Existence and absence of percolation for outdegree-one random graphs

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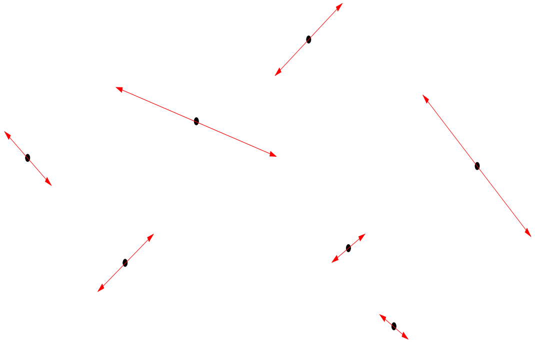
19th Workshop on Stochastic Geometry, Stereology and Image Analysis, Luminy, may-16th 2017

1. The models

Line segment model

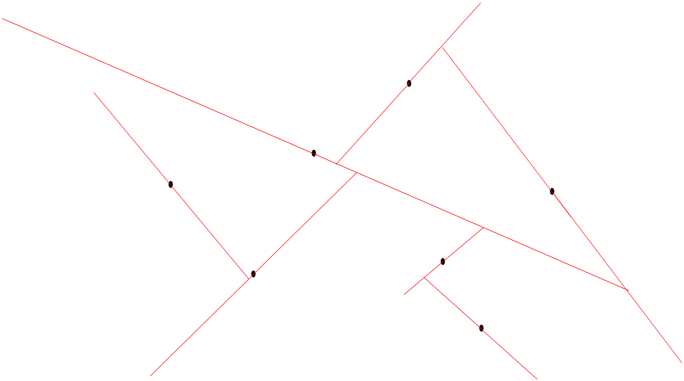
Definition by an independently marked P.P.P. in $\mathbf{R}^2$.

- **Position :** A P.P.P. Λ with intensity $z\,\lambda_2$ $(z > 0)$.
- **Direction :** An uniform distribution Θ on $[0, \pi]$.
- **Speed :** A distribution $\mathbf{V}$ on $[0, +\infty)$ with $\mathbf{P}(\mathbf{V} = 0) = 0$.



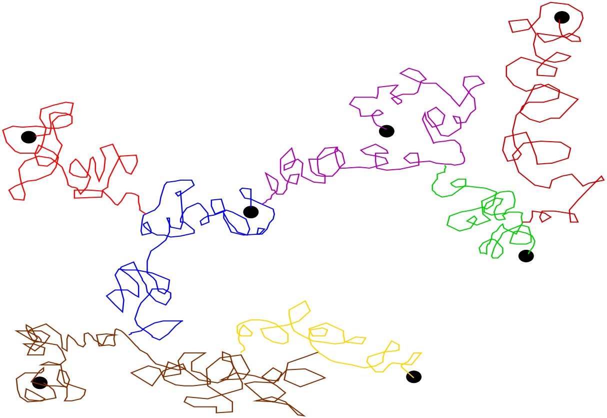
The segments are growing

At time 0, each line segment starts to grow with the corresponding speed and direction. A given line segment is stopped when its extremity hurts another line segment.



The Brownian model

At time 0, a $2D$-Brownian motion is starting to grow at each vertex. A given trajectory is stopped when its extremity hurts another one.



First question

Existence

In each model, does the stopping condition give a final state with probability one ?

- **D.J Daley, S.Ebert and G.Last (2014)** show that the line segment model exists with probability one, if the speed distribution is a Dirac measure.

Second question

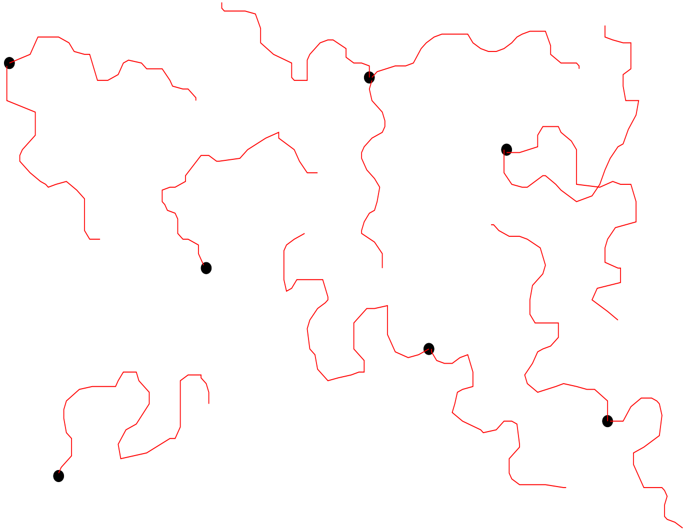
Percolation

Is there an infinite connected component with positive probability ?

