Anisotropy in finite continuum percolation: threshold estimation by Minkowski functionals

Michael Andreas Klatt, Gerd E. Schröder-Turk, Klaus Mecke

Karlsruhe Insitute of Technology (KIT),
Institute for Stochastics

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DFG FORSCHERGRUPPE
Geometry & Physics of Spatial Random Systems

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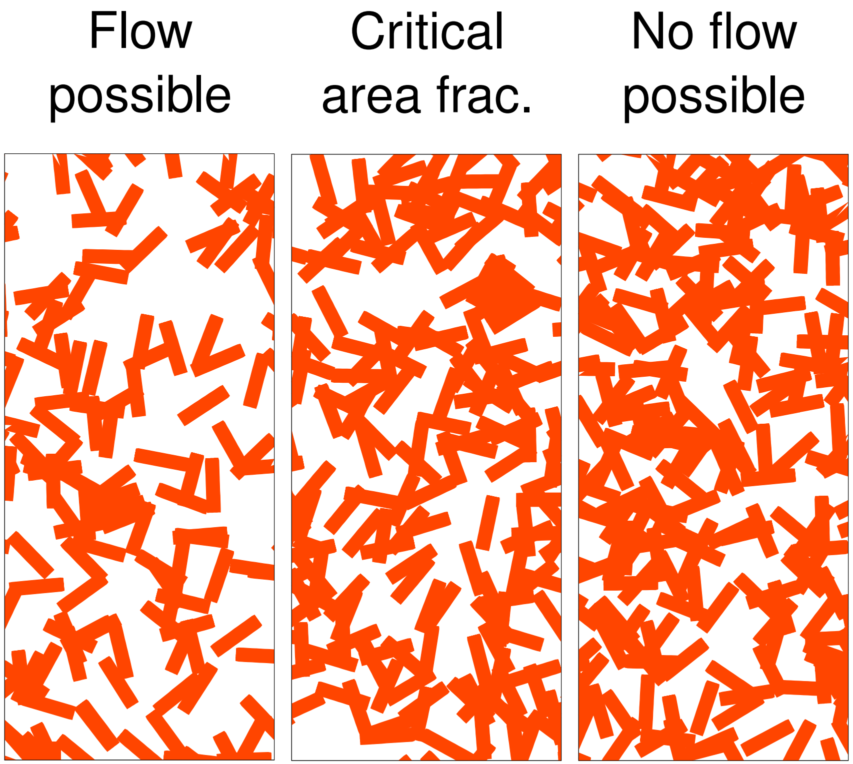
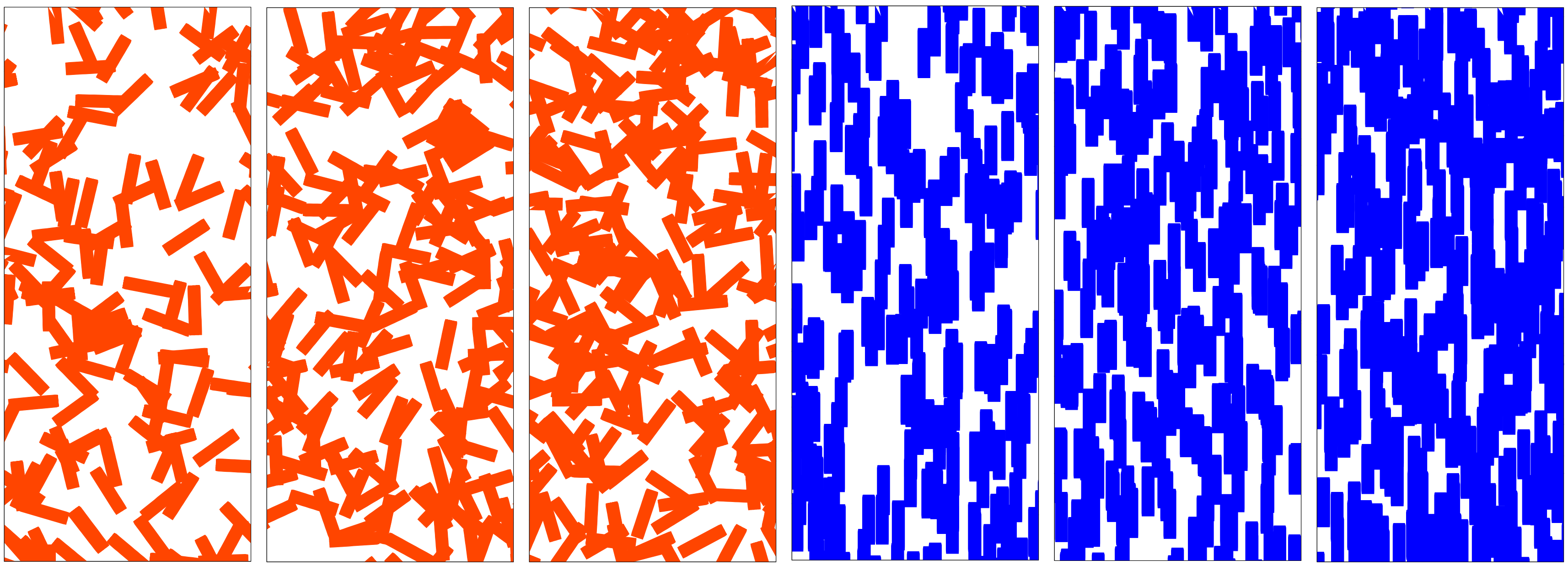
Percolation: Geometric phase transition



K., Schröder-Turk, Mecke, *JSTAT* (2017)

Percolation: Geometric phase transition

Flow possible	Critical area frac.	No flow possible	Flow possible	Critical area frac.	No flow possible



Topology ⟲ Geometry

Universal behavior
Simultaneous percolation in all directions

Non-universal properties
Approximate threshold using Minkowski functionals

K., Schröder-Turk, Mecke, *JSTAT* (2017)

