Hyperplane tessellations in Euclidean and spherical spaces

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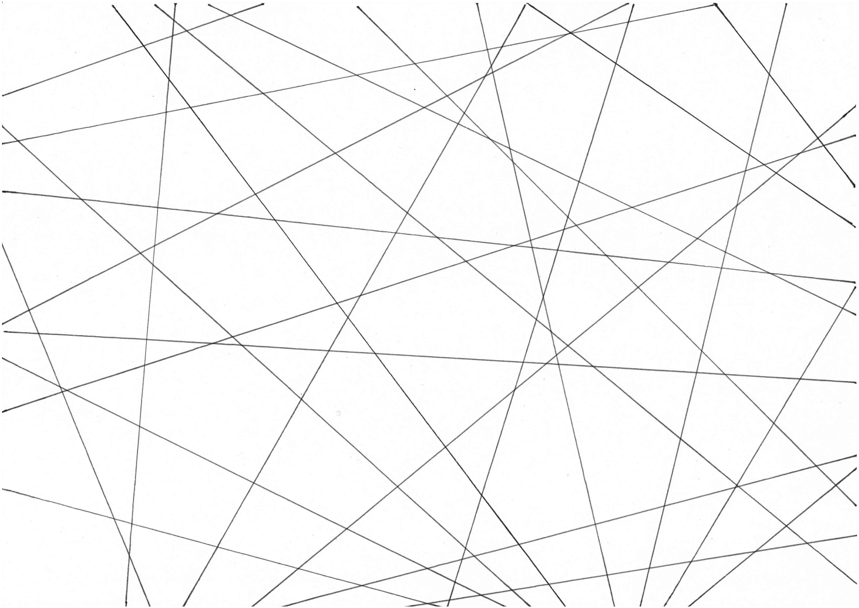
Our topic:

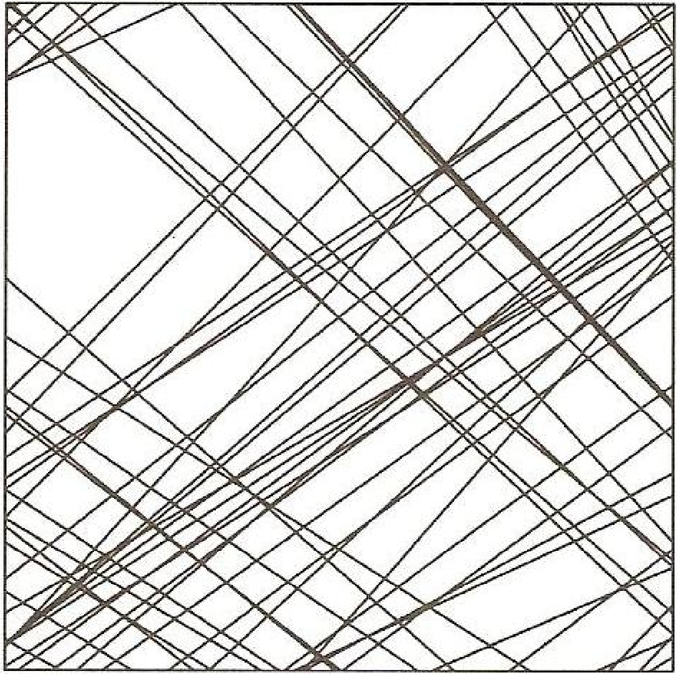
Tessellations of $\mathbb{R}^d$, induced by stationary Poisson hyperplane processes

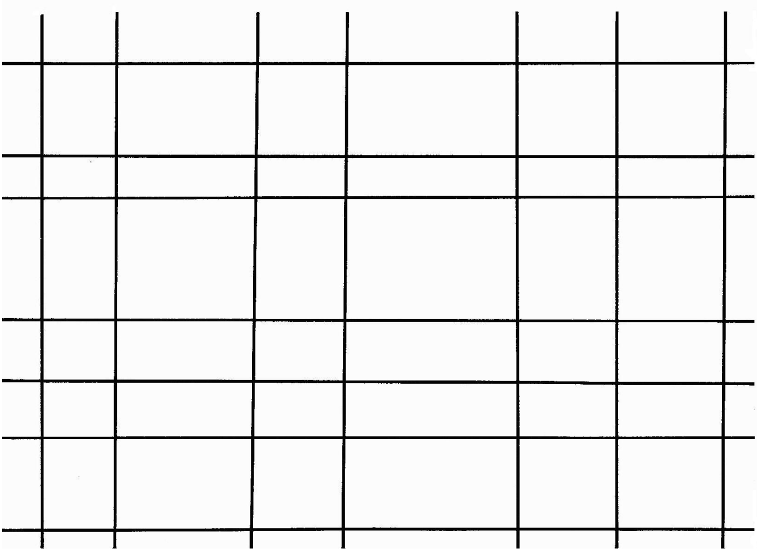
Tessellations of $\mathbb{R}^d$ into polyhedral cones (or of $\mathbb{S}^{d-1}$ into spherical polytopes) by i.i.d. random hyperplanes through the origin