- **D.J Daley, S.Ebert and G.Last (2014)** conjecture the almost surely absence of infinite connected component in the line segment model.
- **C.Hirsch (2016)** proves the absence of percolation for the line segment model with a uniform direction distribution on $\{\frac{k\pi}{2}, 0 \leq k \leq 1\}$.

2 Existence of stopped germs/grains models

Introductive example



The germs/grains model

- Let $\mathbf{F} = F_0(\mathbf{R}_+, \mathbf{R}^2)^{\mathbf{N}} = \{(f_n)_{n\geq 0} \ ; \ \forall n \geq 0,\ f_n(0) = 0\}$ and $\mathcal{F}$ the standard σ-algebra on $\mathbf{F}$.
- Let $\mathcal{L}$ a law on $(\mathbf{F}, \mathcal{F})$ such that the marginals $(\mathcal{L}_n)_{n\geq 0}$ are identically distributed.
- Let δ a probability measure on $\mathbf{N}^*$ such that $\mathbf{E}_\delta(K) < +\infty$.

Definition

A $(z, \delta, \mathcal{L})$-germs/grains model is a Poisson point process of intensity $z\lambda_2 \otimes \delta \otimes \mathcal{L}$.

What is the grain ?

Let $\mathbf{X}$ a $(z, \delta, \mathcal{L})$-germs/grains model.
For a marked point $x = (\xi, k, Y)$, (where $Y = (Y_i)_{i\geq 0}$), the grain of length t associated to x is :

$$\mathbf{Grain}(x,t) = \bigcup_{i=0}^{k-1} \{\xi + Y_i(s) \ ; \ 0 \leq s \leq t\}.$$

Let

$$\mathbf{Grain}(x,+\infty) = \bigcup_{t\geq 0} \mathbf{Grain}(x,t).$$

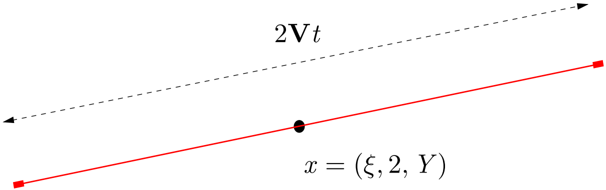
We also introduce the extremity of the grain associated to x at time t :

$$\mathbf{H}(x,t) = \{\xi + Y_i(t) \ ; \ 0 \leq i \leq k-1\}.$$

Is the line segment model a germs/grains model ?

Line segment model :

- The probability measure δ is the Dirac measure $\delta_{\{2\}}$.
- Considering Θ the uniform distribution on $[0, \pi]$ and $\mathbf{V}$ the speed distribution, the probability measure $\mathcal{L}$ is the law of the sequence $(\,Y_1,\, Y_2,\, Y_1,\, Y_1, \dots,\, Y_1, \dots\,)$ where $Y_1(t) = (\mathbf{V}.t\cos(\Theta), \mathbf{V}.t\sin(\Theta))$ and $Y_2 = -\,Y_1$.
- Considering $x = (\xi, 2,\, Y)$, for all $t > 0$ the grain $\mathbf{Grain}(x, t)$ is a segment of length $2\,\mathbf{V}\,t$ with center ξ.



Stopped germs/grains model

Let $\mathbf{X}$ a $(z, \delta, \mathcal{L})$-germs/grains model. $\mathbf{X}$ is **stopped** if :

$$\forall \ a.s \ \mathbf{X}, \exists! \, f_{\mathbf{X}} : \mathbf{X} \longrightarrow \mathbf{R}_{+} \cup \{+\infty\} \text{ such that :}$$

- (i) $\forall x \neq y \in \mathbf{X}, \ \forall t \leq f_{\mathbf{X}}(x), \ \forall s \leq f_{\mathbf{X}}(y),$

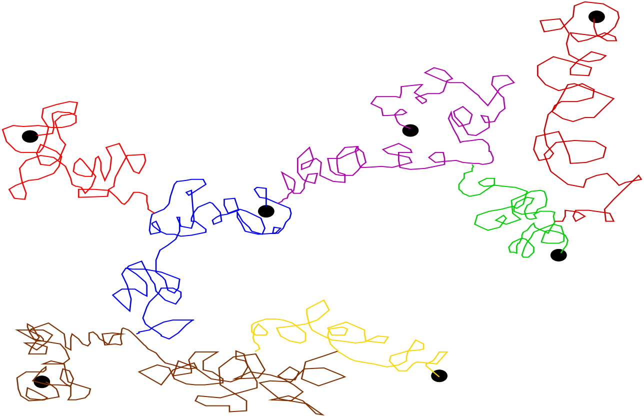
$$\mathbf{Grain}(x, t) \cap \mathbf{Grain}(y, s) = \emptyset.$$