Anisotropy and threshold estimation

1. Anisotropy in finite percolation
   Isotropic percolation threshold

2. Explicit threshold approximations
   1st and 2nd moments of Minkowski functionals

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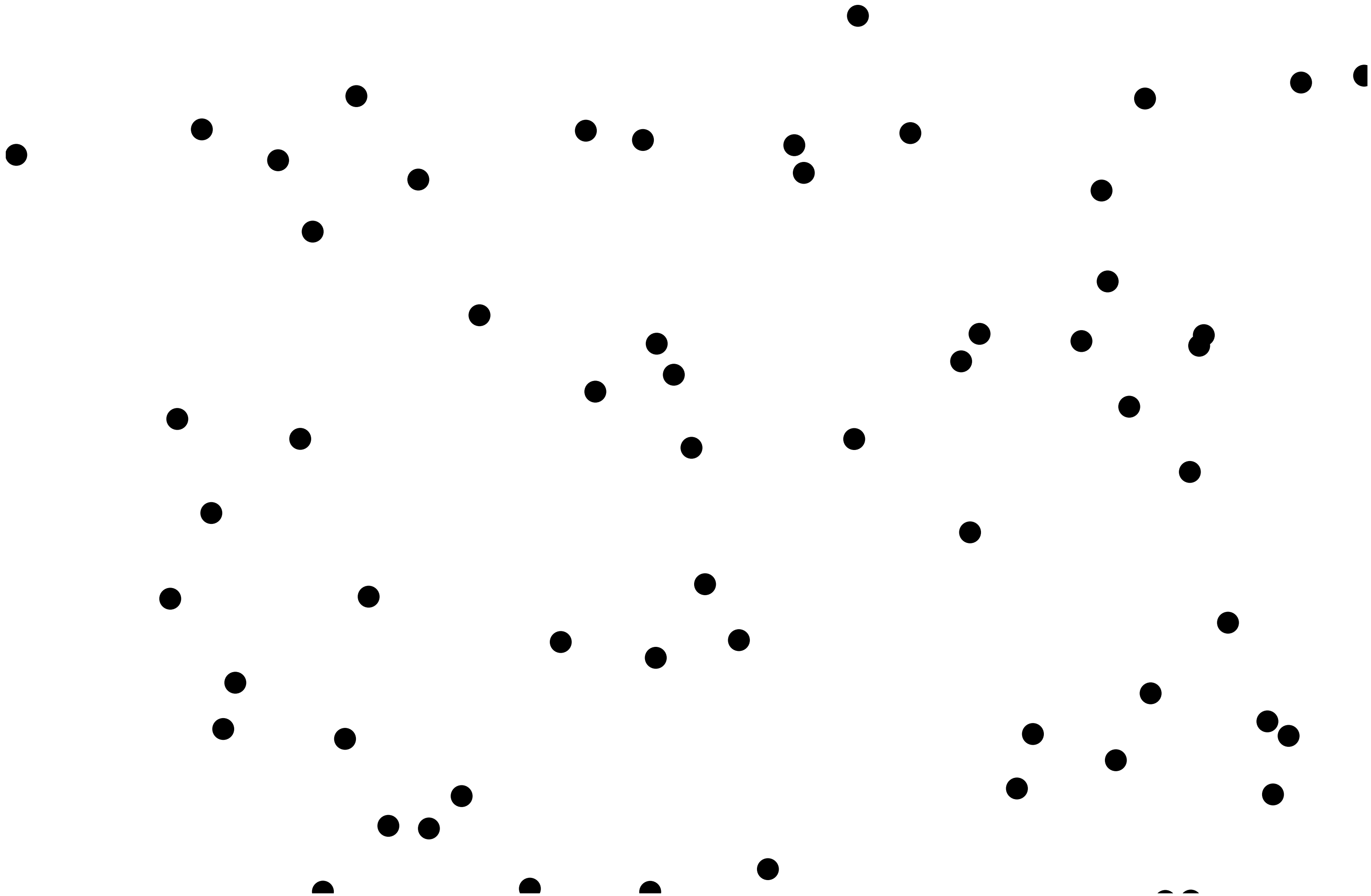
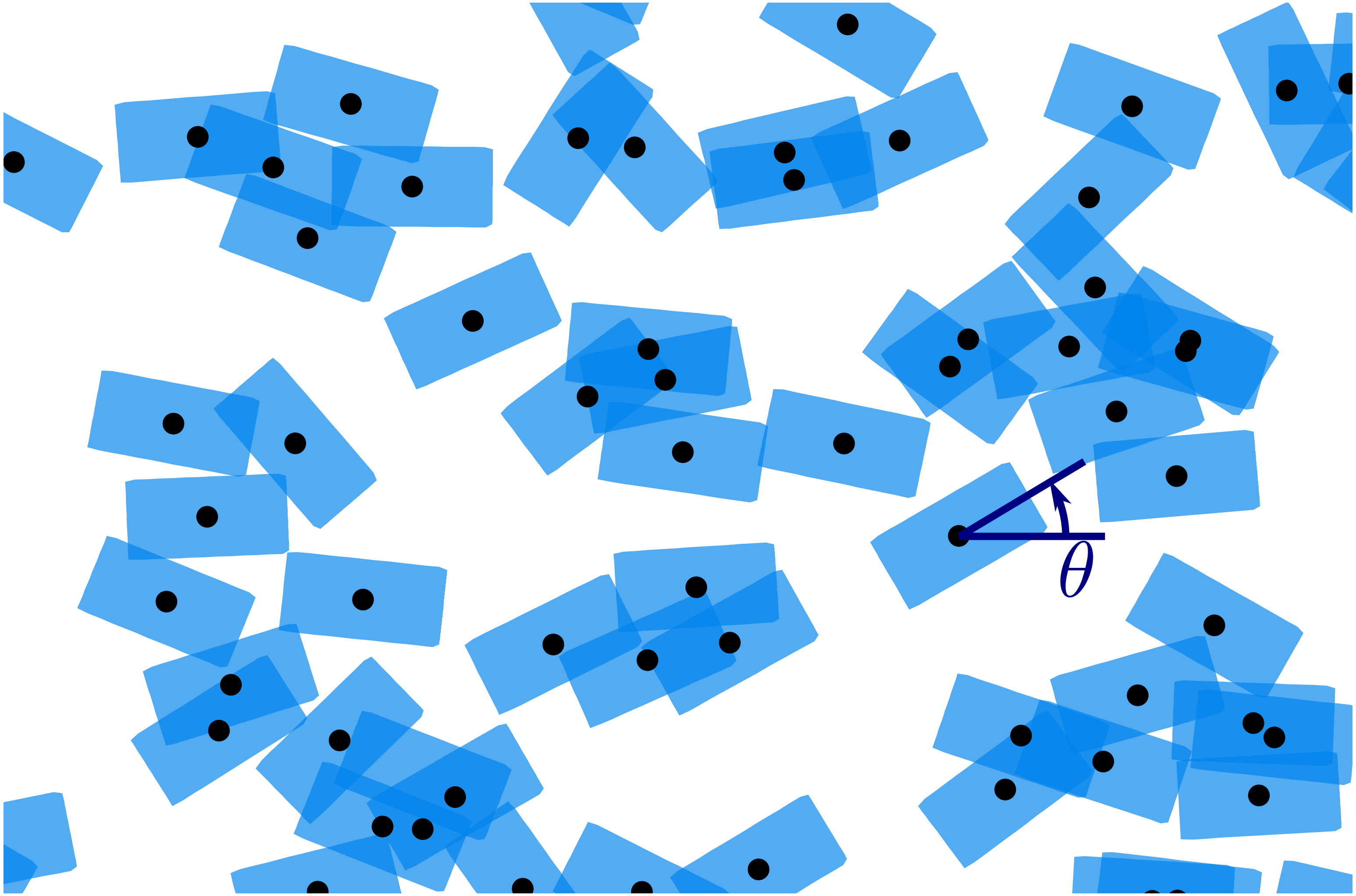
Boolean model as anisotropic porous media