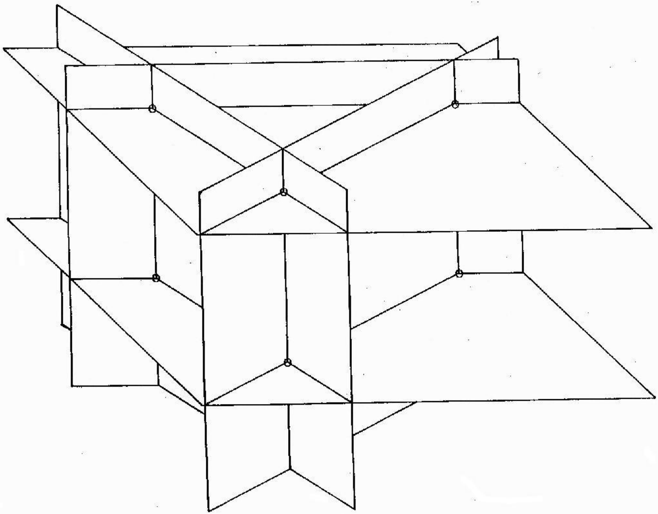
From a geometric viewpoint:

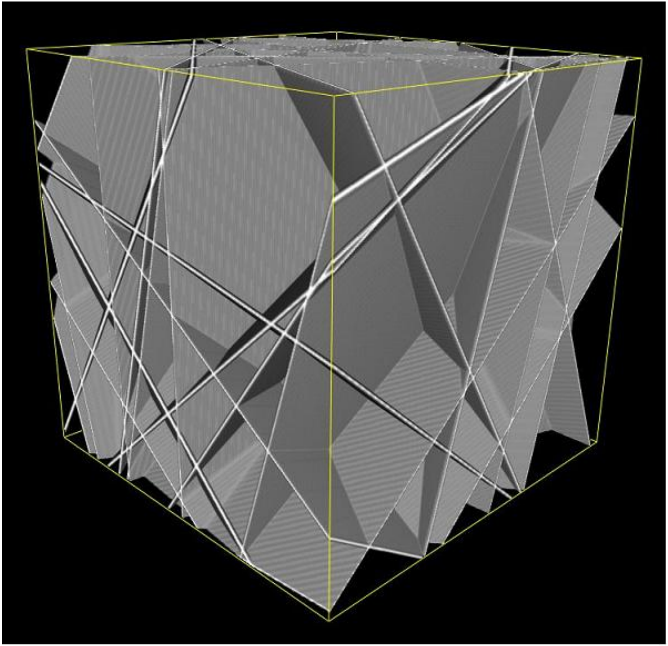
Shapes and sizes (functionals) of the induced random polytopes and cones









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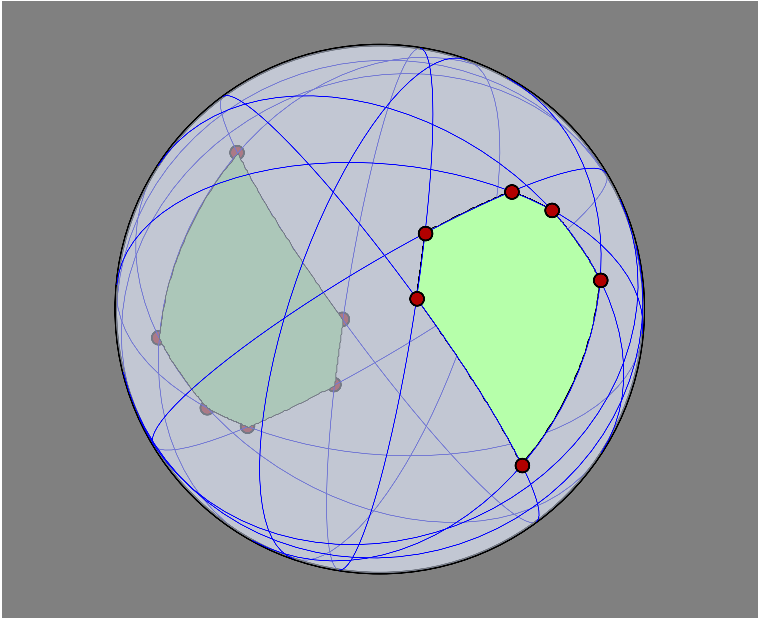


Programme

1. Explanations
2. Which shapes occur?
3. Typical cells and k-faces
4. A very brief survey: shapes of large cells
5. Variances, in particular of the vertex number
6. Random cones

1. Explanations

hyperplane: $u^{\perp} + tu$ with $u \in \mathbb{S}^{d-1}$ (unit sphere), $t \in \mathbb{R}$

$\mathcal{H}^d$ space of hyperplanes in $\mathbb{R}^d$

A hyperplane process X is a measurable mapping from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the measurable space of locally finite subsets of $\mathcal{H}^d$ (with a suitable σ-algebra).

Its intensity measure is defined by

$$\Theta(A) := \mathbb{E}\,\mathrm{card}(X \cap A), \qquad A \in \mathcal{B}(\mathcal{H}^d),$$

($\mathcal{B} =$ Borel sets).

X is a stationary Poisson hyperplane process if Θ is translation invariant, locally finite, $\not\equiv 0$,

$$\mathbb{P}(\mathrm{card}(X \cap A) = n) = e^{-\Theta(A)} \frac{\Theta(A)^n}{n!}, \quad n \in \mathbb{N}_0, \ A \in \mathcal{B}(\mathcal{H}^d),$$

and the restrictions of X to pairwise disjoint sets $A_1, \dots, A_k \in \mathcal{B}(\mathcal{H}^d)$ are stochastically independent.

If Θ is also invariant under rotations, X is called isotropic. In this case, its distribution is invariant under rigid motions.

By stationarity, the intensity measure Θ has a decomposition

$$\Theta(\cdot) = 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_{(\cdot)}(u^\perp + tu)\,\mathrm{d}t\,\varphi(\mathrm{d}u).$$

γ the intensity of X,

φ the directional distribution of X,
an even probability measure on the unit sphere $\mathbb{S}^{d-1}$,
not concentrated on a great subsphere (assumption).

