Asymptotics for random marked closed sets

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Random marked closed set

$\Phi_{\text{usc}} = \{(X, f) : X \subseteq \mathbb{R}^d \text{ is closed}, f : X \to \bar{\mathbb{R}} \text{ is u.s.c.}\}$

$\tau : (X, f) \mapsto \{(x, t) \in X \times \bar{\mathbb{R}} : t \leq f(x)\}, \quad (X, f) \in \Phi_{\text{usc}}$

$\tau(X, f)$ is closed subset of $\mathbb{R}^d \times \bar{\mathbb{R}}$ (hypograph)

$(\Omega, \mathcal{A}, \mathbb{P})$ ... complete probability space

$(\Xi, \Gamma) : \Omega \to \Phi_{\text{usc}}$ is a random marked closed set (RMCS) if

$$\{\omega \in \Omega : \tau((\Xi, \Gamma)(\omega)) \cap K \neq \emptyset\} \in \mathcal{A}$$

for every compact set K in $\mathbb{R}^d \times \bar{\mathbb{R}}$

Ballani, Kabluchko, Schlather (2012)

Special examples

Marked point process

Ξ ... union of points of the point process $\{\xi_i\}$ in $\mathbb{R}^d$

$\Gamma(\xi_i)$ ... real-valued mark corresponding to the point ξ_i

Random field

Ξ ... deterministic closed subset of $\mathbb{R}^d$

$\Gamma(x)$ ... real-valued random variable associated with x

$x \mapsto \Gamma(x)$ upper semicontinuous

Labelled random closed set

Ξ ... random closed set split into several closed subsets Ξ_i

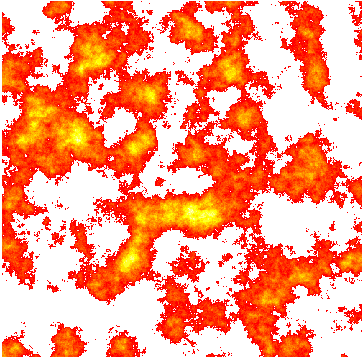
$\Gamma(\Xi_i)$ ... nominal value for labelling of Ξ_i

Molchanov (1984), Ayala and Simó (1995)

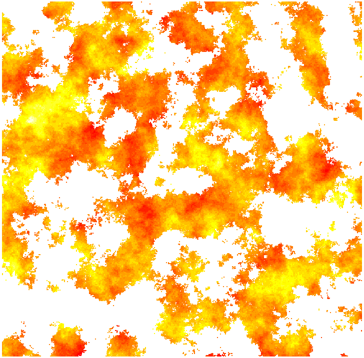
Marked sets generated by excursions of random fields

Nott and Wilson (2000)

