Cancellations in Random Nodal Sets

Giovanni Peccati (Luxembourg University)

Joint works with:
— D. Marinucci, M. Rossi and I. Wigman (GAFA, 2016)
— F. Dalmao, I. Nourdin and M. Rossi (ArXiv, 2016)
— M. Rossi (ArXiv, 2017)
— I. Nourdin and M. Rossi (in preparation, 2017)

Marseille — May 18, 2017

First Model (Berry, 1977)

- Fix $E > 0$. The **Berry random wave model** on $\mathbb{R}^2$ with parameter E, written

$$B_E = \{B_E(x) : x \in \mathbb{R}^2\},$$

is defined as the unique (in law) centred, isotropic Gaussian field on $\mathbb{R}^2$ such that

$$\Delta B_E + 4\pi^2 E \cdot B_E = 0, \text{ where } \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

- Equivalently, $\mathbb{E}[B_E(x)B_E(y)] = J_0(2\pi\sqrt{E}\|x-y\|)$ (J_0 = **Bessel function of the 1st kind**) or

$$B_E(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} e^{2i\pi\sqrt{E}\langle z,x\rangle}\, G(dz),$$

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SECOND MODEL (RUDNICK AND WIGMAN, 2007)

- Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2 \simeq [0,1)^2$ be the 2-dimensional flat torus.
- We are again interested in real (random) **eigenfunctions** of Δ, that is, solutions of the **Helmholtz equation**

$$\Delta f + E f = 0,$$

for some adequate $E > 0$ (**eigenvalue**).
- A L^2-complete orthonormal set of eigenfunctions of Δ is obtained as:

$$(x_1, x_2) \quad \mapsto \quad \exp\left\{2i\pi(\lambda_1 x_1 + \lambda_2 x_2)\right\},$$

with $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$. Each one is associated with the eigenvalue $4\pi^2(\lambda_1^2 + \lambda_2^2)$.

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- ⋆ The eigenvalues of Δ are therefore given by the set

$$\{E_n := 4\pi^2 n : n \in S\},$$

where

$$S = \{n : n = a^2 + b^2;\, a, b \in \mathbb{Z}\}.$$

- ⋆ For $n \in S$, the dimension of the corresponding eigenspace is $\mathcal{N}_n = r_2(n) := \#\Lambda_n$, where $\Lambda_n := \{(\lambda_1, \lambda_2) : \lambda_1^2 + \lambda_2^2 = n\}$.
- ⋆ We define the **arithmetic random wave** of order n as:

$$f_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e^{2i\pi\langle\lambda, x\rangle},\ x \in \mathbb{T},$$

where the a_λ are i.i.d. complex standard Gaussian, except for the relation $a_\lambda = \overline{a_{-\lambda}}$.
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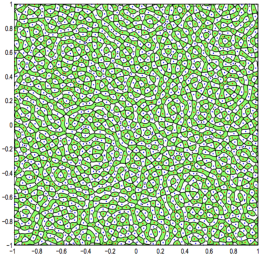
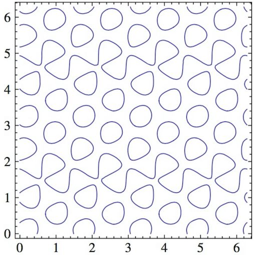
Nodal Sets

We are interested in the high-energy (respectively as $E \to \infty$ and $\mathcal{N}_n \to \infty$) geometry of the **nodal sets** (components are the **nodal lines**):

$$B_E^{-1}(\{0\}) \cap \mathcal{D} := \{x \in \mathcal{D} : B_E(x) = 0\},$$
$$f_n^{-1}(\{0\}) := \{x \in \mathbb{T} : f_n(x) = 0\},$$

where $\mathcal{D}$ is a compact set with piecewise smooth boundary.





a *From: Belyaev (2016) and Bourgain and Rudnick (2013)*

Other Models

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- ⋆ Here, the eigenvalues are $n(n+1)$, $n \in \mathbb{N}$, and the random eigenfunctions are called **random spherical harmonics**.

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Nodal Lengths and Spectral Measures

- Our aim is to characterise the fluctuations of the random **nodal lengths**

$$L_n := \text{length}\, f_n^{-1}(\{0\}), \quad \text{as } \mathcal{N}_n \to \infty$$

$$L_E := \text{length}\, B_E^{-1}(\{0\}) \cap \mathcal{D}, \quad \text{as } E \to \infty.$$

- For L_n, crucial role played by the set of probability measures on $\mathbb{S}^1$

$$\mu_n(dz) := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}}(dz), \quad n \in S$$

(invariant with respect to $z \mapsto \overline{z}$ and $z \mapsto i \cdot z$.)