- (ii) $\forall x \in \mathbf{X}$ s.t $f_{\mathbf{X}}(x) < +\infty, \exists y \in \mathbf{X} \setminus \{x\}$, s.t :

$$\mathbf{H}(x, f_{\mathbf{X}}(x)) \cap \mathbf{Grain}(y, f_{\mathbf{X}}(y)) \neq \emptyset.$$

For the Brownian model

- Given $x \in \mathbf{X}$, $f_{\mathbf{X}}(x)$ is the exploration time of x in the configuration $\mathbf{X}$.



Existence of stopped germs/grains models

Theorem (Coupier, Dereudre, LS)

Let $\mathbf{X}$ *be a* $(z, \delta, \mathcal{L})$*-germs/grains model. Let*

- $\mathbf{K}$ *a random variable following the law* δ, $\mathbf{E}_\delta(\mathbf{K}) < +\infty$.
- $\mathbf{Y} = (Y_0, Y_1, \dots,)$ *following the law* $\mathcal{L}$.

For $t, t' \geq 0$, *we define :*

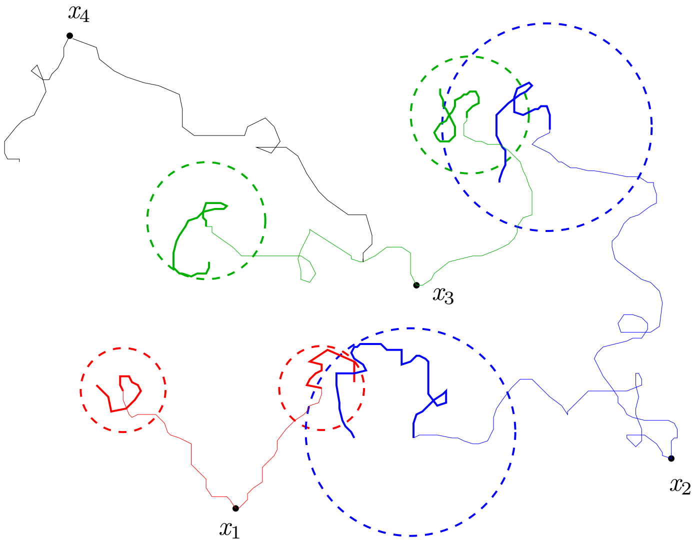
$$M_{t,t'} = \max_{0 \leq k \leq \mathbf{K}-1} \sup_{0 \leq s \leq t'} \| Y_k(t+s) - Y_k(t) \|_2.$$

If we suppose that :

$$\lim_{t' \to 0} \mathbf{E} \left(\sup_{t \geq 0} M^3_{t,t'} \right) = 0,$$

then, $\mathbf{X}$ *is a* ***stopped germs/grains model.***

Some ideas about the proof



Line segment model ? Brownian model ?

Corollary

The following are ***stopped germs/grains models*** *:*

- *the Brownian model,*
- *the line segment model with* $\mathbf{E}(V^3) < +\infty$.

More precisely, they satisfy the following property :
$\forall a.s\ \mathbf{X}, \forall x \in \mathbf{X},\ f_{\mathbf{X}}(x) < +\infty$ *and :*

$$\exists! y \in \mathbf{X} \setminus \{x\}\ ;\ \boldsymbol{H}(x, f_{\mathbf{X}}(x)) \cap \boldsymbol{Grain}(y, f_{\mathbf{X}}(y)) \neq \emptyset. \qquad (1)$$

Equation (1) ensures the **existence** and the **uniqueness** of the stopping grain for each marked point.