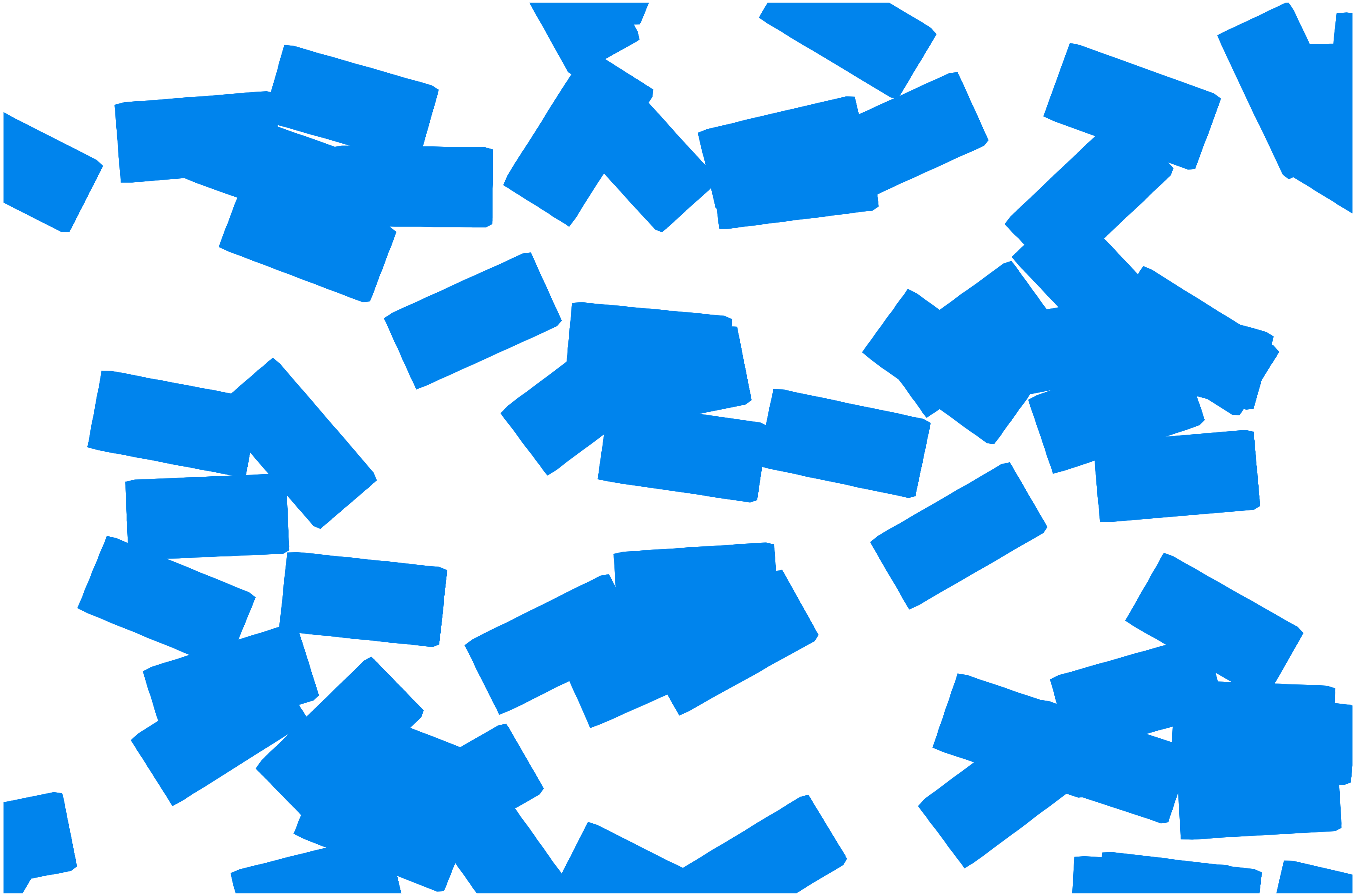
Homogeneous Poisson point process

Boolean model as anisotropic porous media



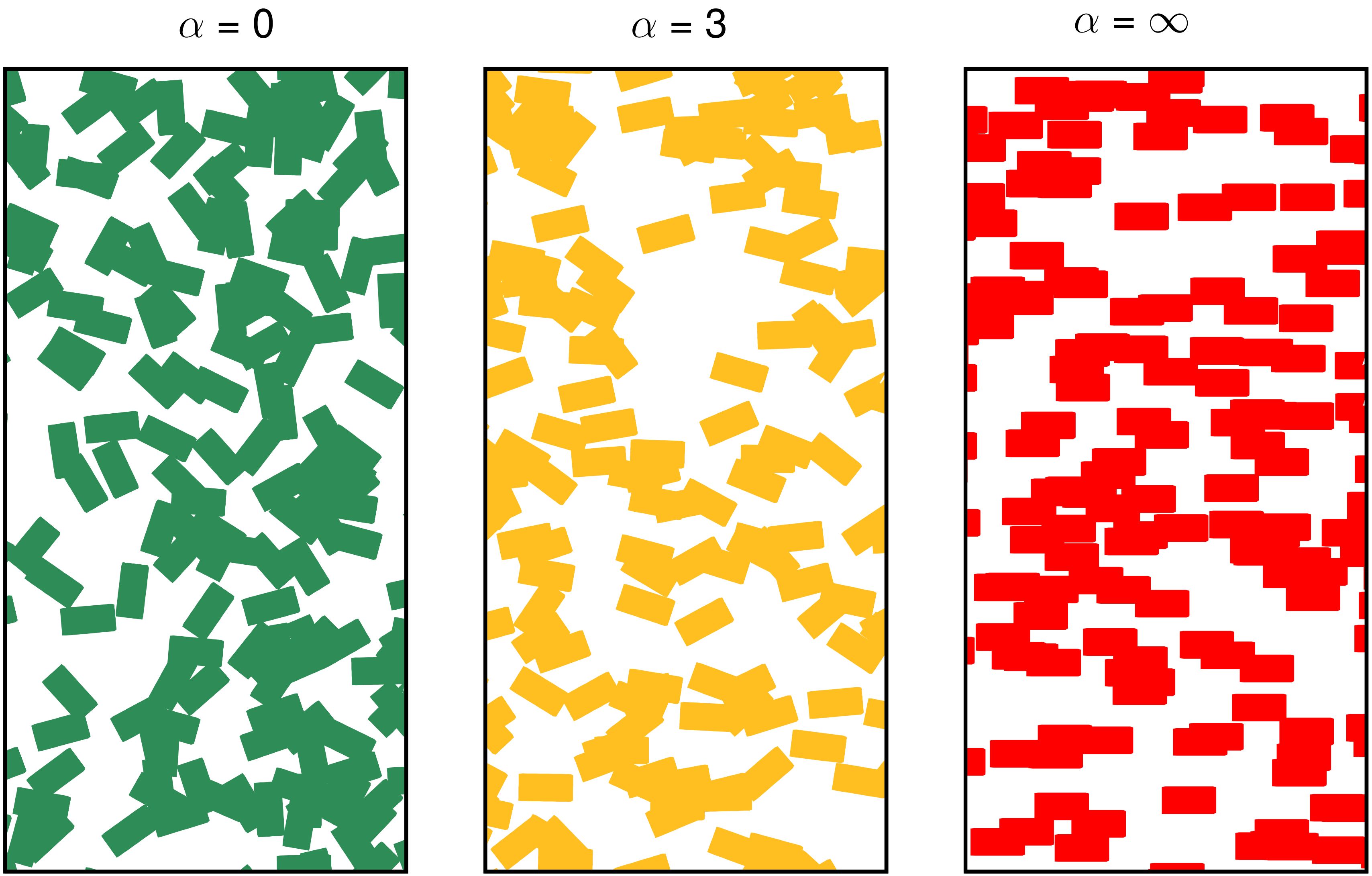
A grain with random orientation $\mathcal{P}(\theta) \propto \cos^{\alpha}\theta$ is assigned to each point