- Note that μ_n is the **spectral measure** of f_n:

$$\begin{aligned} \mathbb{E}[f_n(x) f_n(y)] &= \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} e^{2i\pi\langle \lambda, x-y \rangle} \\ &= \int_{\mathbb{S}^1} e^{2i\pi\langle a, (x-y)\sqrt{n} \rangle} \mu_n(da) := r_n(x-y). \end{aligned}$$

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From $\{\mu_n\}$ to Planar Waves

- ⋆ The set $\{\mu_n : n \in S\}$ is relatively compact and its adherent points are an **infinite strict subset** of the class of invariant probabilities on the circle (see Kurlberg and Wigman (2015)).
- ⋆ Quick demonstration (see Krishnapur, Kurlberg and Wigman (2013)): *the adherent points of the set*

$$\widehat{\mu}_n(4)^2 := \left(\int_{\mathbb{S}^1} z^{-4} \, \mu_n(dz) \right)^2, \quad n \in S,$$

are given by the whole interval $[0,1]$.
- ⋆ Remark: if $\mu_{n_j} \Rightarrow \mu$, then f_{n_j} admits a (non-universal) **local scaling limit**: for $(x,y) \in \mathbb{R}^2$

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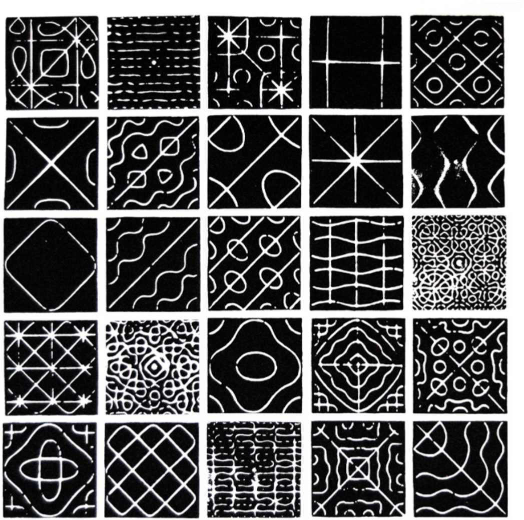
Chladni Plates (1787)



Some Motivations

- ⋆ Geometric study of **excursion sets** of isotropic random fields.
- ⋆ When applied to other manifolds (like e.g. the sphere) high-energy limit theorems can be regarded as **high-resolution limit theorems**. Typical applications in Cosmology (CMB: see Marinucci and Peccati, 2011).
- ⋆ An amplification of **Berry's universality conjecture** (1977) states that the **high-energy** behaviour of Laplace eigenfunctions on a Riemaniann surface coïncides with the average behaviour of the Random Wave Model on a comparable planar domain (see Zelditch, 2009). Used to heuristically test open problems on the geometry of deterministic nodal sets, like e.g. **Yau's conjecture**.

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MEAN AND VARIANCE – PLANAR WAVES

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$$\mathbb{E}[L_E] = \frac{2\pi\sqrt{E}}{2\sqrt{2}}, \quad \mathbf{Var}(L_E) \sim \frac{\text{area}\,\mathcal{D}}{512\pi}\log E,$$

although the natural guess for the order of the variance is $\sim \sqrt{E}$. Such a variance reduction "*... results from a cancellation whose meaning is still obscure...* " (Berry (2002), p. 3032).

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Mean and Variance – Arithmetic Waves

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- ⋆ *Krishnapur, Kurlberg and Wigman* (Ann. Math., 2013): if $\{n_j\} \subset S$ is such that $\mathcal{N}_{n_j} \to \infty$, then

$$\mathbf{Var}(L_{n_j}) = \frac{E_{n_j}}{\mathcal{N}_{n_j}^2} \times c(n_j) + O(E_{n_j} R_5(n_j)),$$

where

$$c(n_j) = \frac{1 + \widehat{\mu}_{n_j}(4)^2}{512};\ R_5(n_j) = \int_{\mathbb{T}} |r_{n_j}(x)|^5 dx = o\left(1/\mathcal{N}_{n_j}^2\right).$$

- ⋆ Two phenomena: (i) **cancellation**, and (ii) **non-universality**.

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- Two phenomena: (i) **cancellation**, and (ii) **non-universality**.

Mean and Variance – Arithmetic Waves

- *Rudnick and Wigman* (Ann. I.H.P., 2008): For every $n \in S$, $\mathbb{E}[L_n] = \frac{\sqrt{E_n}}{2\sqrt{2}}$. Moreover, $\mathbf{Var}(L_n) = O\left(E_n/\mathcal{N}_n^{1/2}\right)$. Conjecture: $\mathbf{Var}(L_n) = O(E_n/\mathcal{N}_n)$.
- *Krishnapur, Kurlberg and Wigman* (Ann. Math., 2013): if $\{n_j\} \subset S$ is such that $\mathcal{N}_{n_j} \to \infty$, then

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Next Step: Second Order Results