Intuitive meaning:

$$\varphi(A) = \frac{\mathbb{E}\,\mathrm{card}\left(\left\{u^\perp + tu \in X : u \in A,\ t \in [0,1]\right\}\right)}{\mathbb{E}\,\mathrm{card}\left(\{u^\perp + tu \in X : t \in [0,1]\}\right)}$$

for $A \subset \mathcal{B}(\mathbb{S}^{d-1})$.

2. Which shapes occur?

We consider a stationary Poisson hyperplane process X, with directional distribution φ.

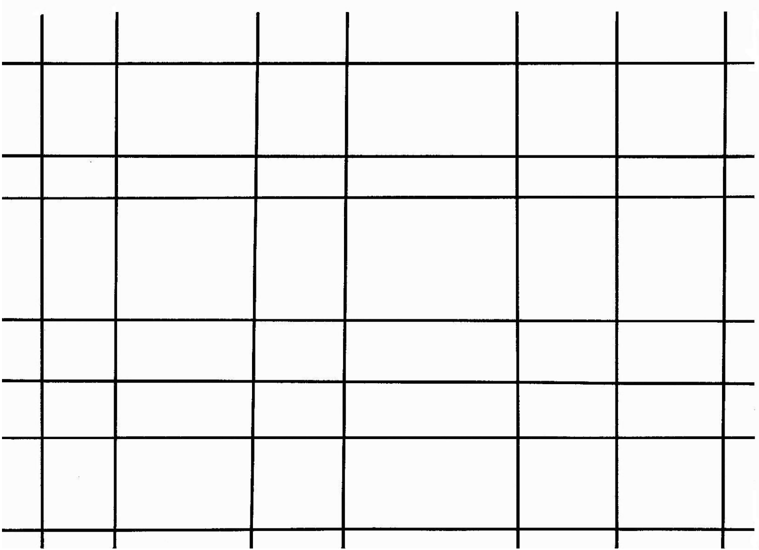
With probability one, X induces a tessellation of $\mathbb{R}^d$ into compact convex polytopes.

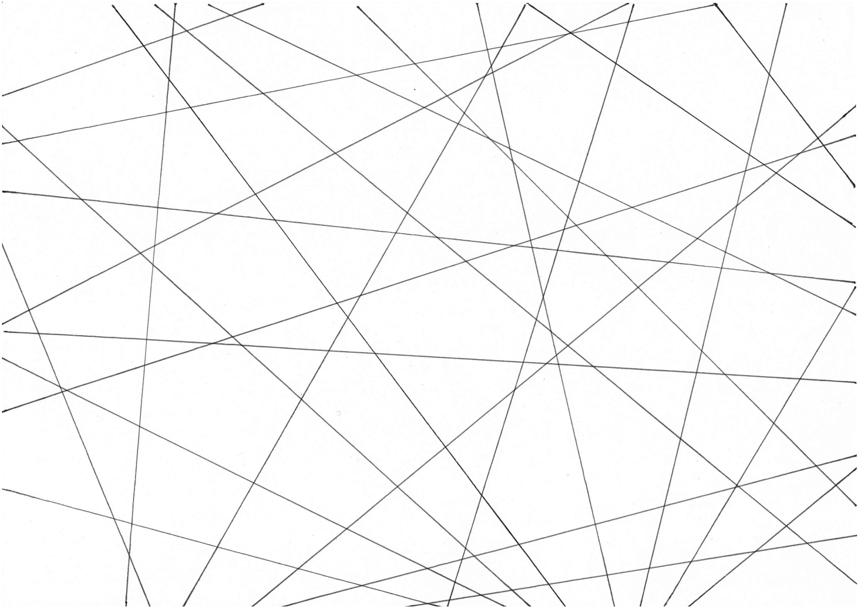
It is called the mosaic induced by X and denoted by M_X.

We are interested in its cells (d-dimensional polytopes) and k-faces.

With probability one, every cell is a simple polytope.

Other restrictions?





M. Reitzner, R. Schneider: On the cells in a stationary Poisson hyperplane mosaic. arXiv:1609.04230

Assumption $(*)$**:** The support of the directional distribution φ is the whole unit sphere $\mathbb{S}^{d-1}$, and φ assigns measure zero to each great subsphere of $\mathbb{S}^{d-1}$.

$\mathcal{K}^d$ denotes the space of convex bodies in $\mathbb{R}^d$ with the Hausdorff metric.

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Theorem 1. *If assumption* $(*)$ *is satisfied, then with probability one the set of all translates of the cells of* M_X *is dense in* $\mathcal{K}^d$*.*

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Theorem 2. *If assumption* $(*)$ *is satisfied, then with probability one, to every simple* d*-polytope* P *there are infinitely many cells of* M_X *that are combinatorially isomorphic to* P*.*

The proof uses some geometric constructions, together with the following extension of the Borel–Cantelli lemma, due to Erdös and Rényi (1959):

Lemma *Let* $E_1, E_2, \ldots$ *be a sequence of events (on some probability space) with* $\sum_{j=1}^{\infty} \mathbb{P}(E_j) = \infty$ *and*

$$\liminf_{n\to\infty} \frac{\sum_{i,j=1,\, i\neq j}^{n} [\mathbb{P}(E_i \cap E_j) - \mathbb{P}(E_i)\mathbb{P}(E_j)]}{(\sum_{j=1}^{n} \mathbb{P}(E_j))^2} = 0.$$

Then $\mathbb{P}(\limsup_{j\to\infty} E_j) = 1$*.*

3. Typical cells and k-faces

$\mathcal{F}_k(X)$ set of k-faces in M_X

One can define a typical k-face of the random mosaic M_X in different natural ways.