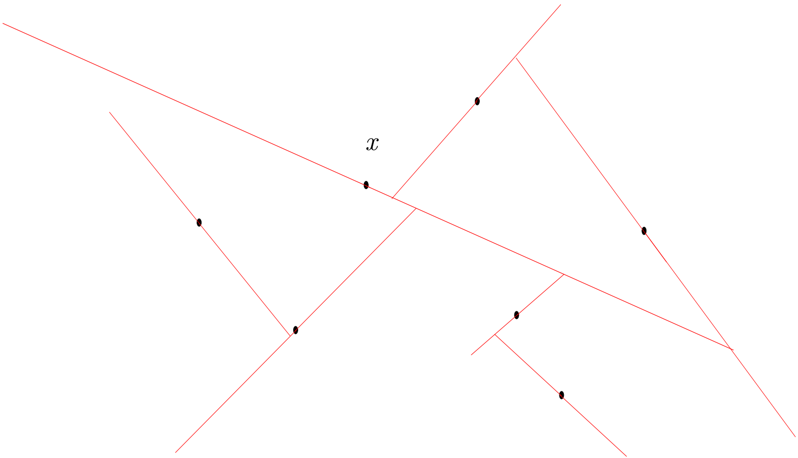
3 Absence of percolation

Poisson outdegree-one graph

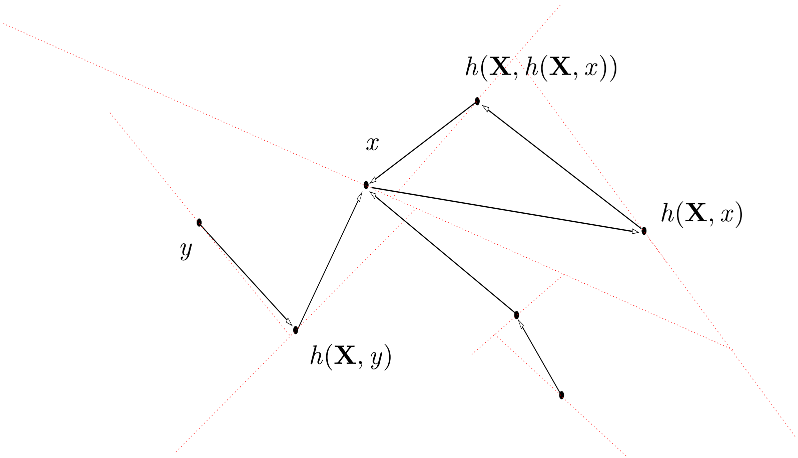
Because of the Corollary, it is possible to observe an **outdegree-one** graph structure on the two germs/grains models studied.

- $\forall$ a.s $\mathbf{X}$, $\forall x \in \mathbf{X} : h(\mathbf{X}, x)$ is defined like the only marked point y of the equation (1).
- The vertices of the random graph $\mathcal{G}(\mathbf{X})$ are the elements of $\mathbf{X}$ and the oriented edges are the pairs $(x, h(\mathbf{X}, x))$, for $x \in \mathbf{X}$.

Line segment model



Line segment model



Cluster

Definition

Let $x \in \mathbf{X}$, we define the forward of x in $\mathbf{X}$:

$$\mathbf{For}(\mathbf{X}, x) := \{x,\ h(\mathbf{X}, x),\ h(\mathbf{X}, h(\mathbf{X}, x))...\}.$$

We also define the backward of x in $\mathbf{X}$:

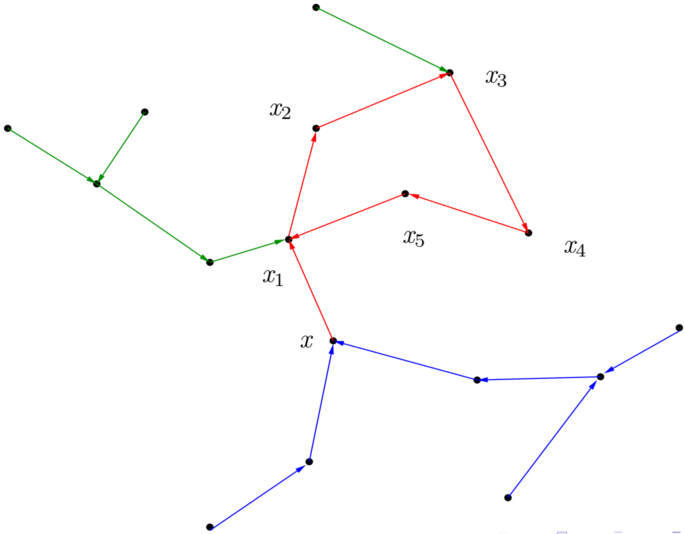
$$\mathbf{Back}(\mathbf{X}, x) = \{y \in \mathbf{X} :\ x \in \mathbf{For}(\mathbf{X}, y)\}.$$