- For $E > 0$ and $n \in S$, define the normalized quantities

$$\widetilde{L}_E := \frac{L_E - \mathbb{E}(L_E)}{\mathbf{Var}(L_E)^{1/2}}, \text{ and } \widetilde{L}_n := \frac{L_n - \mathbb{E}(L_n)}{\mathbf{Var}(L_n)^{1/2}}.$$

- **Task**: Assume that $E, \mathcal{N}_{n_j} \to \infty$; characterise the law of those r.v.'s Y, Z such that

$$\widetilde{L}_E \stackrel{\mathbf{LAW}}{\longrightarrow} Y,$$

and

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STRATEGY

- ⋆ **Step 1**. Let $V = f_n$ or B_E, and $L = L_E$ or L_n. Use the representation (based on the coarea formula)

$$L = \int \delta_0(V(x))\|\nabla V(x)\|\, dx, \quad \text{in } L^2(\mathbb{P}),$$

  to deduce the **Wiener chaos expansion** of L.
- ⋆ **Step 2**. Show that exactly **one** chaotic projection $L(4) := \mathrm{proj}(L \,|\, C_4)$ dominates in the high-energy limit – thus accounting for the cancellation phenomenon.
- ⋆ **Step 3**. Study by "bare hands" the limit behaviour of $L(4)$.
- ⋆ Examples of previous use of Wiener chaos: Sodin and Tsirelson (2002) (Gaussian analytic functions), Azaïs and Leon's proof (2011) of the Granville-Wigman CLT for zeros of trigonometric polynomials.

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Vignette: Wiener Chaos

- ⋆ Consider a generic Gaussian field $\mathbb{G} = \{G(u) : u \in \mathscr{U}\}$.
- ⋆ For every $q = 0, 1, 2...$, set

$$P_q := \overline{\textbf{v.s.}}\Big\{p\big(G(u_1), ..., G(u_r)\big) : d^\circ p \le q\Big\}.$$

Then: $P_q \subset P_{q+1}$.

- ⋆ Define the family of orthogonal spaces $\{C_q : q \ge 0\}$ as $C_0 = \mathbb{R}$ and $C_q := P_q \cap P_{q-1}^{\perp}$; one has

$$L^2(\sigma(\mathbb{G})) = \bigoplus_{q=0}^{\infty} C_q.$$

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A Rigid Asymptotic Structure

For fixed $q \geq 2$, let $\{F_k : k \geq 1\} \subset C_q$ (with unit variance).

- $\star$ *Nourdin and Poly (2013)*: If $F_k \Rightarrow Z$, then Z has necessarily a density (and the set of possible laws for Z does not depend on $\mathbb{G}$).
- $\star$ *Nualart and Peccati (2005)*: $F_k \Rightarrow Z \sim \mathscr{N}(0,1)$ if and only if $\mathbb{E}F_k^4 \to 3 (= \mathbb{E}Z^4)$.
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Main Results – I

Theorem (Nourdin, P., & Rossi, 2017)

1. **(Cancellation)** *For every fixed $E > 0$,*

$$\mathrm{proj}(L_E \mid C_{2q+1}) = 0, \quad q \geq 0,$$

and $\mathrm{proj}(\widetilde{L}_E \mid C_2)$ *reduces to a "negligible boundary term", as* $E \to \infty$.

2. **(4^{th} chaos dominates)** *Let $E \to \infty$. Then,*

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IDEA OF THE PROOF

- $\star$ Write $L_n(u) = \text{length}\, f_n^{-1}(u)$. One has that

$$\begin{aligned}\mathrm{proj}(L_n(u)\,|\,C_2) &= ce^{-u^2/2}u^2\int_{\mathbb{T}}(f_n(x)^2-1)dx\\ &= c\frac{e^{-u^2/2}u^2}{\mathcal{N}_n}\sum_{\lambda\in\Lambda^n}(|a_\lambda|^2-1)\end{aligned}$$

  (this is the dominating term for $u \neq 0$; it verifies a CLT).
- $\star$ Prove that $\mathrm{proj}(L_n\,|\,C_4)$ has the form

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- ⋆ The cancellation of the second chaos and the dominance of the fourth seems to be a **general phenomenon**, valid for more general manifolds and more general geometric functionals (nodal intersections, critical points, Euler Poincaré characteristics).
- ⋆ **Quantitative versions** are available: e.g. (Peccati and Rossi, 2017)

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- ⋆ **Quantitative versions** are available: e.g. (Peccati and Rossi, 2017)

$$\mathbf{Wass}_1(\widetilde{L}_n, M(\widehat{\mu}_n(4))) = \inf_{X\sim L, Y\sim M} \mathbb{E}|X-Y| = O\left(\frac{1}{\mathcal{N}_n^{1/4}}\right).$$

- ⋆ **Phase singularities** in complex random waves (Dalmao, Nourdin, Peccati and Rossi, 2016).

THANK YOU FOR YOUR ATTENTION!