Boolean model as anisotropic porous media

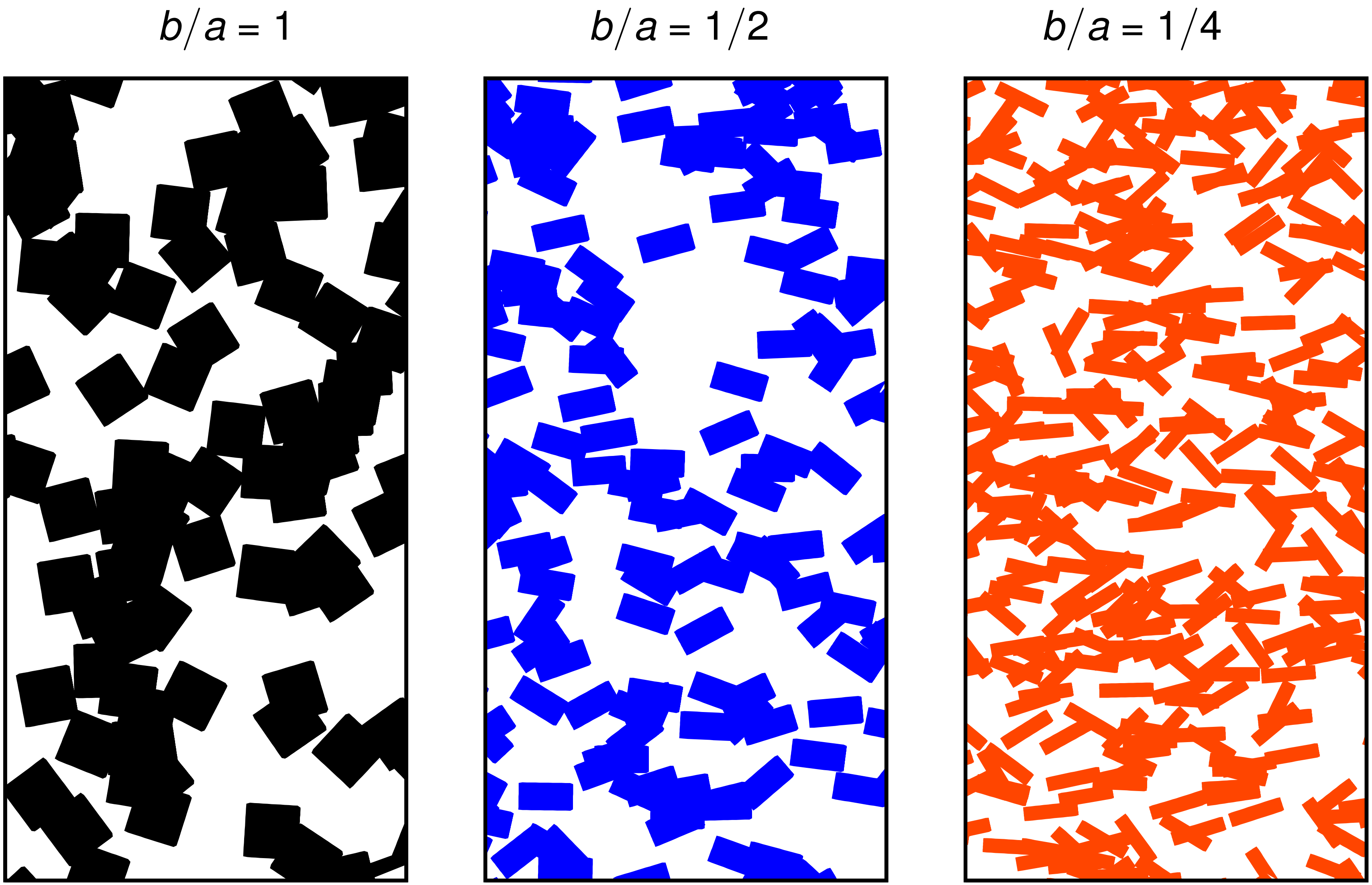


The union of all grains forms Boolean model

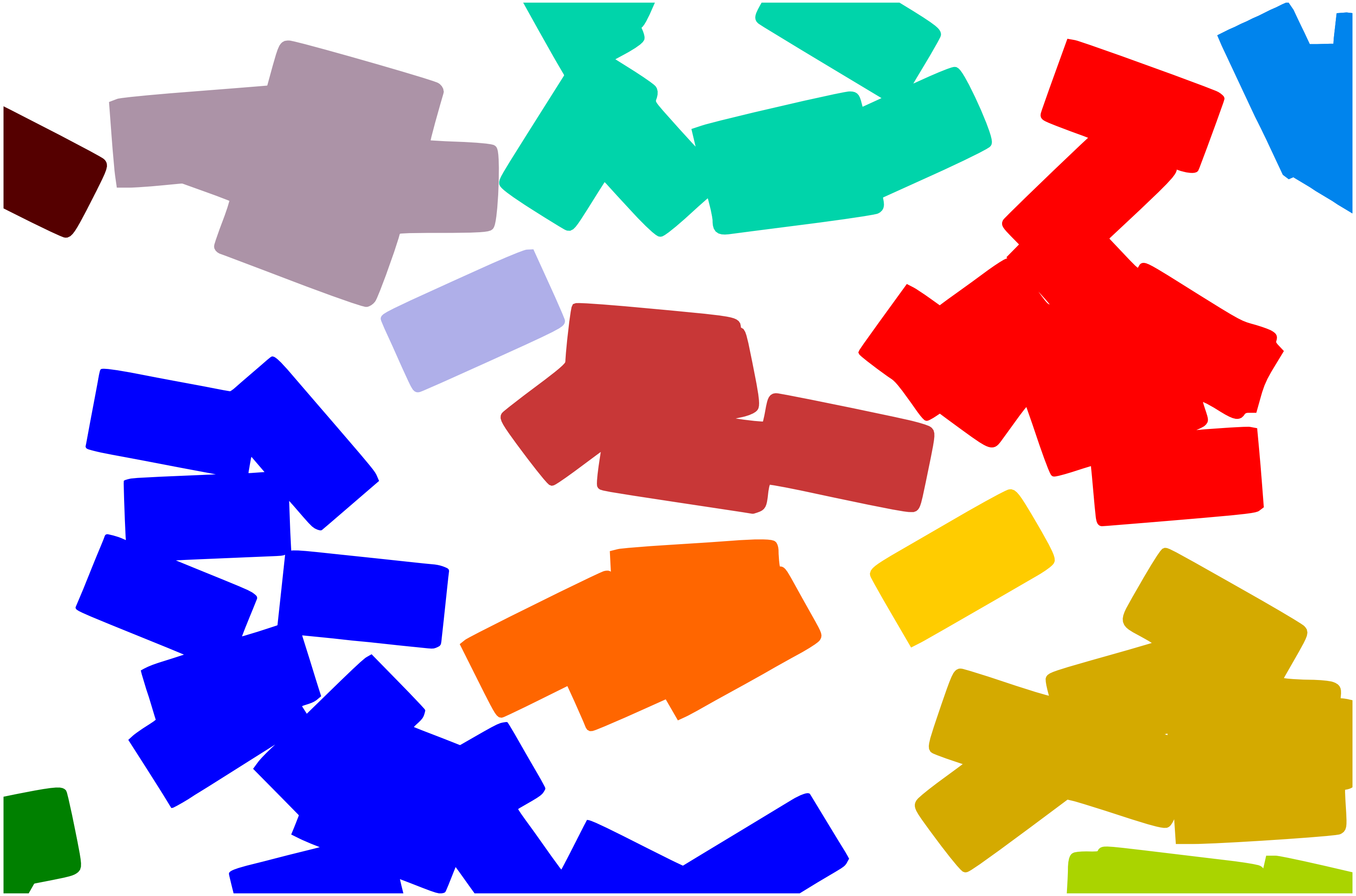
Anisotropy parameters: orientation bias $\mathcal{P}(\theta) \propto \cos^{\alpha}\theta$