To finish, we introduce

$$\mathbf{Cluster}(\mathbf{X}, x) = \mathbf{For}(\mathbf{X}, x) \cup \mathbf{Back}(\mathbf{X}, x), \quad (2)$$

$$\mathbf{ConComp}(\mathbf{X}, x) = \bigcup_{y \in \mathbf{For}(\mathbf{X}, x)} \mathbf{Back}(\mathbf{X}, y). \quad (3)$$

Example of a finite connected component



Loops

Definition

Let $n \geq 2$ be an integer. A Loop of size n in $\mathcal{G}(\mathbf{X})$ is a subset $\{x_1, \ldots, x_n\} \in \mathbf{X}^n$ such that :

- $\forall i \in \{1, \ldots, n-1\},\ h(\mathbf{X}, x_i) = x_{i+1}$,
- $h(\mathbf{X}, x_n) = x_1$.

Remark

- *There is at most one Loop in a given connected component and precisely one Loop in a finite connected component.*
- *The abundance of Loop in the graph $\mathcal{G}(\mathbf{X})$ will have an important place in the absence of percolation results.*

Percolation

Definition

We say that the Poisson outdegree-one graph $\mathcal{G}(\mathbf{X})$ does not percolate if, almost surely, all its clusters are finite :

$$\mathbf{P}\left(\forall x \in \mathbf{X},\ \#\mathbf{Cluster}(\mathbf{X}, x) < +\infty\right) = 1.$$

Remark

It is not difficult to observe that the absence of percolation corresponds to the almost sure absence of infinite ***connected component****.*

The line segment model does not percolate

Theorem (Coupier, Dereudre, LS)

If we suppose that the speed distribution $\mathbf{V}$ *satisfies :*

$$\mathbf{E}(\mathbf{V}^3) < +\infty,$$

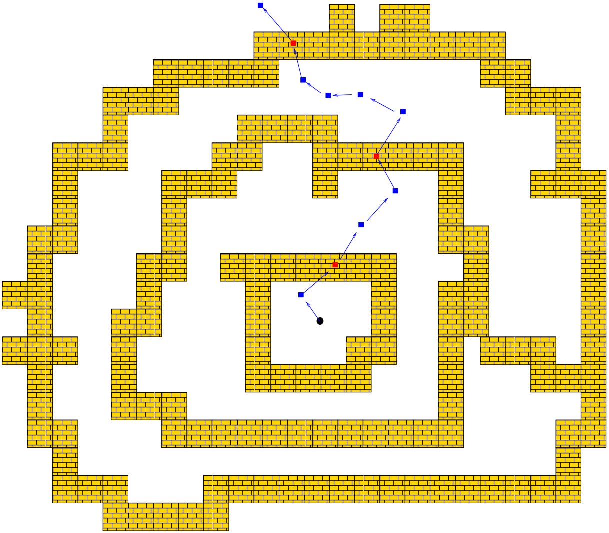
*then, **the line segment model does not percolate**.*

Some ideas about the proof

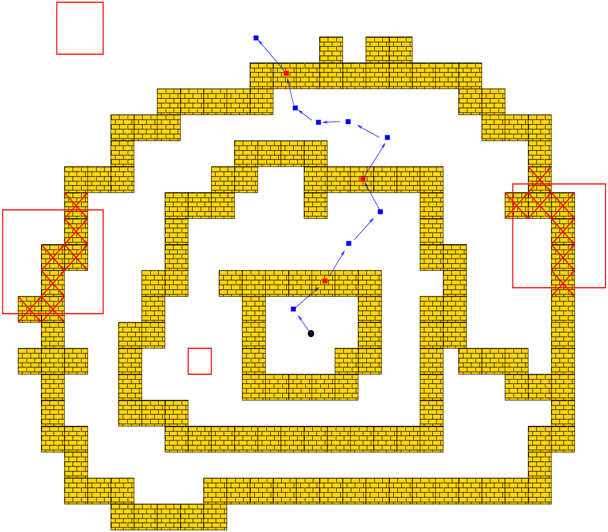
- Thanks to the **mass transport principle**, it is sufficient to show that

$$\mathbf{P}\left(\forall x \in \mathbf{X},\ \#\mathrm{For}(\mathbf{X}, x) < +\infty\right) = 1.$$

Some ideas about the proof



Some ideas about the proof



Some References

- **D. Daley, S. Ebert and G. Last**. Two lilyponds systems of finite line-segments. *To appear in Probability and Mathematical Statistics, 2014.*
- **C.Hirsch**. On the absence of percolation in a line segment based lilypond model. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(1) :127-145, 2016.
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Thank you for your attention