Excursion set

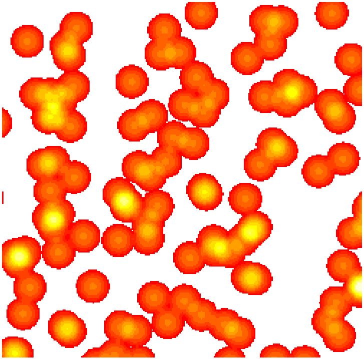


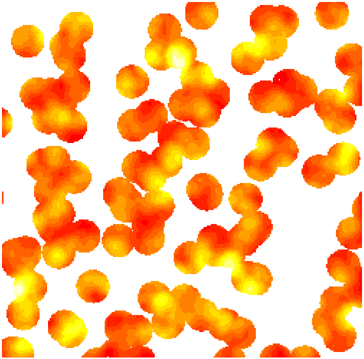
$\Gamma(x) = Z(x)$



Γ independent of Ξ

Marked Boolean model of balls





$$\Gamma(x) = c_\Gamma \sum_{i \geq i} k\left(\frac{\|x - \xi_i\|}{R_i}\right)$$

Γ independent of Ξ

Random field model

Ξ ... random closed set in $\mathbb{R}^d$

Γ ... random u.s.c. function on $\mathbb{R}^d$, independent of Ξ

then (Ξ, Γ) is called a random field model

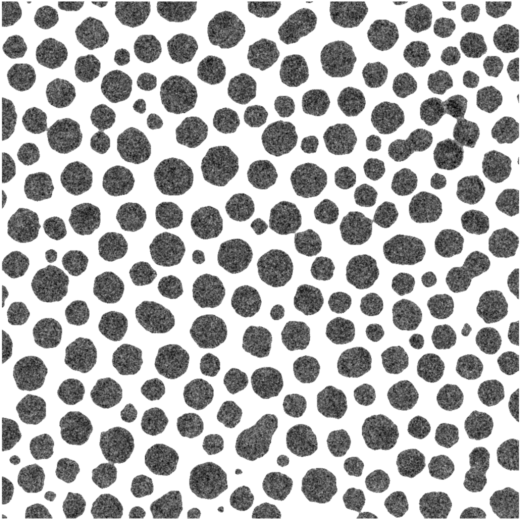
non-parametric test of independence

Koubek, Pawlas, Brereton, Kriesche and Schmidt (2016)

based on second-order summary characteristics

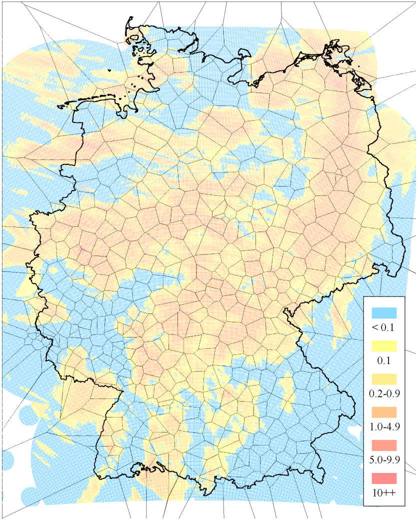
Material data

Molecular Materials and Nanosystems, Eindhoven University of Technology



Radar data

Deutscher Wetterdienst (DWD)



Weighted random measure

Ψ …random measure in $\mathbb{R}^d$

$C(A \times \mathcal{U}) = \mathbb{E}\Psi(A)\mathbf{1}\{\Psi \in \mathcal{U}\}$ …Campbell measure of Ψ

$w : \text{supp}\, C \to W$ …weight function

Then the tuple (Ψ, w) is called a weighted random measure in $\mathbb{R}^d$ with weight space W.

Special case: Ψ point process, (Ψ, w) marked point process

(Ψ, w) induces a random measure $\tilde{\Psi}$ on $\mathbb{R}^d \times W$:

$$\tilde{\Psi}(B \times D) = \Psi(\{x \in B : w(x, \Psi) \in D\}), \quad B \in \mathcal{B}^d,\ D \in \mathcal{B}(W).$$

Stoyan and Ohser (1984)

Random measure generated by random closed set

Ψ_d ...random volume measure generated by Ξ:

$$\Psi_d(B) = |\Xi \cap B|, \quad B \in \mathcal{B}^d,$$

or $\Psi_k(B) = \mathcal{H}^k(\Xi \cap B)$ if Ξ is a random $\mathcal{H}^k$-set

$$w(x, \Psi_k) = \Gamma(x),\ x \in \Xi \quad (W = \mathbb{R})$$

(Ψ_k, w) is weighted random measure

$$\tilde{\Psi}_k(B \times D) = \mathcal{H}^k(\{x \in B \cap \Xi : \Gamma(x) \in D\})$$

Stationary RMCS

A RMCS (Ξ, Γ) is called stationary if $\tau(\Xi, \Gamma) + (x, 0)$ and $\tau(\Xi, \Gamma)$ have the same distribution for all $x \in \mathbb{R}^d$.