Anisotropy parameters: aspect ratio b/a

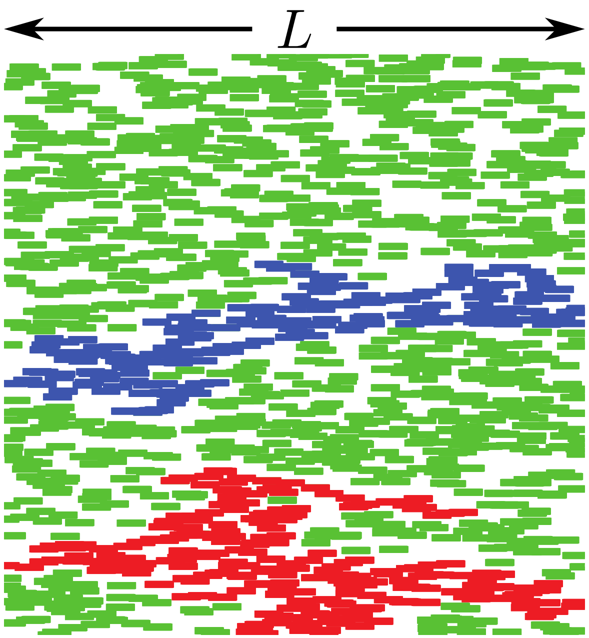


Continuum percolation



Clusters of connected components, that is, intersecting grains

Effective percolation in finite systems



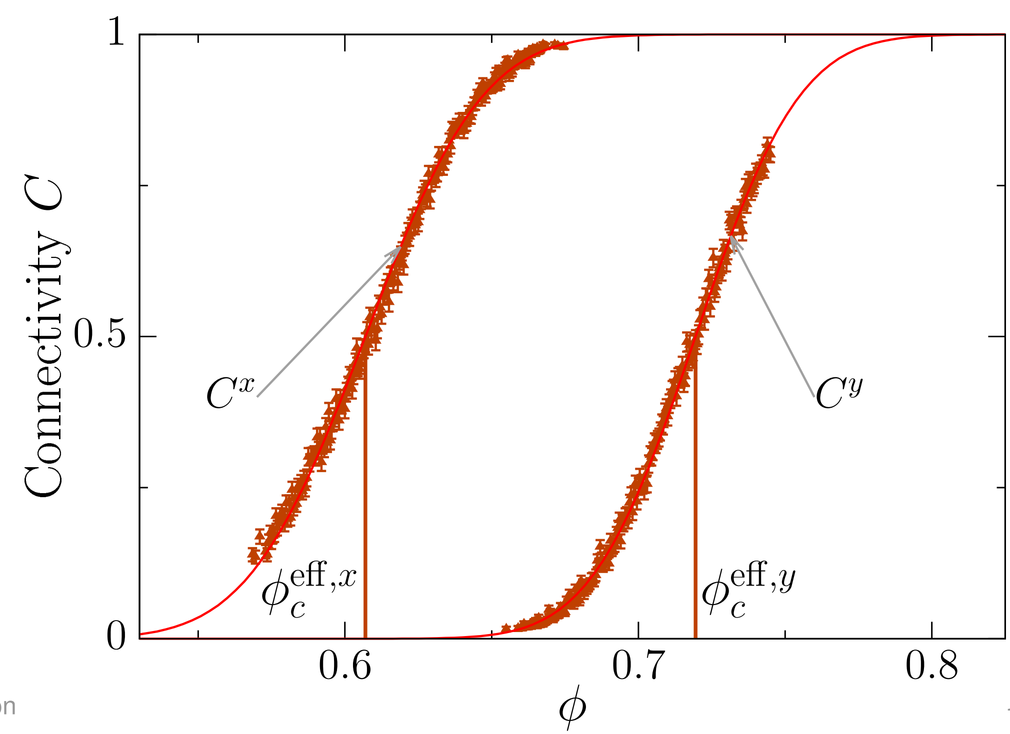
Connectivity C:
probability that the system percolates;
depends on area fraction ϕ occupied by grains

$$C(\phi, L) \approx \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{\phi - \phi_c^{\mathrm{eff}}(L)}{D(L)}\right) .$$

Finite "width" $D(L)$

Effective percolation threshold $\phi_c^{\mathrm{eff}}(L)$

- There is no phase transition in finite systems
- Percolating cluster spans the system in x- or in y-direction



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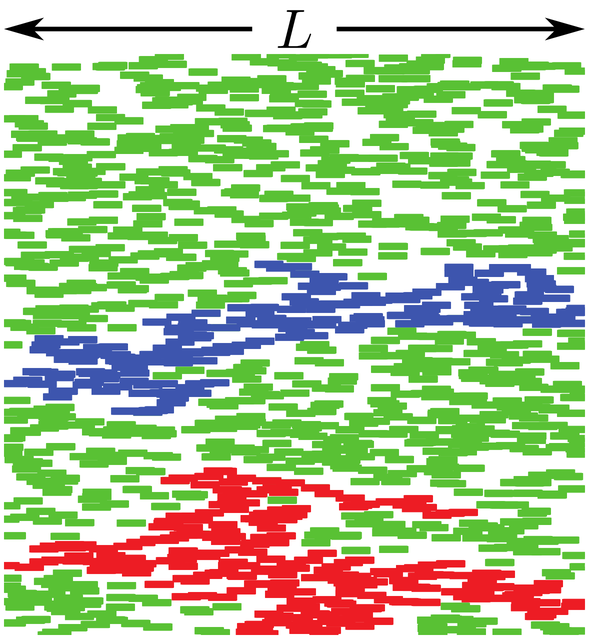

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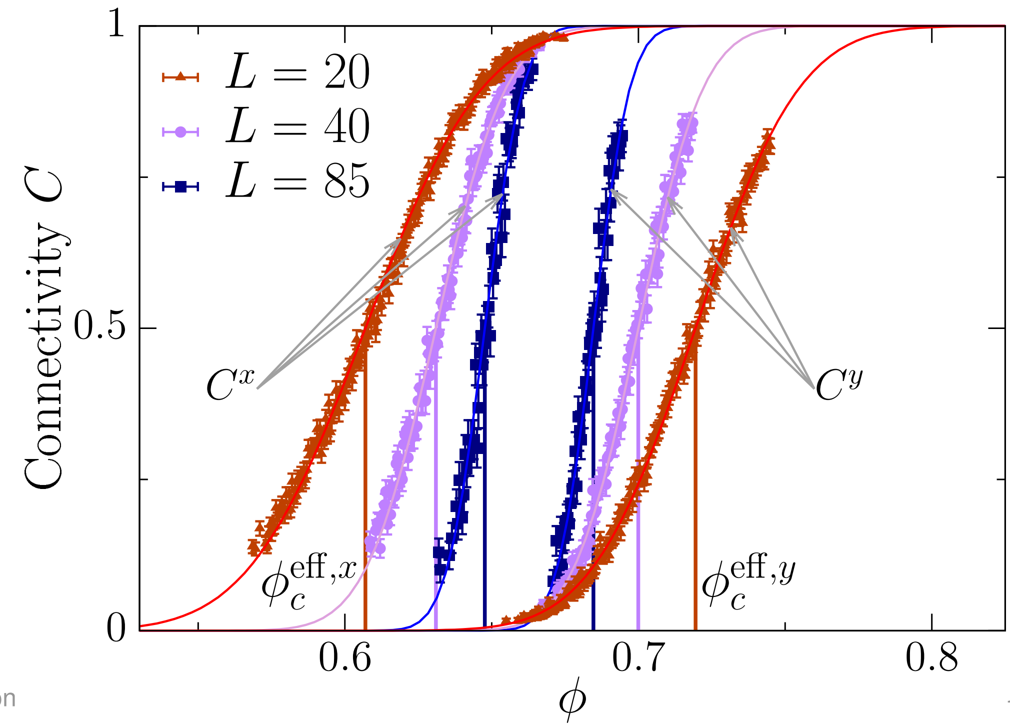
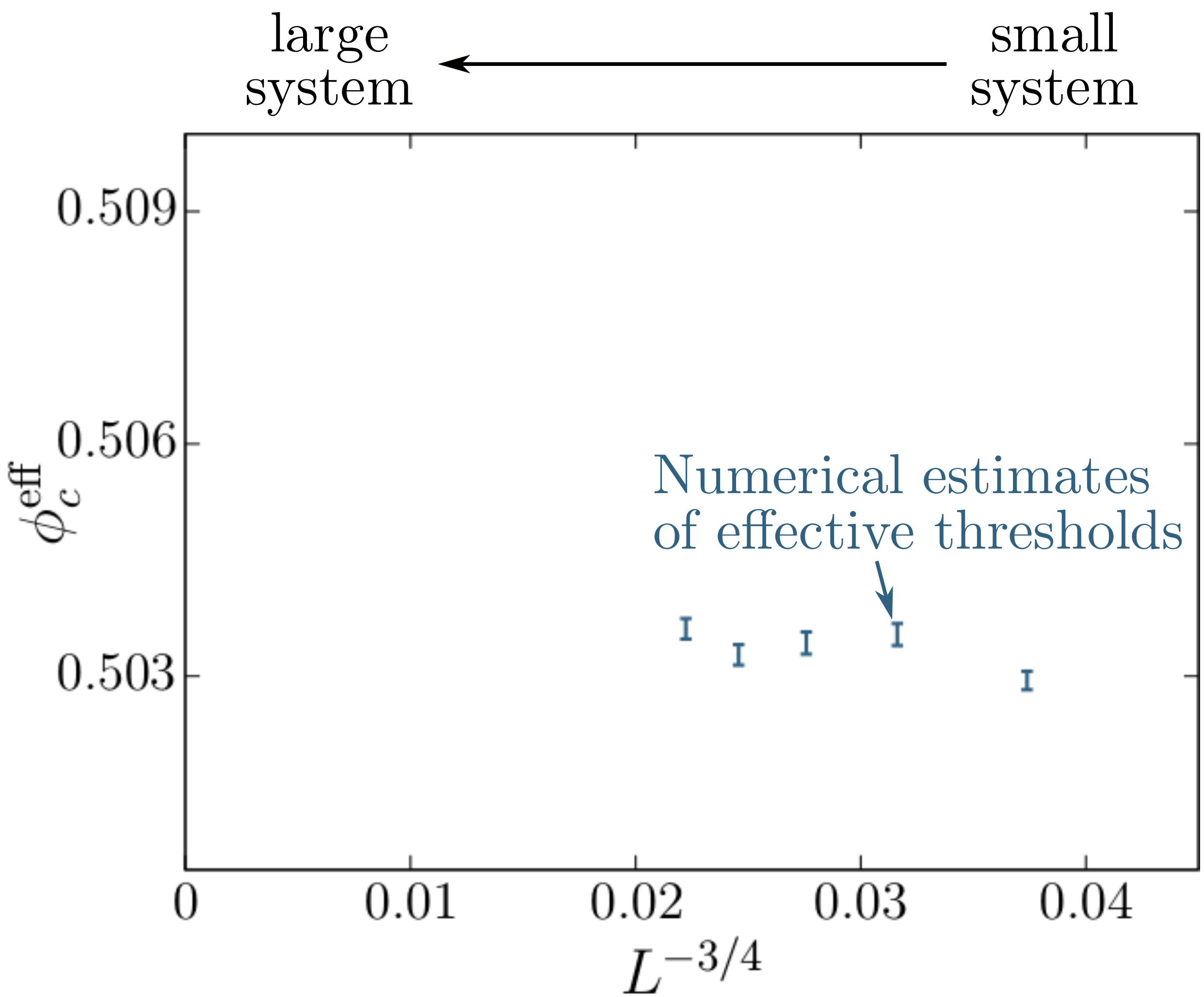


