-Nakamura-Nakanishi-Ozawa'05].

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- (linear) stability/instability of solitary waves [Comech-Guan-Gustafson'14, Contreras-Pelinovsky-Shimabukuro'16, Boussaid-Comech'16...]

Basic Linear Bounds

$$\left.\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= 0 \\ \psi(0) &= f \end{aligned}\right\} \quad \text{on } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

- Energy Estimate

$$\|\psi\|_{L^\infty_t H^s_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{H^s(\mathbb{R}^n)} + \|(-i\gamma^\mu\partial_\mu + M)\psi\|_{L^1_t H^s_x(\mathbb{R}^{1+n})}$$

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- L^∞ Strichartz
  Let $\frac{1}{q} < \min\{\frac{n-1}{4}, \frac{1}{2}\}$. Then

$$\|\psi\|_{L^q_t L^\infty_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{H^{\frac{n}{2}-\frac{1}{q}}(\mathbb{R}^n)} + \|(-i\gamma^\mu\partial_\mu + M)\psi\|_{L^1_t H^{\frac{n}{2}-\frac{1}{q}}_x(\mathbb{R}^{1+n})}$$

(see [Strichartz'77],[Ginibre-Velo'89],[Escobedo-Vega '97]...).

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$$\|\psi\|_{L^\infty_t H^s_x} \lesssim \|f\|_{H^s} + \|\overline{\psi}\psi\psi\|_{L^1_t H^s_x}$$

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- To improve need two further ingredients:
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  2. Need to exploit null frames introduced by Tataru in the study of the wave maps equation.

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Null Structure I

Let $-i\gamma^\mu \partial_\mu \psi = 0$ and consider the bilinear term $\overline{\psi}\psi$.

- Introduce potential

$$-i\gamma^\mu \partial_\mu \varphi = \psi.$$

Then $\Box \varphi = 0$ and

$$\overline{\psi}\psi = Q(\varphi, \varphi)$$

where Q is sum of classical null forms

$$Q_{\mu\nu}(u, v) = \partial_\mu u \partial_\nu v - \partial_\nu u \partial_\mu v, \qquad Q_0(u, v) = \partial^\mu u \partial_\nu v$$

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- These bilinear forms have improved regularity/decay properties and have been well-studied [KLAINERMAN-MACHEDON'93], [KLAINERMAN-FOSCHI'00], [LEE-VARGAS'08]
- As a consequence, we get the bound

$$\left\| \overline{\psi}\psi \right\|_{L^2_{t,x}} \lesssim \left\| \psi(0) \right\|_{L^2_x} \|\psi(0)\|_{H^{\frac{n-1}{2}}}.$$

Null structure II

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$$(\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi = \begin{cases} (\overline{\psi}\psi)\psi & n=2 \\ (\overline{\psi}\psi)\psi - (\overline{\psi}\gamma^5\psi)\gamma^5\psi & n=3 \end{cases}$$

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- To close an iteration argument, now requires exploiting the above null structure observation in the adapted null frame spaces used in the wave map theory (if $M = 0$), and constructing null frame spaces adapted to the hyperboloid (if $M > 0$).

Dirac-Klein-Gordon system on $\mathbb{R}^{1+3}$.

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= \phi\psi \\ \Box\phi + m^2\phi &= \overline{\psi}\psi \end{aligned}$$

with $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ and $\psi : \mathbb{R}^{1+n} \to \mathbb{C}^N$. Masses satisfy $M, m \geqslant 0$.

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- Have small data global well-posedness and scattering for critical data with $\sigma > 0$ angular derivatives and $M, m > 0$ [C.-Herr'16], results in non-resonant case $2M > m > 0$ [Bejenaru-Herr'15], [Wang'13]

Large Data gwp

Theorem (C.-Herr'17)

Let $z \in \mathbb{C}$, $|z| = 1$. *Let* $M, m > 0$ *and* $\sigma > 0$. *For any* $A \geqslant 0$, *there exists* $\epsilon = \epsilon(A) > 0$ *such that if*

$$\|\langle\Omega\rangle^\sigma \phi(0)\|_{H^{\frac{1}{2}}} + \|\langle\Omega\rangle^\sigma \partial_t \phi(0)\|_{H^{-\frac{1}{2}}} + \|\langle\Omega\rangle^\sigma \psi(0)\|_{L^2} \leqslant A,$$

and

$$\big\|\langle\Omega\rangle^\sigma \big(\psi(0) + z\gamma^2\psi^*(0)\big)\big\|_{L^2_x} \leqslant \epsilon$$

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- Related work in the smooth case [Chadam-Glassey'74], [Bachelot'89].

Sketch of proof

$$-i\gamma^\mu \partial_\mu \psi + M\psi = \phi\psi$$
$$\Box\phi + m^2\phi = \overline{\psi}\psi$$

- From previous work [C.-Herr'16] given an interval $I \subset \mathbb{R}$, have a norm $F^{s,\sigma}$ and a bilinear estimate

$$\|\psi\|_{F^{0,\sigma}(I)} \lesssim \|\langle\Omega\rangle^\sigma \psi(0)\|_{L^2_x} + \|\phi\|_{F^{\frac{1}{2},\sigma}(I)} \|\psi\|_{F^{0,\sigma}(I)}.$$

  Problem is that we have no smallness here (assumption only implies $\overline{\psi}\psi$ is small).

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- Moreover, the norms $F^{s,\sigma}(I)$ don't become small as I shrinks (they are essentially V^2 type norms with some modulation gain). Instead need to prove stronger bound, for some $\delta > 0$

$$\|\psi\|_{F^{0,\sigma}(I)} \lesssim \|\langle\Omega\rangle^\sigma \psi(0)\|_{L^2_x} + \|\langle\Omega\rangle^\sigma \phi\|^{\delta}_{L^4_{t,x}(I\times\mathbb{R}^3)} \|\phi\|^{1-\delta}_{F^{\frac{1}{2},\sigma}(I)} \|\psi\|_{F^{0,\sigma}(I)}$$

  as the $L^4_{t,x}(I \times \mathbb{R}^3)$ does become small as I shrinks.

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Improved Bilinear Estimate

The key additional ingredient is the following bilinear restriction type estimate at multiple scales.

Theorem (C.'17)

Let $n \geqslant 2$, $1 \leqslant q, r \leqslant 2$, $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$, *and* $0 \leqslant m_j \lesssim \lambda_j$ *for* $j = 1, 2$. *Let* $\alpha > 0$ *and define* $\beta = (\frac{m_1}{\alpha\lambda_1} + \frac{m_2}{\alpha\lambda_2} + 1)^{-1}$. *If the supports of* $\widehat{f}$ *and* $\widehat{g}$ *are* α*-transverse, and at frequencies* λ_1 *and* λ_2 *respectively, then*

$$\left\| e^{it\langle\nabla\rangle_{m_1}} f e^{it\langle\nabla\rangle_{m_2}} g \right\|_{L^q_t L^r_x} \lesssim \alpha^{n-1-\frac{n-1}{r}-\frac{2}{q}} \beta^{1-\frac{1}{r}} \lambda_{min}^{n-\frac{n}{r}-\frac{1}{q}} \left(\frac{\lambda_{max}}{\lambda_{min}}\right)^{\frac{1}{q}-\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}$$

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- This is Klein-Gordon version of the Wave bilinear restriction estimates of [Lee-Vargas'08], [Tao'01], [Wolf'01]...