(Ξ, Γ) stationary $\Rightarrow$ Ξ stationary $\Rightarrow$ Ψ_k stationary $\Rightarrow$

$$\mathbb{E}\Psi_k(B) = \lambda_k |B|$$

$\lambda_k = \mathbb{E}\mathcal{H}^k(\Xi \cap [0,1]^d)$ is called intensity – assumed to be positive

for $k = d$:

$$\lambda_d = \mathbb{E}|\Xi \cap [0,1]^d| = \mathbb{P}(o \in \Xi)$$

intensity of Ψ_d = volume fraction of Ξ

$$\mathbb{E}\widetilde{\Psi}_k(B \times D) = \lambda_k \, |B| \, \mathbb{Q}(D), \quad B \in \mathcal{B}^d, \ D \in \mathcal{B}$$

$\mathbb{Q}$ is called the mark distribution

Estimation of mark distribution

a single realization of (Ξ, Γ) observed within a bounded convex window $W \subseteq \mathbb{R}^d$

$$\widehat{\lambda}_k = \frac{\mathcal{H}^k(\Xi \cap W)}{|W|} = \frac{\Psi_k(W)}{|W|}$$

$$\widehat{\mathbb{Q}(D)} = \frac{\widetilde{\Psi}_k(W \times D)}{\widehat{\lambda}_k |W|} = \frac{\widetilde{\Psi}_k(W \times D)}{\widetilde{\Psi}_k(W \times \mathbb{R})}$$

increasing domain asymptotics

If $\widetilde{\Psi}_k$ is ergodic, then $\widehat{\mathbb{Q}(D)}$ is strongly consistent estimator of $\mathbb{Q}(D)$.

Palm distribution

Ψ stationary random measure in $\mathbb{R}^d$ with intensity $\lambda > 0$

Palm distribution of Ψ

$$P_o(\mathcal{U}) = \frac{1}{\lambda|B|}\mathbb{E}\int_B \mathbf{1}\{\Psi - x \in \mathcal{U}\}\,\Psi(\mathrm{d}x)$$

if $\Psi = \Psi_d$ is a random volume measure generated by Ξ

$$\begin{aligned}
P_o(\mathcal{U}) &= \frac{1}{\lambda_d|B|}\mathbb{E}\int_B \mathbf{1}\{\Psi_d - x \in \mathcal{U}, x \in \Xi\}\,\mathrm{d}x \\
&= \frac{1}{\lambda_d|B|}\int_B \mathbb{P}(\Psi_d - x \in \mathcal{U}, x \in \Xi)\,\mathrm{d}x \\
&= \frac{1}{\lambda_d}\mathbb{P}(\Psi_d \in \mathcal{U}, o \in \Xi) = \mathbb{P}(\Psi_d \in \mathcal{U} \mid o \in \Xi)
\end{aligned}$$

Reduced second-order moment measure

Ψ stationary random measure in $\mathbb{R}^d$ with intensity $\lambda > 0$ and Palm distribution P_o

reduced second-order moment measure of Ψ

$$\mathcal{K}(B) = \frac{1}{\lambda} \int \mu(B \setminus \{o\})\, P_o(\mathrm{d}\mu), \quad B \in \mathcal{B}^d$$

K-function of Ψ

$$K(r) = \mathcal{K}(b(o, r)), \quad r > 0$$

$$\mathcal{K}(B) = \frac{1}{\lambda^2 |A|} \mathbb{E} \int_A \Psi((B \setminus \{o\}) + x)\, \Psi(\mathrm{d}x)$$

Second-order characteristics of stationary random closed sets

two-point probability function:

$$C(h) = \mathbb{P}(o \in \Xi, h \in \Xi), \quad h \in \mathbb{R}^d$$

K-function of Ψ_d (or Ξ):

$$K_\Xi(r) = \frac{1}{\lambda_d} \mathbb{E}_o \Psi_d(b(o, r))$$

$\lambda_d K_\Xi(r)$ is the mean volume of Ξ within a ball of radius r centred at a 'typical' point of Ξ

$$K_\Xi(r) = \frac{1}{\lambda_d^2} \mathbb{E} \int_{b(o,r)} \mathbf{1}\{o \in \Xi, h \in \Xi\} \, \mathrm{d}h = \frac{1}{\lambda_d^2} \int_{b(o,r)} C(h) \, \mathrm{d}h$$

Random measure generated by stationary RMCS

We assume non-negative marks: $\Gamma(x) \geq 0$.