Finite size scaling of effective percolation threshold ϕ_c^{eff}



Assume independent perc. thresholds ϕ_c^x, ϕ_c^y in limit $L \to \infty$

$$\phi_c^{\mathrm{eff},x}(L) - \phi_c^x \propto L^{-3/4}$$
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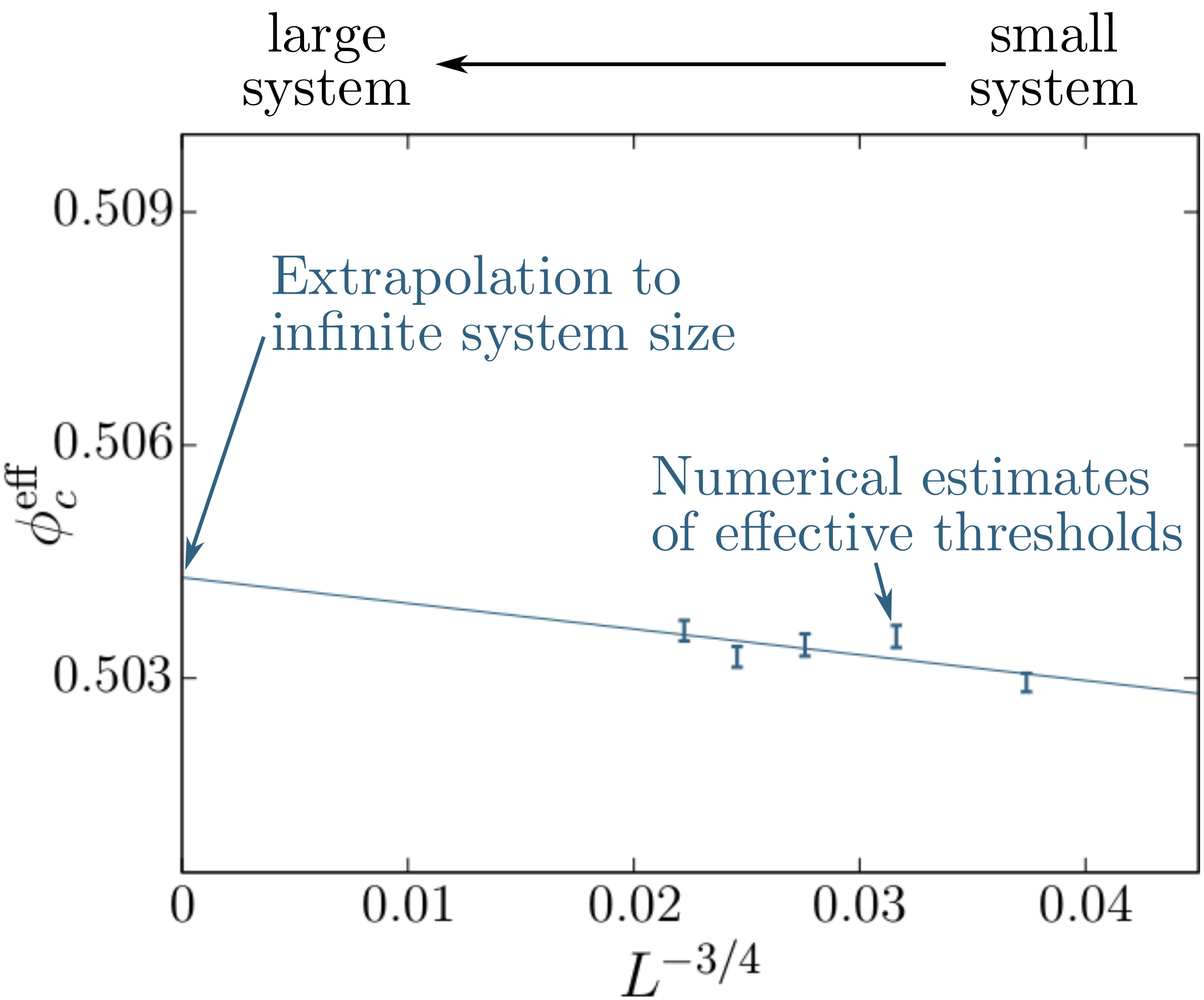

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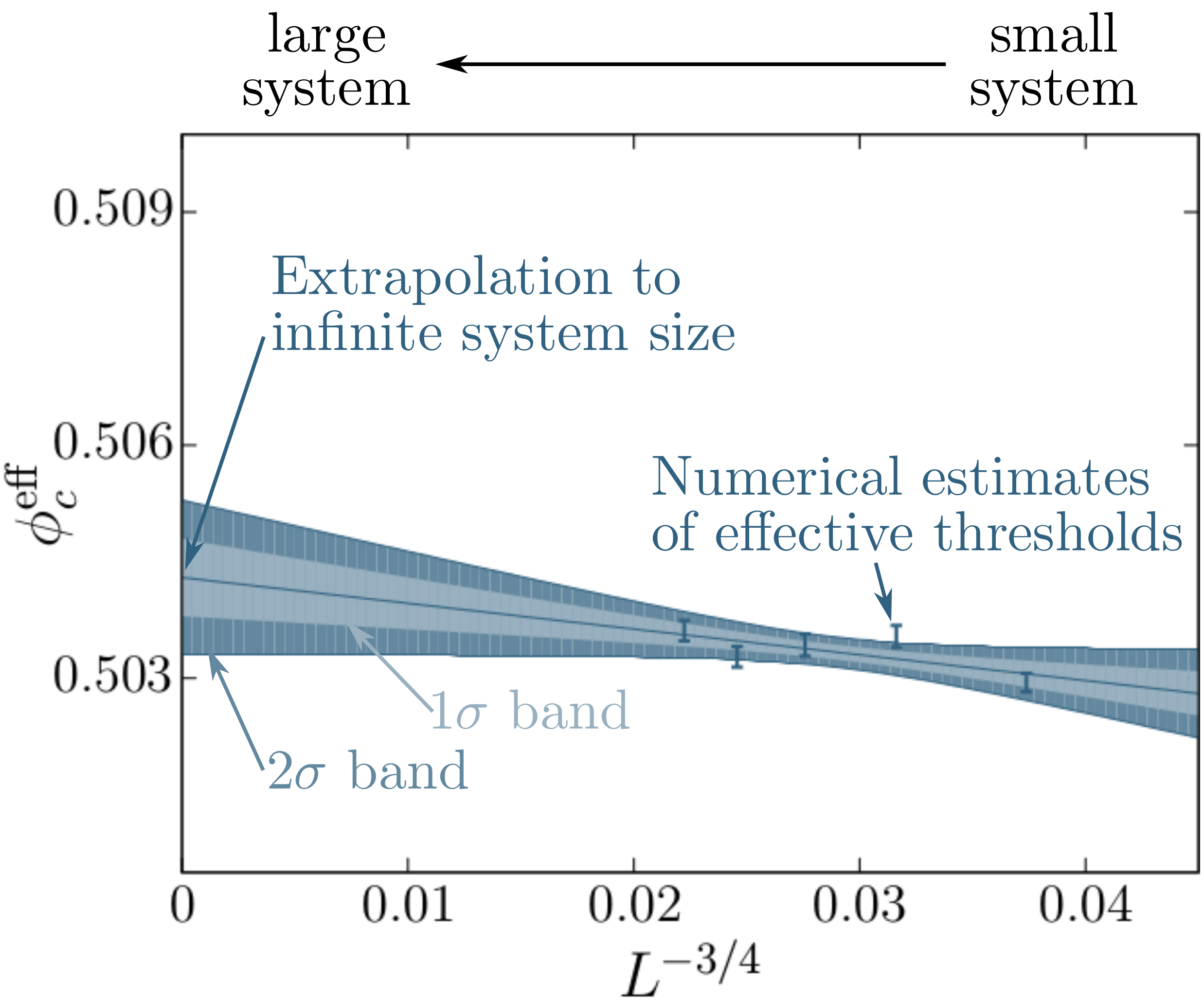

