Wireless network signals with moderately correlated shadowing still appear Poisson

Dominic Schuhmacher

Institute for Mathematical Stochastics

University of Göttingen

Joint work with Nathan Ross

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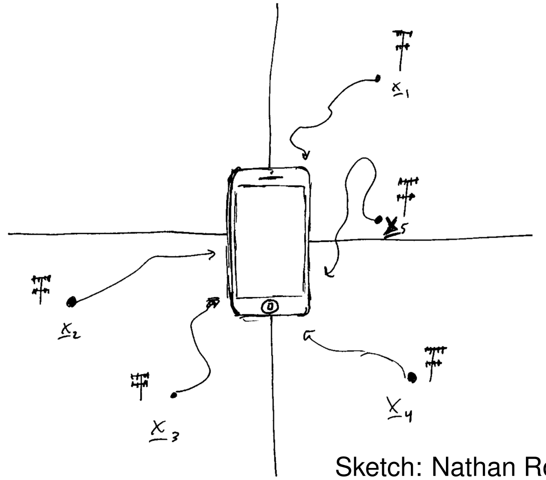
Problem statement

Goal: Understand the pattern of signal strengths received from base stations (e.g. cell towers) by a “typical” user (e.g. cell phone).



Sketch: Nathan Ross

Standard model for densely populated areas

- Base stations (BS) are placed as a simple point process $\Xi = \sum_{i \in \mathcal{I}} \delta_{X_i}$ in $\mathbb{R}^2$. All transmit with the same power P.
- Power received at origin from BS at x is

$$P_x^{\text{rec}} = \frac{S_x}{g(x)} P,$$

where
distance loss: $g(x) = (K\|x\|)^\beta$, $\beta > 2$ (finite interference).
shadow-fading effect (what remains): $S_x = \exp\bigl(\sigma Z_x - \sigma^2/\beta\bigr)$, $\sigma > 0$.
where $Z_x \sim \mathcal{N}(0,1)$ and $(Z_x)_x$ independent of Ξ.

β and σ have to do with transmission frequency and environment.

- Propagation loss for signal from x is

$$Y_x = \frac{P}{P_x^{\text{rec}}} = \frac{g(x)}{S_x}.$$

Point process of propagation losses: $N_\sigma = \sum_{i \in \mathcal{I}} \delta_{Y_i}$, where $Y_i = Y_i^{(\sigma)} = Y_{X_i}$.

Regime and consequences

Regime:

$\beta > 2$ fixed,
$\sigma \to \infty$ (“strong shadowing effects”, i.e. heterogeneous environment; $\sigma \geq 2$ ok).

Consequences:

- $S_x = \exp\bigl(\sigma Z_x - \sigma^2/\beta\bigr) \to 0$ a.s., $Y_x = g(x)/S_x \to \infty$ a.s.
- S_x is normalized so that $\mathbb{E} S_x^{2/\beta} = 1$.
- If $\mathbb{E}\Xi(A) = \kappa \mathit{Leb}^d(A)$ for every $A \in \mathcal{B}^2$, then

$$\mathbb{E}N_\sigma((0,t]) = \mathbb{E}\int_{\mathbb{R}^2} 1\{Y_x \leq t\}\,\Xi(dx) = \kappa \int_{\mathbb{R}^2} \mathbb{P}(Y_x \leq t)\,dx = \kappa \int_{\mathbb{R}^2} \mathbb{P}(g(x)/S \leq t)\,dx$$
$$= \kappa \mathbb{E}\int_{\mathbb{R}^2} 1\{(K\|x\|)^\beta/S \leq t\}\,dx = \frac{\kappa\pi t^{2/\beta}}{K^2}\mathbb{E}(S^{2/\beta}) = \frac{\kappa\pi t^{2/\beta}}{K^2} =: \mu((0,t])$$

- If Ξ is deterministic, satisfying $\Xi(\bar{\mathbb{B}}(0,r)) \sim \kappa\pi r^2$ as $r \to \infty$, then

$$\mathbb{E}N_\sigma((0,t]) = \sum_{i \in \mathcal{I}} \mathbb{P}(Y_{x_i} \leq t) \to \frac{\kappa\pi t^{2/\beta}}{K^2} = \mu((0,t]) \quad \text{as } \sigma \to \infty.$$

Independent shadowing results

Suppose that Z_{X_i}, $i \in \mathcal{I}$, are independent.