Stationary random measure generated by (Ξ, Γ):

$$\Psi_\Gamma(B) = \int_B \Gamma(x)\, \Psi_d(\mathrm{d}x), \quad B \in \mathcal{B}^d.$$

Its intensity is

$$\lambda_\Gamma = \mathbb{E}\Gamma(\mathrm{o})\mathbf{1}\{\mathrm{o} \in \Xi\} = \lambda_d \mathbb{E}_\mathrm{o}\Gamma(\mathrm{o}).$$

Palm distribution becomes

$$P_\mathrm{o}(\mathcal{U}) = \frac{\mathbb{E}\mathbf{1}\{\Psi_\Gamma \in \mathcal{U}\}\Gamma(\mathrm{o})\mathbf{1}\{\mathrm{o} \in \Xi\}}{\mathbb{E}\Gamma(\mathrm{o})\mathbf{1}\{\mathrm{o} \in \Xi\}} = \frac{\mathbb{E}_\mathrm{o}\Gamma(\mathrm{o})\mathbf{1}\{\Psi_\Gamma \in \mathcal{U}\}}{\mathbb{E}_\mathrm{o}\Gamma(\mathrm{o})}.$$

Estimation of mean mark

$$\hat{\lambda}_d = \frac{|\Xi \cap W|}{|W|}$$

$$\widehat{\mathbb{E}_o \Gamma(o)} = \frac{\Psi_\Gamma(W)}{\hat{\lambda}_d |W|} = \frac{\Psi_\Gamma(W)}{\Psi_d(W)}$$

under ergodicity assumption, it is strongly consistent estimator of $\mathbb{E}_o \Gamma(o)$

Reduced second-order moment measure of Ψ_Γ

$$\mathcal{K}_\Gamma(B) = \frac{\mathbb{E}\Psi_\Gamma(B \setminus \{o\})\Gamma(o)\mathbf{1}\{o \in \Xi\}}{\lambda_\Gamma \mathbb{E}\Gamma(o)\mathbf{1}\{o \in \Xi\}} = \frac{1}{\lambda_\Gamma^2}\int_B C_\Gamma(h)\,\mathrm{d}h,$$

where $C_\Gamma(h) = \mathbb{E}\Gamma(o)\Gamma(h)\mathbf{1}\{o \in \Xi, h \in \Xi\}$

mark-weighted K-function of Ψ_Γ: $K_\Gamma(r) = \mathcal{K}_\Gamma(b(o,r))$

mark-weighted multiparameter K-function

$$K_\Gamma(r_1,\dots,r_d) = \mathcal{K}_\Gamma([-r_1,r_1]\times\cdots\times[-r_d,r_d]), \quad r_1,\dots,r_d > 0$$

f-weighted K-function

f-weighted reduced second-order moment measure

$$\mathcal{K}_f(B) = \frac{\mathbb{E}_o \int_B f(\Gamma(o), \Gamma(h))\, \Psi_d(\mathrm{d}h)}{\lambda_d \int \int f(\gamma_1, \gamma_2)\, \mathbb{Q}(\mathrm{d}\gamma_1)\, \mathbb{Q}(\mathrm{d}\gamma_2)}, \quad r > 0,$$

where $\mathbb{Q}(\cdot) = \mathbb{P}(\Gamma(o) \in \cdot \mid o \in \Xi)$ is the mark distribution

$f(\gamma_1, \gamma_2) = \gamma_1\gamma_2$ yields $\mathcal{K}_f(B) = \mathcal{K}_\Gamma(B)$

$f(\gamma_1, \gamma_2) = \gamma_1$ yields

$$\mathcal{K}_{\gamma\bullet}(B) = \frac{\mathbb{E}_o \Gamma(o) \Psi_d(B)}{\lambda_d \mathbb{E}_o \Gamma(o)}$$

$K_{\gamma\bullet}(r) = \mathcal{K}_{\gamma\bullet}(b(o, r))$

random field model $\Rightarrow$ $K_{\gamma\bullet}(r) = K_\Xi(r)$

Estimation of mark-weighted K-function

a single realization of (Ξ, Γ) observed within a bounded window $W \subseteq \mathbb{R}^d$

select test points $\xi_1, \ldots, \xi_N \in W$

$$\widehat{\lambda_d \mathcal{K}_{\gamma\bullet}}(B) = \frac{\sum_{i=1}^N \Gamma(\xi_i) \mathbf{1}\{\xi_i \in \Xi\} |(B + \xi_i) \cap \Xi|}{\sum_{i=1}^N \Gamma(\xi_i) \mathbf{1}\{\xi_i \in \Xi\}}$$