Heuristic explanation:

• The typical k-face $Z^{(k)}$ is obtained (up to translation) if we select a k-face of the mosaic at random, with equal chances for each k-face.

• Let w be a translation-invariant, positive, measurable function on k-polytopes (e.g., k-volume).

The w-weighted typical k-face $Z_w^{(k)}$ is obtained (up to translation) if we select a k-face of the mosaic at random, with chances proportional to the value of w.

There are several precise definitions, e.g., using ergodic means, or grain distributions of stationary particle processes, or Palm distributions.

The volume-weighted cell is stochastically equivalent, up to translations, to the zero cell, the (a.s. unique) cell containing the origin.

Let s denote the Steiner point (or any other center function) and $B(o, r)$ the ball with center o and radius r.

The distribution of the weighted typical k-face $Z_w^{(k)}$ is given by

$$\mathbb{P}\{Z_w^{(k)} \in A\} = \lim_{r\to\infty} \frac{\mathbb{E}\sum_{F\in\mathcal{F}_k(X),\, F\subset B(o,r)} \mathbf{1}_A\{F - s(F)\} w(F)}{\mathbb{E}\sum_{F\in\mathcal{F}_k(X),\, F\subset B(o,r)} w(F)}$$

for Borel sets A in the space of polytopes.

4. A very brief survey: shapes of large cells

In the early 1940s, D.G. Kendall conjectured that the shape of the zero cell of the random mosaic generated by a stationary and isotropic Poisson line process in the plane tends to circularity, given that the area of the cell tends to ∞.

A proof was given by I.N. Kovalenko (1997).

A brief survey on later extensions
(higher dimensions, non-isotropic, different interpretations of 'large', typical k-faces)

D. Hug, M. Reitzner, R. Schneider, The limit shape of the zero cell in a stationary Poisson hyperplane tessellation. *Ann. Probab.* **32** (2004), 1140–1167.

The Blaschke body B_X of X is the unique o-symmetric convex body with

$$S_{d-1}(B_X, \cdot) = \varphi.$$

Z_0 denotes the zero cell and Z the typical cell of M_X,

ϑ measures the homothetic deviation of two convex bodies,

V_d is the volume,

Then, for $\varepsilon > 0$,

$$\mathbb{P}\{\vartheta(Z_0, B_X) \geq \varepsilon \mid V_d(Z_0) \geq a\} \leq c \exp[-c_0 \varepsilon^{d+1} \gamma a^{1/d}],$$

with constants c, c_0 independent of a.

Similary for the typical cell.

Later developments:

D. Hug, M. Reitzner, R. Schneider, Large Poisson–Voroni cells and Crofton cells. *Adv. Appl. Prob. (SGSA)* **36** (2004), 667–690.

Zero cell in the isotropic case, size measured by kth intrinsic volume ($k \geq 2$)

D. Hug, R. Schneider, Asymptotic shapes of large cells in random tessellations. *Geom. Funct. Anal.* **17** (2007), 156–191.

Zero cells of not necessarily stationary mosaics, general size functionals (axiomatic), limit shapes as extremal bodies of certain isoperimetric inequalities

D. Hug, R. Schneider, Typical cells in Poisson hyperplane tessellations. *Discrete Comput. Geom.* **38** (2007), 305–319.

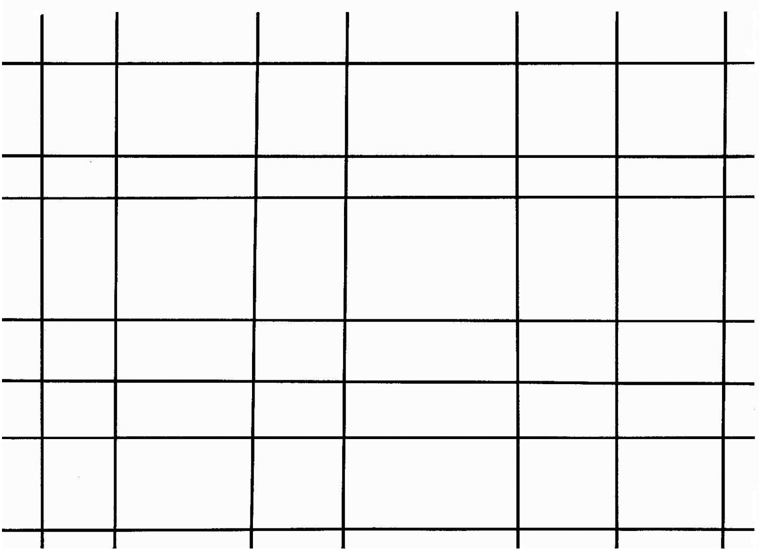
Typical cells in the isotropic case, size measured by kth intrinsic volume ($k \geq 2$) or diameter