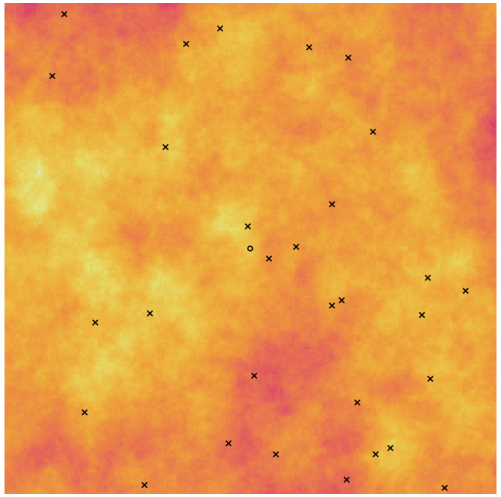
Błaszczyszyn, Karray and Klepper (2010):
To obtain N_σ, the points of Ξ are subjected to the independent random transform $[x \mapsto g(x)/S_x]$.
So if $\Xi \sim \mathrm{Pop}(\kappa Leb^d)$, then $N_\sigma \sim \mathrm{Pop}(\mu)$.

Błaszczyszyn, Karray and Keeler (2013):
For deterministic Ξ, we have $N_\sigma \xrightarrow{\mathscr{D}} \mathrm{Pop}(\mu)$ as $\sigma \to \infty$.

Keeler, Ross and Xia (to appear, presented at SGSIA 2015):
For very general Ξ, distance loss functions and independent shadowing models:
under the compatibility condition $\int \mathbb{P}(Y_x \le t)\,(\mathbb{E}\Xi)(dx) \to \mu((0,t])$ as $\sigma \to \infty$,
we still have $N_\sigma \xrightarrow{\mathscr{D}} \mathrm{Pop}(\mu)$.

But independent shadowing is unrealistic!

Dependent shadowing model



Still $S_x = \exp(\sigma Z_x - \sigma^2/\beta)$,

but now $(Z_x)_{x\in\mathbb{R}^2}$ is Gaussian random field with
mean 0, variance 1, and
general correlation function ϱ,

i.e. $\varrho(x, y) = \mathrm{corr}(Z_x, Z_y)$.

Dependent shadowing results

Template of a general result, assuming Ξ is in some sense stationary:
$N_\sigma \xrightarrow{\mathscr{D}} \mathrm{Pop}(\mu)$ as $\sigma \to \infty$ still holds if

1. Ξ is orderly enough, i.e. $\frac{1}{\varepsilon^d}\mathbb{P}\big(\Xi(\mathbb{B}(x,\varepsilon)) \ge 2\big) \to 0$ sufficiently fast as $\varepsilon \to 0$;
2. Ξ satisfies a (very weak) mixing condition ("no long-range pos. correlations");
3. $\rho(x, y) \to 0$ fast enough as $\|x - y\| \to \infty$;
4. $\rho(x, y) \to 1$ not too fast as $\|x - y\| \to 0$.

Dependent shadowing results

Concrete result for hard core process (Ross and S, 2017):
Let Ξ second-order stationary
$\Longrightarrow$ $\exists$ constant intensity κ, reduced 2nd factorial moment measure $\check{\lambda}_{[2]}$.
Assume further

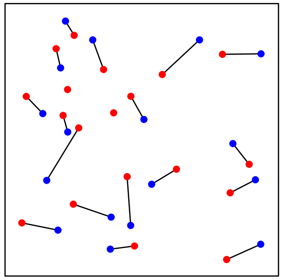
1. Ξ has a hard core, i.e. there is a $\varepsilon_* > 0$ such that $\inf_{\{x,y\}\subset\Xi}\|x - y\| \ge \varepsilon_*$ a.s.;
2. Ξ is B_2^+-mixing, in the sense that the reduced covariance measure $\check{\gamma}_{[2]}$, given by
$$\check{\gamma}_{[2]}(B) = \check{\lambda}_{[2]}(B) - \kappa|B|,$$
for any $B \in \mathcal{B}^2$ bounded can be extended to a $[-\infty, \infty)$-valued signed measure (i.e. of finite positive variation);
3. There is a non-increasing $\tilde{\varrho}\colon \mathbb{R}_+ \to [0, 1]$ satisfying $\varrho(x, y) \le \tilde{\varrho}(\|x - y\|)$, such that $\tilde{\varrho}(r) < 1$ for $r > 0$, $\tilde{\varrho}(r) = O(r^{-(1+a)})$ for some $a > 0$, and $[r \mapsto r\tilde{\varrho}^2(r)]$ non-increasing on some interval $[r_0, \infty)$;
4. ϱ is *uniformly positive definite (u.p.d.)*:
$\forall \varepsilon > 0\colon \exists \delta = \delta(\varepsilon) > 0\colon \forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in \mathbb{R}^2$ with $\min_{i\ne j}\|x_i - x_j\| \ge \varepsilon$ and $\forall v \in \mathbb{R}^n\colon \sum_{i,j=1}^n v_i v_j \rho(x_i, x_j) \ge \delta\|v\|^2$.