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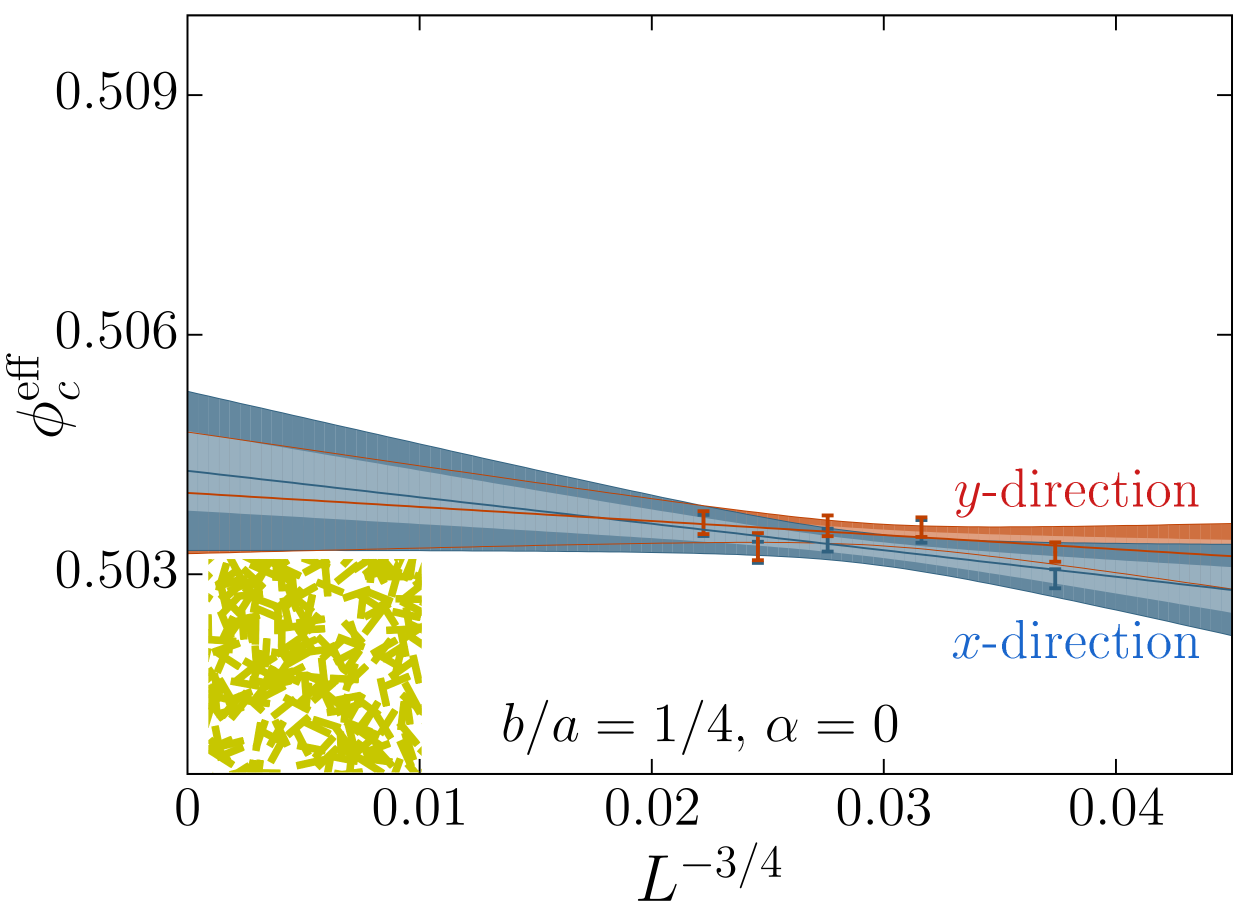

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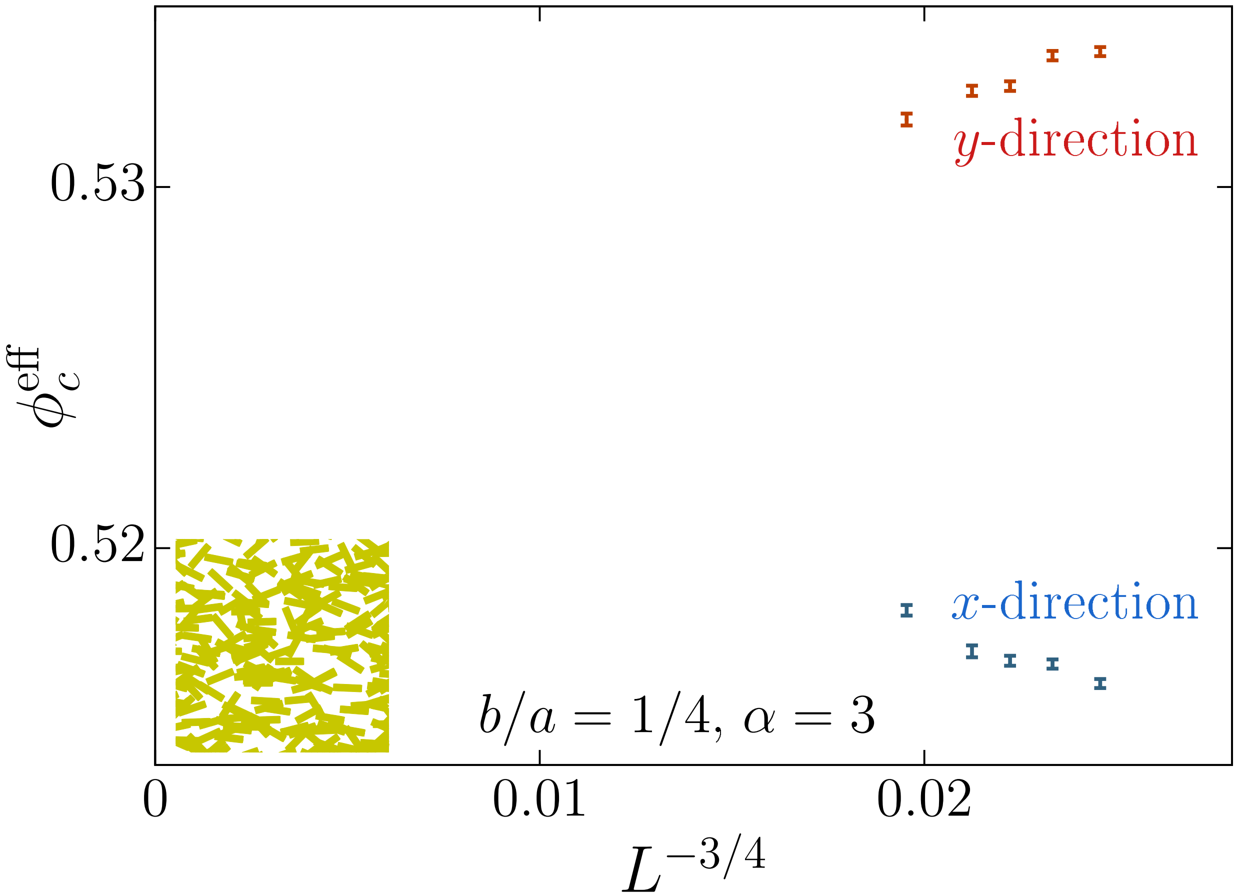