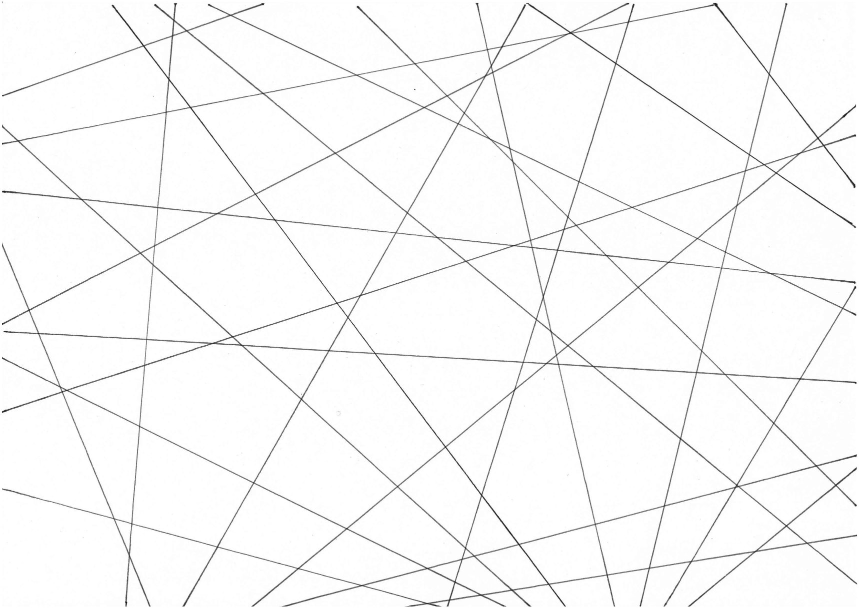
D. Hug, R. Schneider, Large faces in Poisson hyperplane mosaics. *Ann. Probab.* **38** (2010), 1320–1344.

D. Hug, R. Schneider, Faces with given directions in anisotropic Poisson hyperplane mosaics. *Adv. Appl. Prob.* **43** (2011), 308–321.

Shapes of large (weighted) typical k-faces, depending on the direction of the face

5. Variances, in particular of the vertex number





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The expected vertex number of the typical cell is 2^d.

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J. Mecke (1984):

The expected number of j-faces of the typical k-face,

$$\mathbb{E}\, f_j(Z^{(k)}) = 2^{k-j}\binom{k}{j},$$

is essentially independent of the distribution of the underlying stationary hyperplane process (because it is of a combinatorial nature).

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This changes drastically if we ask for the variance.

Theorem 3. *Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$ ($d \geq 2$), and let $Z^{(k)}$ be the typical k-face of its induced mosaic M_X ($k \in \{2, \ldots, d\}$). Then*

$$0 \leq \operatorname{var} f_0(Z^{(k)}) \leq 2^k k! \left(\sum_{j=0}^{k} \frac{\kappa_j^2}{4^j (d-j)!} \right) - 2^{2k}.$$

($\kappa_j =$ volume of the j-dimensional unit ball)
Equality on the left side holds if and only if X is a parallel process.
Equality on the right side holds if X is isotropic with respect to a suitable scalar product on $\mathbb{R}^d$, and for $k = d$ it holds only in this case.

R. Schneider, Second moments related to Poisson hyperplane tessellations. *J. Math. Anal. Appl.* **434** (2016), 1365–1375.

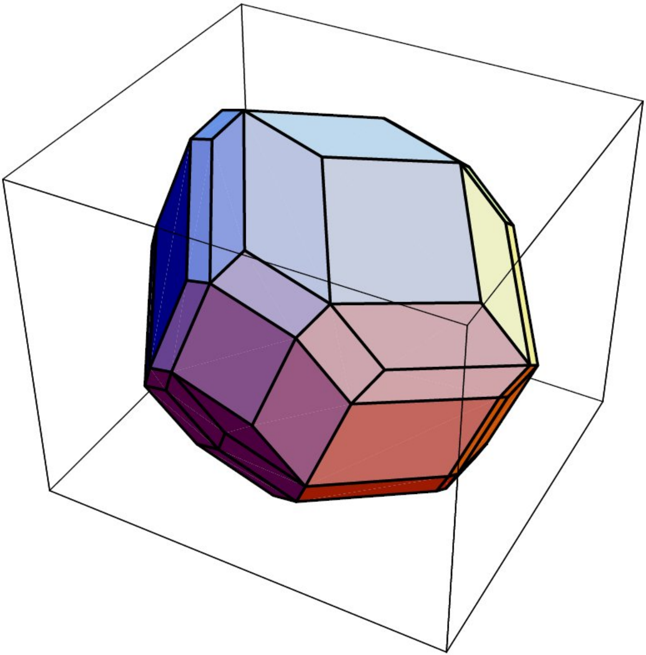
Some ideas of the proof (for $k = d$)

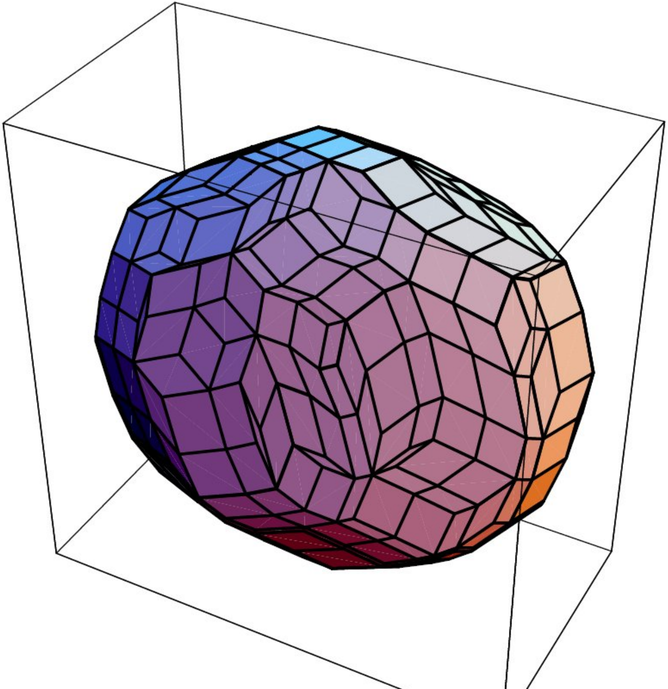
A preliminary remark:

The associated zonoid of X is the convex body Π_X with support function

$$h(\Pi_X, u) = \frac{\gamma}{2} \int_{\mathbb{S}^{d-1}} |\langle u, v \rangle| \, \varphi(\mathrm{d}v), \quad u \in \mathbb{R}^d.$$

This strange, but very useful construction was invented by Georges Matheron (1930–2000).





