Covariograms and geometric functionals of random excursions

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- **Intrinsic volume** V_k : Homogeneous **additive** functionals in the convex ring :

$$V_k(A \cup B) = V_k(A) + V_k(B) - V_k(A \cap B)$$

$$\begin{aligned} k &= 0 : \text{ Euler characteristic} \\ &\ldots \\ k &= d-1 : \text{surface measure (``Perimeter'' here)} \\ k &= d : \text{ Volume} \end{aligned}$$

- Can be extended to larger class of sets

Covariogram (Galerne 2011)

Let $A \subset \mathbb{R}^d$ be a "nice set". Then

$$\text{Perimeter}(A) = \lim_{\varepsilon \to 0} \int_{\mathcal{S}^{d-1}} \frac{\text{Lebesgue}_d(A + \varepsilon \mathbf{u} \setminus A)}{\varepsilon} \sigma_{d-1}(d\mathbf{v})$$

- $\mathcal{S}^{d-1}$: d-Sphere
- σ_{d-1} : renormalized $(d-1)$-Hausdorff measure
- Valid for any measurable set $A \subset \mathbb{R}^d$ if the perimeter is defined variationnally :

$$\text{Perimeter}(A) = \sup_{\varphi \in \mathcal{C}_c^1(\mathbb{R}^d, \mathbb{R}^d) : \|\varphi\|_\infty \leqslant 1} \int_{\mathbb{R}^d} \mathbf{1}_{\{A\}}(x) \text{div}(\varphi)(x) dx.$$

- Only concerns points carrying the mass of A

Relation between covariograms intrinsic volumes

- Let $0 \leqslant k \leqslant d$
- k-th order variogram :

$$\gamma_{x_1,\dots,x_k}(A) = \text{Lebesgue}_d(A \cap (A + x_1) \cap \dots \cap (A + x_k))$$

- For instance, for a "nice set" A, $V_{d-1}(A) = \text{Perimeter}(A)$ can be expressed with $\gamma_{\cdot}(A)$

$$\text{Lebesgue}_d(A + \varepsilon\mathbf{u} \setminus A) = \gamma_0(A) - \gamma_{\varepsilon\mathbf{u}}(A), \mathbf{u} \in \mathcal{S}^{d-1}$$

- Trivial : $\gamma(A) = \text{Lebesgue}_d(A) = V_d(A)$

Expression of V_{d-1} linearly in function of the 1t-order variograms

- Let $A \subset \mathbb{R}^2$,
- $CC(A)$: bounded connected components.
- If $CC(A)$ and $CC(A^c)$ are finite : **Euler characteristic**

$$\chi(A) = \#CC(A) - \#CC(A^c).$$

- $\chi = V_0$: Intrinsic volume of order 0 (when defined)
- Additive functional in the polyconvex ring :

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

Extract topological information from covariograms :

Theorem

$A \subset \mathbb{R}^2$ compact such that

- *∂A is a $\mathcal{C}^1$ manifold*
- *The normal vector $x \in \partial A \mapsto n_A(x)$ is Lipschitz ("$\mathcal{C}^{1,1}$ manifold")*

Let $\mathbf{v}_1, \mathbf{v}_2$ be two orthogonal unit vectors. Then for ε suff. small

$$\chi(A) = \frac{1}{\varepsilon^2}\left[\gamma_0^{\varepsilon \mathbf{v}_1, \varepsilon \mathbf{v}_2}(A) - \gamma^0_{-\varepsilon \mathbf{v}_1, -\varepsilon \mathbf{v}_2}(A)\right].$$