Then $N_\sigma \xrightarrow{\mathscr{D}} \mathrm{Pop}(\mu)$ as $\sigma \to \infty$, and we can give a rate in Wasserstein distance.

Wasserstein metric between point process distrib.

We measure distances between point patterns (finite point measures) on $[0, t]$ by the OSPA metric (S, Vo and Vo, 2008; S and Xia, 2008):

Π_n set permutations on $\{1, 2, \ldots, n\}$.



For point patterns $\xi = \sum_{i=1}^m \delta_{t_i}$ and $\eta = \sum_{j=1}^n \delta_{s_j} \in \mathfrak{N}$ with $m \le n$,

$$d(\xi, \eta) = \min_{\pi \in \Pi_n} \frac{1}{n} \sum_{i=1}^m \left| t_i - s_{\pi(i)} \right| + \frac{t}{n}(n - m).$$

Consider Wasserstein metric between point process distributions P and Q:

$$d_W(P, Q) = \min_{\substack{\Xi \sim P \\ \mathrm{H} \sim Q}} \mathbb{E}\, d(\Xi, \mathrm{H}).$$

Bound for hard core process

- $b_r := \frac{\beta}{\sigma} \log((Kr)/t) + \frac{\sigma}{\beta}$. (Significance: $\mathbb{P}(Y_x \le t) = \mathbb{P}(Z_x \ge b_{\|x\|})$)
- Fix $d, R, C > 0$ with $d, R \le C$, $b := b_d > 0$ and $B := b_C > 1$.
- Let $\varepsilon_* > 0$ be the hard core distance, $T^* = T^*(R) = \left(\frac{R + \varepsilon_*/2}{\varepsilon_*/2}\right)^2$ upper bound for the maximal number of points in any R-ball.
- $F := F(R, T^*, \varepsilon_*) = \frac{1}{\delta(\varepsilon_*)}(4\pi + 1) T^* \left[\tilde{\varrho}^2(R) + \frac{1}{\sqrt{3}R^2} \int_R^\infty s \tilde{\varrho}^2(s)\, ds\right]$ Assume that $B^2 F \le 1$.
- $M(s) = \mu((0, s]) = \mathbb{E} N_\sigma((0, s])$ and $M^\Xi(s) = \mu^\Xi((0, s]) = \mathbb{E}\big(N_\sigma((0, s]) \mid \Xi\big)$.

Then,

$$\begin{aligned} & d_W\big(\mathcal{L}(N_\sigma|_{[0,t]}), \mathrm{Pop}(\mu|_{[0,t]})\big) \\ & \le 2\kappa \int_{\|x\| > C} \mathbb{P}(Z \ge b_{\|x\|})\, dx + \mathbb{E}\big|M^\Xi(t) - M(t)\big| + \mathbb{E} \int_0^t \big|M^\Xi(s) - M(s)\big|\, ds \\ & \quad + M(t) T^* \left[\mathbb{P}(Z \ge b_d) + 5 e^{-b^2(1 - \tilde{\varrho}(\varepsilon_*))/4}\right] + \kappa \pi d^2 \\ & \quad + (t + 1) M(t) \left[\frac{8(B + \sigma^{-1})\sqrt{F}}{\sqrt{1 - F^2}} + (1 + b^{-2})\sqrt{F} e^{-b^2(F^{-1} - 1)/2}\right] = O(\sigma^{-z}) \end{aligned}$$

for any $z > 0$.

Proof idea

1. $d_W(\mathcal{L}(N_\sigma|_{[0,t]}), \mathrm{Pop}(\mu|_{[0,t]}))$
   $\le d_W(\mathcal{L}(N_\sigma|_{[0,t]}), \mathrm{Cox}(\mu^\Xi|_{[0,t]})) + d_W(\mathrm{Cox}(\mu^\Xi|_{[0,t]}), \mathrm{Pop}(\mu|_{[0,t]}))$.
2. Condition first term on Ξ and dispose of nasty events whose probability is small (and nasty bits of Ξ whose influence is small).
3. Apply Stein's method to the rest.
4. Control weak dependence between one BS and all other BSs that are far apart by using conditional distribution of multivariate Gaussian vector and uniform positive definiteness.

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