The expected vertex number of the zero cell Z_0 can be expressed in terms of the associated zonoid.
(J.A. Wieacker (1986))

$$\mathbb{E}\, f_0(Z_0) = \frac{d!}{2^d} V_d(\Pi_X) V_d(\Pi_X^\circ).$$

Here Π_X° denotes the polar body of Π_X.

A side remark: Inequalities from convex geometry, due to Blaschke–Santaló and Reisner, yield:

$$2^d \leq \mathbb{E}\, f_0(Z_0) \leq \frac{d!}{2^d} \kappa_d^2,$$

with known equality cases.

Let g be a translation invariant, nonnegative, measurable real function on d-polytopes. Then

$$\mathbb{E}(gf_0)(Z) = c\int_{(\mathbb{S}^{d-1})^d} \mathbb{E} \sum_{0\in C\in\mathcal{F}_d(X\cup\{u_1^\perp,\dots,u_d^\perp\})} g(C) \times[u_1,\dots,u_d]\,\varphi^d(\mathrm{d}(u_1,\dots,u_d)).$$

with

$$c = \frac{\gamma^d}{d!\gamma^{(d)}}.$$

Here $\mathcal{F}_d(\cdot)$ is the set of d-cells in the induced mosaic, and $[u_1,\dots,u_d]$ is the volume of the parallelepiped spanned by the vectors $u_1,\dots,u_d$.

The proof uses:

Campbell's theorem,

the Slivnyak–Mecke formula,

the stationarity of X,

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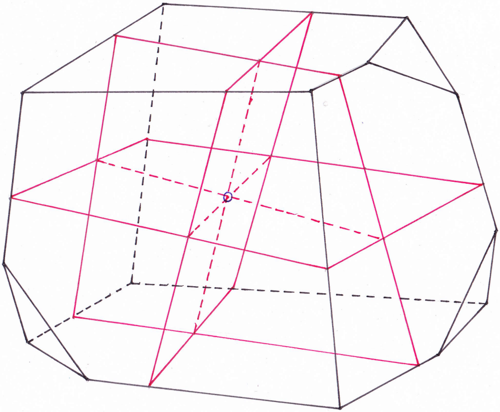
We need this with $g = f_0$. This requires a vertex count:

$$\sum_{0 \in C \in \mathcal{F}_d(X \cup \{u_1^\perp, \ldots, u_d^\perp\})} f_0(C)$$



Let Z_0 be the zero cell of our mosaic, and define

$$\mathcal{C}_j = \{Z_0 \cap u_{i_1}^\perp \cap \cdots \cap u_{i_{d-j}}^\perp : 1 \le i_1 < \cdots < i_{d-j} \le d\}, \quad j = 0, \ldots, d.$$

It is clear form the picture that

$$\sum_{0 \in C \in \mathcal{F}_d(X \cup \{u_1^\perp, \ldots, u_d^\perp\})} f_0(C) = \sum_{j=0}^{d} 2^{d-j} \sum_{P \in \mathcal{C}_j} f_0(P).$$

Inserting this, and with some symmetry considerations, we obtain

$$\mathbb{E}\, f_0^2(Z) = c(d,\gamma) \sum_{j=0}^{d} 2^{d-j} \binom{d}{j} \int_{(\mathbb{S}^{d-1})^d} \mathbb{E}\, f_0(Z_0 \cap u_1^\perp \cap \cdots \cap u_{d-j}^\perp)$$

$$[u_1, \ldots, u_d]\, \varphi^d(\mathrm{d}(u_1, \ldots, u_d)).$$

In the integrand, there appear terms

$$\mathbb{E}\, f_0(Z_0 \cap L),$$

where L is a j-dimensional subspace.

Now we recall the preliminary remark, which says that

$$\mathbb{E}\, f_0(Z_0) = \frac{d!}{2^d}\mathrm{vp}(\Pi_X),$$

where the volume product is defined by

$$\mathrm{vp}(K) = V_d(K)V_d(K^\circ)$$

for a 0-symmetric convex body with interior points, and Π_X is the associated zonoid of X.

We apply this to X_L, the $(j-1)$-plane process in L obtained by intersecting the hyperplanes of X with L.

Fortunately,

$$\Pi_{X_L} = \Pi_X | L,$$

where $|$ denotes orthogonal projection.

Result:

$$\mathbb{E}\, f_0^2(Z) = \sum_{j=0}^{d} 2^{d-2j} \frac{d!}{(d-j)!} \frac{\gamma^d}{\gamma^{(d)} d!} \int_{(\mathbb{S}^{d-1})^d} \mathrm{vp}(\Pi_X | u_1^\perp \cap \cdots \cap u_{d-j}^\perp)$$

$$[u_1, \ldots, u_d]\, \varphi^d(\mathrm{d}(u_1, \ldots, u_d)).$$

For the integrands, we have the sharp estimates

$$\frac{4^j}{j!} \leq \mathrm{vp}(\Pi_X | u_1^\perp \cap \cdots \cap u_{d-j}^\perp) \leq \kappa_j^2.$$

Fortunately,

$$\frac{\gamma^d}{\gamma^{(d)} d!} \int_{(\mathbb{S}^{d-1})^d} [u_1, \ldots, u_d] \, \varphi^d(\mathrm{d}(u_1, \ldots, u_d)) = 1.$$

This gives

$$2^{2d} \leq \mathbb{E}\, f_0^2(Z) \leq \sum_{j=0}^{d} \frac{d!}{(d-j)!} 2^{d-2j} \kappa_j^2,$$

with known equality cases.