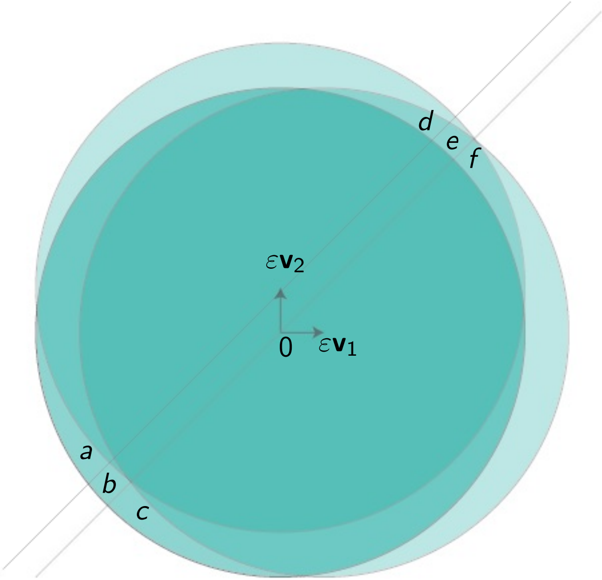
$$\gamma^{y_1,\dots,y_m}_{x_1,\dots,x_q}(A) = \mathrm{Lebesgue}_d(A \cap \dots (A + x_i) .. \cap \dots (A + y_j)^c \dots)$$

- Linear expression in terms of k-th order variograms :

$$\gamma_0^{\varepsilon \mathbf{v}_1, \varepsilon \mathbf{v}_2} = \gamma - \gamma_{\varepsilon \mathbf{v}_1} - \gamma_{\varepsilon \mathbf{v}_2} + \gamma_{\varepsilon \mathbf{v}_1, \varepsilon \mathbf{v}_2}$$

- Extension to dimension d : χ in function of $(d+1)$-order variograms.

Unit disc



- $\gamma_0^{\varepsilon\mathbf{v}_1,\varepsilon\mathbf{v}_2} = a + b + c$
- $\gamma^0_{\pm\varepsilon\mathbf{v}_1,\pm\varepsilon\mathbf{v}_2} = d + e + f$
- $a = f$
- $a + b = c$
- $f = d + e$
- $\chi(A) = a + b + c - d - e - f = 2b \sim 2\frac{\varepsilon^2}{2}$

Random excursion

- For $f : W \to \mathbb{R}, u \in \mathbb{R}$, note

$$\{f \geqslant u\} = \{x \in W : f(x) \geqslant u\}.$$

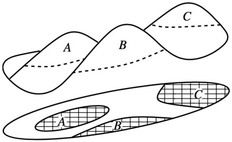


FIGURE – Adler & Taylor

- Functionals of interest :

$$\begin{aligned} \mathrm{Vol}(\{f \geqslant u\}) \\ \mathrm{Perimeter}(\{f \geqslant u\}) \\ \chi(\{f \geqslant u\}) \\ \ldots \end{aligned}$$

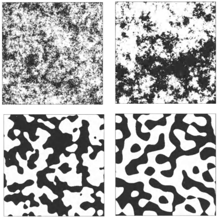


FIGURE – Lantuejoul

- f : Gaussian field, Shot noise field, infinitely divisible field, ...

Mean values

- **Mean perimeter** (with covariograms) : need to compute

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbf{E}\left[\mathrm{Lebesgue}_d(\{x : f(x)\geqslant u,\ f(x-\varepsilon\mathbf{v})\geqslant u\})\right]$$
$$=\lim_{\varepsilon\to 0}\varepsilon^{-1}\int \mathbf{P}(f(x)\geqslant u; f(x-\varepsilon\mathbf{v})<u)dx$$

  and integrate over $\mathbf{v}$.
- **Mean Euler characteristic** (Worsley, Adler, Taylor, ...) : For f random field almost surely **Morse** :

$$\chi(\{f\geqslant u\})=\lim_{\varepsilon\to 0}\frac{(-1)^d}{(2\varepsilon)^d}\int_W \det(\mathrm{Hess}_f(x))\mathbf{1}_{\{f(x)\geqslant u\}}\mathbf{1}_{\{\|\nabla f(x)\|_\infty\leqslant\varepsilon\}}dx$$
$$+\text{ boundary terms}$$

- **Expectations** : Requires bounded conditional density

Random fields excursions (LR-2015)

- W : compact non-degenerate union of rectangles
- f : real $\mathcal{C}^{1,1}$ random field : $\mathbb{R}^2 \to \mathbb{R}$
- $u \in \mathbb{R}$

Assume

- some **conditions on the density** of $(f(x), \partial_1 f(x), \partial_2 f(x))$ (not necessarily bounded around u)
- some **moment conditions** on $\mathrm{Lip}(\partial_i f), i = 1, 2$.