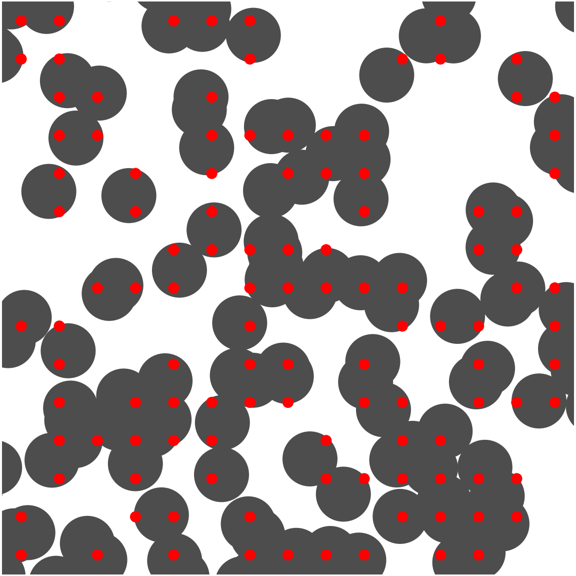
is a ratio-unbiased estimator of $\lambda_d \mathcal{K}_{\gamma\bullet}(B)$

$$\frac{\mathbb{E} \sum_{i=1}^N \Gamma(\xi_i) \mathbf{1}\{\xi_i \in \Xi\} \Psi_d(B + \xi_i)}{\mathbb{E} \sum_{i=1}^N \Gamma(\xi_i) \mathbf{1}\{\xi_i \in \Xi\}} = \frac{N \mathbb{E}_o \Gamma(o) \Psi_d(B)}{N \mathbb{E}_o \Gamma(o)}$$
$$= \lambda_d \mathcal{K}_{\gamma\bullet}(B)$$

$\widehat{\lambda_d \mathcal{K}_{\gamma\bullet}}(B)$ may require information from outside W

edge corrections – minus sampling

Test points – regular grid



Continuous version of the estimator

$$\widehat{\lambda_d \mathcal{K}_{\gamma \bullet}}(B) = \frac{\int_{\Xi \cap W} \Gamma(y) |(B + y) \cap \Xi| \, \mathrm{d}y}{\int_{\Xi \cap W} \Gamma(y) \, \mathrm{d}y}$$

is a ratio-unbiased estimator of $\lambda_d \mathcal{K}_{\gamma \bullet}(B)$

$$\widehat{\lambda_d \mathcal{K}_{\gamma \bullet}}(B) = \frac{\int_{\Xi \cap W} \int_{\Xi} \Gamma(y) \mathbf{1}\{x - y \in B\} \, \mathrm{d}x \, \mathrm{d}y}{\int_{\Xi \cap W} \Gamma(y) \, \mathrm{d}y}$$

translation edge correction:

$$\widehat{\lambda_d \mathcal{K}_{\gamma \bullet}}(B) = \frac{\int_{\Xi \cap W} \int_{\Xi \cap W} \Gamma(y) \frac{\mathbf{1}\{x - y \in B\} |W|}{|(W - x) \cap (W - y)|} \, \mathrm{d}x \, \mathrm{d}y}{\int_{\Xi \cap W} \Gamma(y) \, \mathrm{d}y}$$

is a ratio-unbiased estimator of $\lambda_d \mathcal{K}_{\gamma \bullet}(B)$

Weak consistency

$W_n \nearrow \mathbb{R}^d$ sequence of compact and convex windows

$$\widehat{\mathcal{K}}_n(B) = \frac{|W_n|}{|W_n \cap \Xi|} \frac{\sum\limits_{\xi \in \mathbb{Z}^d \cap W_n \cap \Xi} \Gamma(\xi) |(B + \xi) \cap \Xi|}{\sum\limits_{\xi \in \mathbb{Z}^d \cap W_n \cap \Xi} \Gamma(\xi)}$$