What else is possible?

The vertex number of the typical cell was only a very special case.

1. The typical cell Z can be extended to the typical k-face, denoted by $Z^{(k)}$ ($k = 1, \ldots, d$).

2. The vertex number f_0 can be generalized to the functional L_r, the total r-dimensional volume of the r-faces ($r = 0, \ldots, d$) of a polytope.
Thus, L_0 is the vertex number, L_1 is the total edge length, L_{d-1} is the surface area, and L_d is the volume.

All second moments $\mathbb{E}(L_r L_s)(Z^{(k)})$ can be expressed by sums of integrals involving the associated zonoid.

For an isotropic Poisson hyperplane process, the result is completely explicit, namely

$$\mathbb{E}(L_r L_s)(Z^{(k)})$$

$$= \frac{2^k \pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}[r+1]\right)\Gamma\left(\frac{1}{2}[s+1]\right)} \left\{ \frac{\Gamma\left(\frac{1}{2}[d+1]\right)}{\Gamma\left(\frac{1}{2}d\right)\gamma} \right\}^{r+s}$$

$$\times \sum_{j=\max\{r,s\}}^{k} \binom{k}{j} \left(\frac{\pi}{2}\right)^j \frac{\Gamma\left(\frac{1}{2}[j+1]\right)}{\Gamma\left(\frac{1}{2}j+1\right)} \frac{j!}{(j-r)!} \frac{j!}{(j-s)!}$$

For $k = d$, this is a formula of Miles (1961).

6. Random cones

Recently, there has been increased interest in random polyhedral cones.

For example:

"When does a randomly oriented cone strike a fixed cone?"

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“When does a randomly oriented cone strike a fixed cone?”

Quoted from:

M.B. McCoy, J.A. Tropp, Sharp recovery bounds for convex demixing, with applications. *Found. Comput. Math.* **14** (2014), 503–567.

Various aspects of this question have recently been considered, motivated by applications in convex programming.

Papers by Amelunxen, Bürgisser, Lotz, McCoy, Tropp, Goldstein, Nourdin, Peccati

What does the question mean?

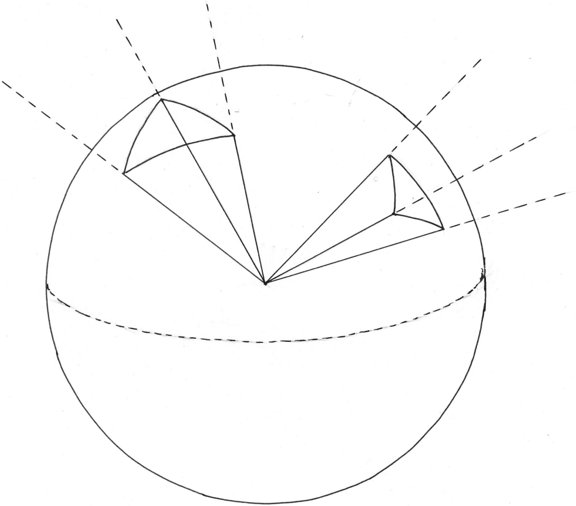
Let $C, D \subset \mathbb{R}^d$ be closed convex polyhedral cones, not both subspaces.

Let θ be a uniform random rotation: a random element of SO_d, with distribution given by the normalized Haar measure ν.

The question asks for the probability

$$\mathbb{P}\{C \cap \theta D \neq \{o\}\}.$$

Instead of cones, we could consider their intersections with the unit sphere.



Note:

As early as 1896,

Henri Poincaré, in his *“Calcul des probabilités”* (p. 118)

considered a fixed and a moving curve on the 2-sphere and asked for the expected number of their intersection points.

He found that it is proportional to the product of the lengths of the two curves.

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What do we need to answer the question about general polyhedral cones?

We need:

(a) The conic intrinsic volumes,

(b) The spherical kinematic formula,

(c) The spherical Gauss–Bonnet formula.

Brief explanations:

(a) The conic intrinsic volumes

Let $C \in \mathcal{P}C^d$ (the polyhedral convex cones in $\mathbb{R}^d$), let $\mathcal{F}_k(C)$ be the set of its k-faces.

Let Π_C be the nearest-point map (metric projection) of C.

A short approach to the conic intrinsic volumes: Define

$$\mathrm{skel}_k C := \bigcup_{F \in \mathcal{F}_k(C)} \mathrm{relint}\, F.$$

Let $\mathbf{g}$ be a standard Gaussian random vector in $\mathbb{R}^d$ (i.e., with probability distribution $f(x) = (2\pi)^{-d/2} e^{-\|x\|^2/2}$), and define

$$V_k(C) := \mathbb{P}\{\Pi_C(\mathbf{g}) \in \mathrm{skel}_k C\}$$

for $k = 0, \ldots, d$.

More explicitly: For $F \in \mathcal{F}_k(C)$,

$\beta(o, F)$ the internal angle of F at o,

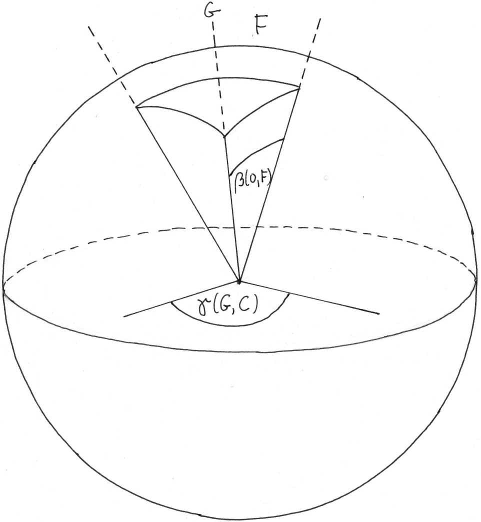
$\gamma(F, C)$ the external angle of C at F.