Then $\mathbf{E}\chi(\{f \geqslant u\} \cap W) =$

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \int_W \Big[\mathbf{P}(f(x) \geqslant u, f(x + \varepsilon \mathbf{v}_1) < u, f(x + \varepsilon \mathbf{v}_2) < u) - \mathbf{P}(f(x) < u, f(x - \varepsilon \mathbf{v}_1) \geqslant u, f(x - \varepsilon \mathbf{v}_2) \geqslant u) \Big] dx + \text{boundary terms}$$

Number of connected components

- Need domination to apply Lebesgue's theorem. For f $\mathcal{C}^{1,1}$ such that $\nabla f \neq 0$ on $\{f = u\}$:

$$\#CC(\{f \geqslant u\}) \leqslant 2^d \kappa_d^{-1} \max(\mathrm{Lip}(\partial_i f)) \times \int_W \frac{dx}{\max(|f(x) - u|, |\partial_i f(x)|, i = 1, \dots, d)^d} + \text{ boundary terms}$$

- $\mathbf{E}[\text{right hand term}]$ does not require bounded density of $(f(x) - u, \partial_1 f(x), \dots, \partial_d f(x))$.
- Finite expectation actually allows to switch $\int_W$ and $\lim_{\varepsilon \to 0}$
- Allows to compute expectation in higher dimensions ?

Application to Gaussian fields

- $f : \mathcal{C}^{1,1}, \mathbb{R}^2 \to \mathbb{R}^2$, **stationary isotropic Gaussian field** with unit variance
- $(f(0), \partial_1 f(0), \partial_2 f(0))$ non-degenerate
- $\mathbf{E}\mathrm{Lip}(\partial_1 f, W)^{6+\eta} < \infty$ for some $\eta > 0$ Then

$$\mathbf{E}\chi(\{f \geqslant u\} \cap W) = \mathrm{Vol}(W) \times e^{-u^2/2} \frac{\mu u}{(2\pi)^{3/2}} + \mathrm{Per}(W) \times e^{-u^2/2} \frac{\sqrt{\mu}}{4\pi} + \chi(W) \times \frac{1}{\sqrt{2\pi}} \Phi(u).$$

- Φ : Gaussian distribution function

Alternative (heuristic) proof of the coarea formula

$$\boxed{\int_{\mathbb{R}} h(t)\mathrm{Per}(\{f \geqslant t\})dt =} \int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \lim_{\varepsilon \to 0} \varepsilon^{-1} \Big| \{f \geqslant t\} \setminus \{f \geqslant t + \varepsilon \mathbf{u}\} \Big| \sigma(d\mathbf{v})$$

$$= \int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\mathbb{R}^d} \mathbf{1}_{\{f(x) \geqslant t, f(x-\varepsilon \mathbf{v}) < t\}} dx \sigma(du) dt$$

$$\approx \int_{\mathcal{S}^{d-1}} \int_{\mathbb{R}^d} \lim_{\varepsilon \to 0} \varepsilon^{-1} \begin{cases} \int_{f(x)}^{f(x-\varepsilon \mathbf{v})} h(t) dt \sigma(d\mathbf{v}) dx \\ 0 \text{ if } f(x) > f(x - \varepsilon \mathbf{v}) \end{cases}$$

$$\approx \int_{\mathbb{R}^d} \int_{\mathcal{S}^{d-1}} h(f(x)) |\partial_{\mathbf{u}} f(x)|^+ \sigma(d\mathbf{v}) dx$$

$$= \boxed{\int_{\mathbb{R}^2} h(f(x)) \|\nabla f(x)\| dx.}$$

Theorem

- *Let $h : \mathbb{R} \to \mathbb{R}$ smooth with compact support,*
- *Let f be Morse over the support of h*