if

$$\sum_{z \in \mathbb{Z}^d} \left| \operatorname{cov}(\Gamma(o)\mathbf{1}\{o \in \Xi\}, \Gamma(z)\mathbf{1}\{z \in \Xi\}) \right| < \infty,$$

$$\sum_{z \in \mathbb{Z}^d} \left| \operatorname{cov}(\Gamma(o)\Psi_d(B)\mathbf{1}\{o \in \Xi\}, \Gamma(z)\Psi_d(B + z)\mathbf{1}\{z \in \Xi\}) \right| < \infty,$$

$$\int |C(h) - \lambda_d^2| \, \mathrm{d}h < \infty,$$

then

$$\widehat{\mathcal{K}}_n(B) \xrightarrow[n \to \infty]{\mathbb{P}} \mathcal{K}_{\gamma^\bullet}(B)$$

Weak consistency for marked Boolean model

$$\Xi = \bigcup_{i \geq 1} b(\xi_i, R_i)$$

$$\Gamma(x) = c_\Gamma \sum_{i \geq i} k\left(\frac{\|x - \xi_i\|}{R_i}\right)$$

k is a bounded probability density function with support $[0, 1]$

if $\mathbb{E} R_i^{4d} < \infty$, then

$$\widehat{\mathcal{K}}_n(B) \xrightarrow[n \to \infty]{\mathbb{P}} \mathcal{K}_{\gamma \bullet}(B)$$

m-dependent RMCS

We say that RMCS (Ξ, Γ) is m-dependent for some $m > 0$ if $\tau(\Xi \cap A, \Gamma)$ and $\tau(\Xi \cap B, \Gamma)$ are independent for any bounded $A, B \in \mathcal{B}^d$ such that $d(A, B) > m$.

Examples:

- Ξ Boolean model with bounded grains and Γ m-dependent random field
- Ξ excursion set of an m-dependent random field Γ (e.g. Gaussian random field with finite dependence range)

Asymptotic normality

$W_n = [-(n+1/2), n+1/2]^d$ sequence of observation windows

$$\widehat{\lambda_d \mathcal{K}_n}(B) = \frac{\sum\limits_{\xi \in \mathbb{Z}^d \cap W_n \cap \Xi} \Gamma(\xi) \big| (B+\xi) \cap \Xi \big|}{\sum\limits_{\xi \in \mathbb{Z}^d \cap W_n \cap \Xi} \Gamma(\xi)}$$

assume that (Ξ, Γ) is m-dependent stationary RMCS and

$$\operatorname{var} \Gamma(\mathrm{o}) \mathbf{1}\{\mathrm{o} \in \Xi\} < \infty, \quad \operatorname{var} \Gamma(\mathrm{o}) \Psi_d(B) \mathbf{1}\{\mathrm{o} \in \Xi\} < \infty,$$

then

$$\sqrt{|W_n|} \left(\widehat{\lambda_d \mathcal{K}_n}(B) - \lambda_d \mathcal{K}_{\gamma\bullet}(B) \right) \xrightarrow[n \to \infty]{\mathrm{d}} N(0, \sigma_B^2)$$

Approximation by m-dependent random fields

marked Boolean model of balls, $\mathbb{E}R_i^{4d} < \infty$

$$\sqrt{|W_n|}\left(\widehat{\lambda_d\mathcal{K}_n}(B) - \lambda_d\mathcal{K}_{\gamma\bullet}(B)\right) \xrightarrow[n\to\infty]{\mathrm{d}} N(0,\sigma_B^2)$$

$$\sum_{\xi\in\mathbb{Z}^d\cap W_n} U_\xi = \sum_{\xi\in\mathbb{Z}^d\cap W_n} U_\xi^{(m)} + \sum_{\xi\in\mathbb{Z}^d\cap W_n} R_\xi^{(m)}$$

$\{U_\xi^{(m)} : \xi\in\mathbb{Z}^d\}$ m-dependent

$\frac{1}{|W_n|}\sup_{n\in\mathbb{N}}\mathbb{E}\left(\sum_{\xi\in\mathbb{Z}^d\cap W_n} R_\xi^{(m)}\right)^2 \to 0$ as $m\to\infty$

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