Then

$$V_k(C) = \sum_{F \in \mathcal{F}_k(C)} \beta(o, F)\gamma(F, C).$$

P. McMullen, Non-linear angle-sum relations for polyhedral cones and polytopes. *Math. Proc. Camb. Phil. Soc.* **78** (1975), 247–261.

Note that, restricted to the sphere, this yields the exact analogues (up to normalizing factors) of the Euclidean intrinsic volumes.


G
F
β(0,F)
γ(G,C)

(b) The spherical (conic) kinematic formula

It says that

$$\mathbb{E} V_k(C \cap \theta D) = \sum_{i=k}^{d} V_i(C) V_{d+k-i}(D)$$

for $k = 1, \ldots, d$. Here, of course,

$$\mathbb{E} V_k(C \cap \theta D) = \int_{\mathrm{SO}_d} V_k(C \cap \vartheta D)\, \nu(\mathrm{d}\vartheta).$$

Differential-geometric (spherical) versions go back to Santaló.

Convex-geometric versions to

S. Glasauer, Integralgeometrie konvexer Körper im sphärischen Raum. Dissertation, Universität Freiburg, 1995.

However, we didn't ask for

$$\mathbb{E} V_k(C \cap \theta D),$$

but for

$$\mathbb{P}\{C \cap \theta D \neq \{o\}\} = \mathbb{E}\mathbf{1}\{C \cap \theta D \neq \{o\}\}.$$

In Euclidean space, it helps that V_0 is constant on convex bodies, hence proportional to the Euler characteristic.

This is a version of the Gauss–Bonnet theorem
(total curvature = Euler characteristic)

(c) The spherical Gauss–Bonnet formula

Formulated for polyhedral cones, it says that

$$2\sum_{k\geq 0} V_{2k+1}(C) = 1, \quad \text{if } C \text{ is not a subspace.}$$

Since $C \cap \theta D$ is, with probability one, either $\{o\}$ (in which case $V_k(C \cap \theta D) = 0$ for $k \geq 1$) or not a subspace, we get

$$\mathbf{1}\{C \cap \theta D \neq \{o\}\} = 2\sum_{k\geq 0} V_{2k+1}(C \cap \theta D) \quad \text{almost surely.}$$

Hence

$$\mathbb{P}\{C \cap \theta D \neq \{o\}\} = 2\sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{i=2k+1}^{d} V_i(C) V_{d+2k+1-i}(D).$$

The conic kinematic formula

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expresses the probability of non-trivial intersection of a fixed cone with a random cone.

The randomness of the cone comes only from the random rotation, which is applied to a fixed cone.

Question: Are there models of random cones, where also the shape is random, that allow for an explicit determination of the intersection probability?

Random Schläfli cones

n hyperplanes through o in general position in $\mathbb{R}^d$ divide the space into

$$C(n,d) := 2\sum_{r=0}^{d-1}\binom{n-1}{r}$$

d-dimensional cones (Schläfli).

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Theorem 4. *If C is a fixed polyhedral cone, then*

$$\mathbb{P}\{C \cap S_n \neq \{o\}\} = \frac{2}{C(n,d)}\sum_{j=1}^{n}\sum_{k=0}^{\lfloor\frac{j-1}{2}\rfloor}\binom{n}{j-2k-1}V_j(C).$$

For the proof, we need the expected conic intrinsic volumes

$$\mathbb{E}V_j(S_n).$$

These and other expectations and moments were determined in

D. Hug, R. Schneider, Random conical tessellations. *Discrete Comput. Geom.* **56** (2016), 395–426.

Expected numbers of k-faces were determined earlier by

T.M. Cover, B. Efron, Geometrical probability and random points on a hypersphere. *Ann. Math. Stat.* **38** (1967), 213–220.

For a polyhedral cone C we define

$$\Lambda_k(C)$$

as the total $(k-1)$-dimensional spherical volume of the $(k-1)$-faces of the spherical polytope $C \cap \mathbb{S}^{d-1}$.

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For general (not necessarily isotropic) random Schläfli cones C, we determined

$$\begin{aligned}
\mathbb{E} f_k(S_n) &= \frac{2^{d-k}\binom{n}{d-k} C(n-d+k,k)}{C(n,d)}, \\
\mathbb{E} V_k(S_n) &= \frac{\binom{n}{d-k}}{C(n,d)}, \\
\mathbb{E} \Lambda_k(S_n) &= \frac{2^{d-k}\binom{n}{d-k}}{C(n,d)}.
\end{aligned}$$

For isotropic random Schläfli cones, we determined all mixed second moments

$$\mathbb{E}(\Lambda_r \Lambda_s)(S_n)$$
$$= \frac{1}{C(n,d)} \sum_{p\in\mathbb{N}} 2^{d-p} \binom{n}{d-p} \binom{n-d+p}{p-r, p-s, n-d-p+r+s}$$
$$\times \theta(n-d-p+r+s, p),$$

where

$$\theta(n,d) := \frac{(d-1)\kappa_{d-1}}{d\kappa_d} \int_0^\pi \left(1 - \frac{x}{\pi}\right)^n \sin^{d-2} x \, \mathrm{d}x.$$

This is the spherical counterpart to the mentioned result of Roger Miles.

Thank you for your attention!