Call

$$Q_1 = \{(x, y) \in \mathbb{R}^2 : x < y < 0\}$$
$$Q_2 = \{(x, y) \in \mathbb{R}^2 : y < x < 0\}$$

Then

$$\int_{\mathbb{R}} h(u)\chi(\{f \geqslant u\})du$$
$$= -\int_{\mathbb{R}^2} \sum_{i=1}^{2} \mathbf{1}_{\{\nabla f(x) \in Q_i\}} \left[\partial_i f(x)^2 h'(f(x)) + \partial_{ii} f(x) h(f(x))\right] dx$$
$$+ \text{boundary terms}$$

Stationary $\mathcal{C}^2$ isotropic fields (**Biermé & Desolneux 2016 :**)

- For f $\mathcal{C}^2$ random with $\mathbf{E}\mathrm{Hess}_f(x) < \infty$,

$$\mathbf{E}\left[\int_W h(u)\chi(\{f \geqslant u\})du\right] = -|W|\mathbf{E}\left[h(f(0))\partial_{ii}f(0)\right]$$

- Obtained via an approach involving Gauss-Bonnet theorem

$$\int_{\mathbb{R}} h(u)\chi(\{f \geqslant u\})du$$

$$= -\int h(f(x))\mathrm{Hess}_f(x)\left[\frac{\nabla f(x)}{\|\nabla f(x)\|}, \frac{\nabla f(x)}{\|\nabla f(x)\|}\right]\mathbf{1}_{\{\nabla f(x) \neq 0\}}dx$$

$$+ \text{boundary terms}$$

Application to shot noise random fields

- Let W be a compact set
- $\eta = \{x_1, x_2, \dots\}$ homogeneous Poisson process restricted to W
- $g_1, g_2, \dots$ random iid integrable functions, belong to the class of SBV for considerations on the perimeter

$$f_\eta(x) = \sum_i g_i(x_i - x)$$

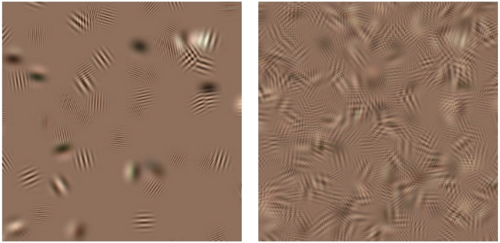


FIGURE – Galerne

Second order results (work in progress)

- Consider for h a test function

$$F_{W,h}^{(1)} = \int_{\mathbb{R}} h(u)\mathrm{Perimeter}(\{f \geqslant u\})du$$
$$F_{W,h}^{(2)} = \int_{\mathbb{R}} h(u)\mathrm{Volume}(\{f \geqslant u\})du$$

- Assume h is such that $F_{W,h}^{(i)}$ is not "trivial"

- Assume polynomial decaying of the kernels g_i : for some $\varepsilon > 0$

$$\mathbf{E}[g_1(x)^4] \leqslant C(1+\|x\|)^{-8d-\varepsilon} \text{ for the volume}$$

$$\text{additionally, } \mathbf{E}[\|\nabla g_1(x)\|^4] \leqslant C(1+\|x\|)^{-8d-\varepsilon} \text{ for the perimter}$$

  plus a moment assumption on the $(d-1)$-Haussdorf measure of the discontinuiy set
- Then, as $\frac{|\partial W|}{|W|} \to 0$

$$\left| \frac{\mathbf{Var}(F^{(i)}_{W,h})}{|W|} - \sigma_0 \right| \to 0$$

$$\sup_{t\in\mathbb{R}} \left| \mathbf{P}\left(\frac{F^{(i)}_{W,h} - \mathbf{E}F^{(i)}_{W,h}}{\sqrt{\mathbf{Var}(F^{(i)}_{W,h})}} \leqslant t \right) - \mathbf{P}\left(\mathcal{N}(0,1 \leqslant t)\right) \right| \leqslant \kappa |W|^{-1/2}.$$

- Application of Stein-Malliavin methods to non-stabilizing functionals
- Requires to study the density of the shot noise field to have similar results at a fixed level u, or for the Euler characteristic of excursions

Thank you
for your attention !