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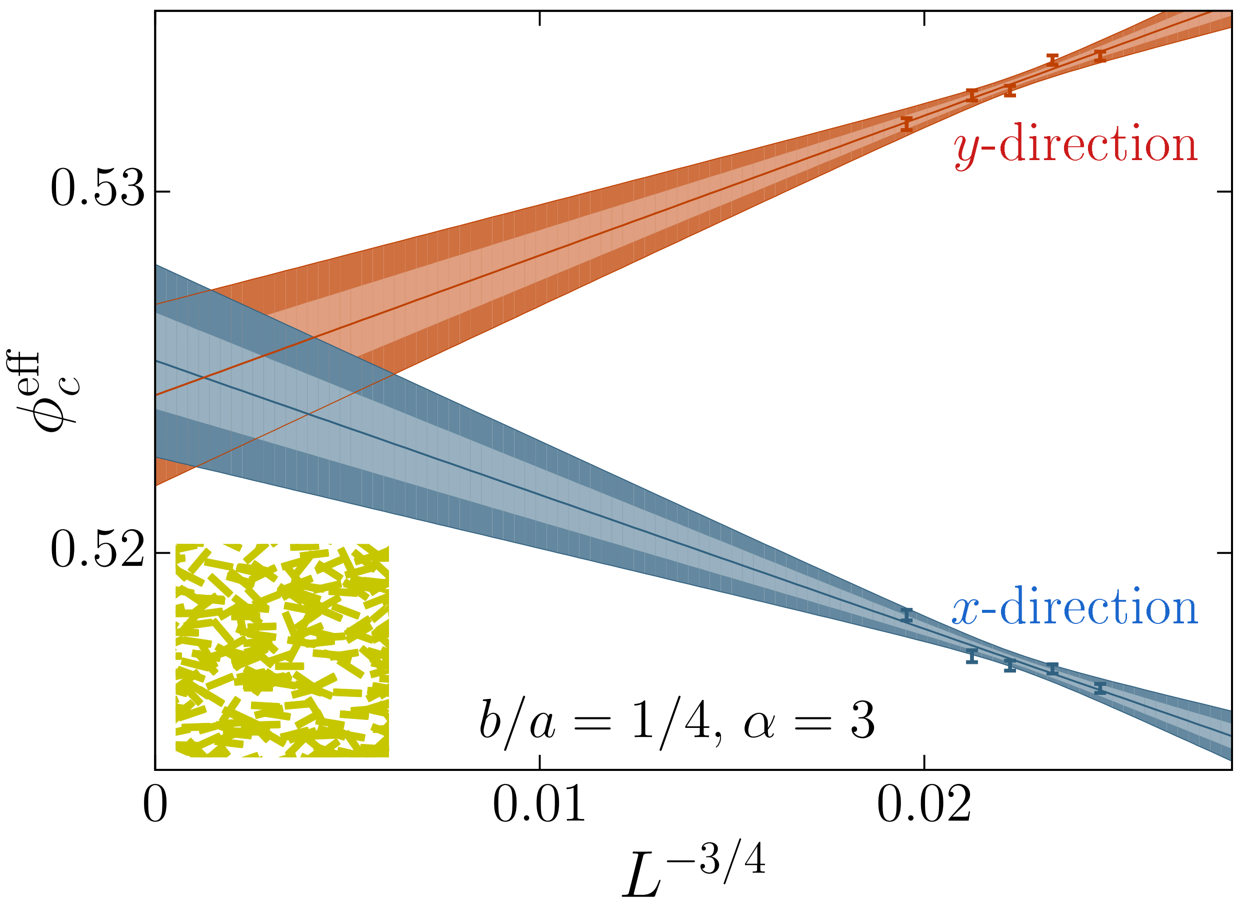

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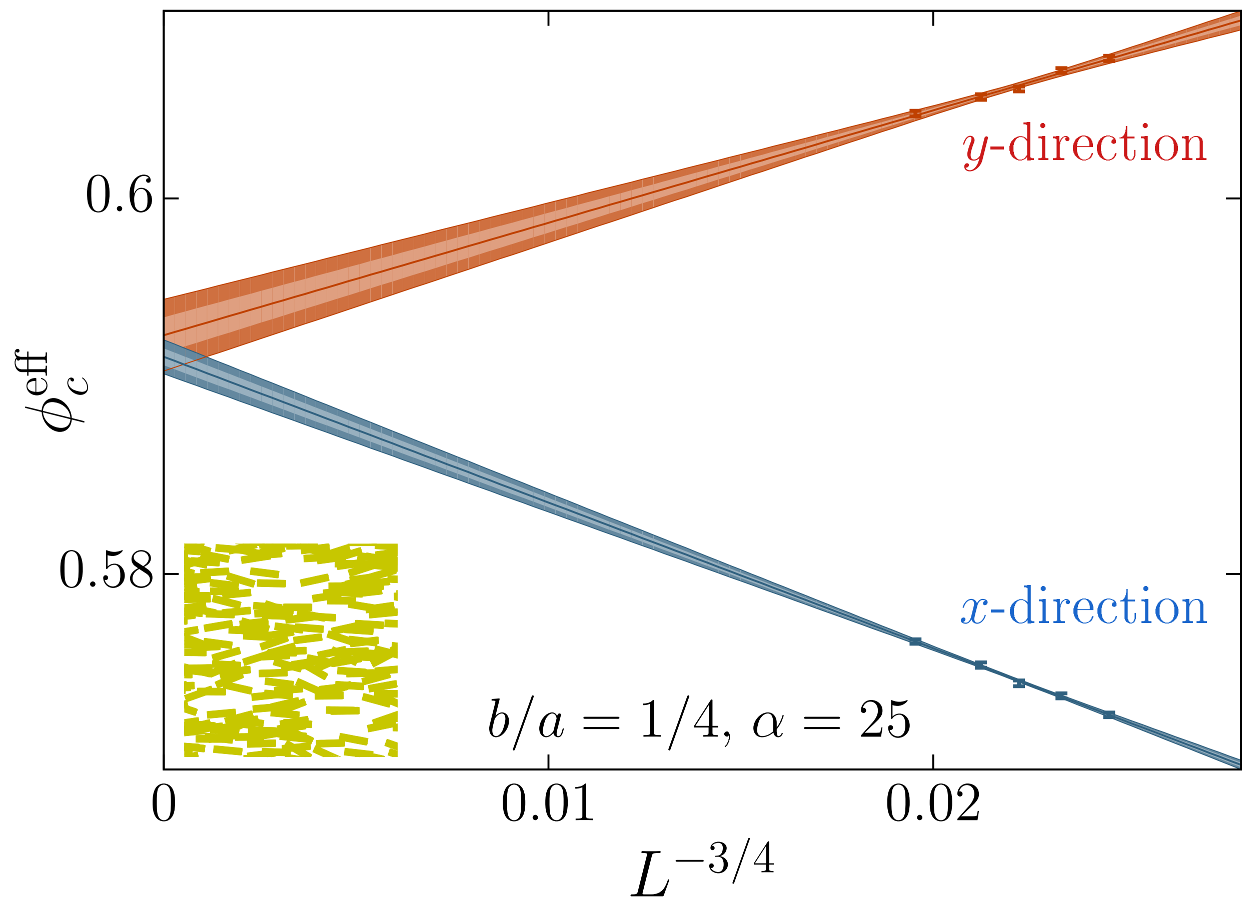

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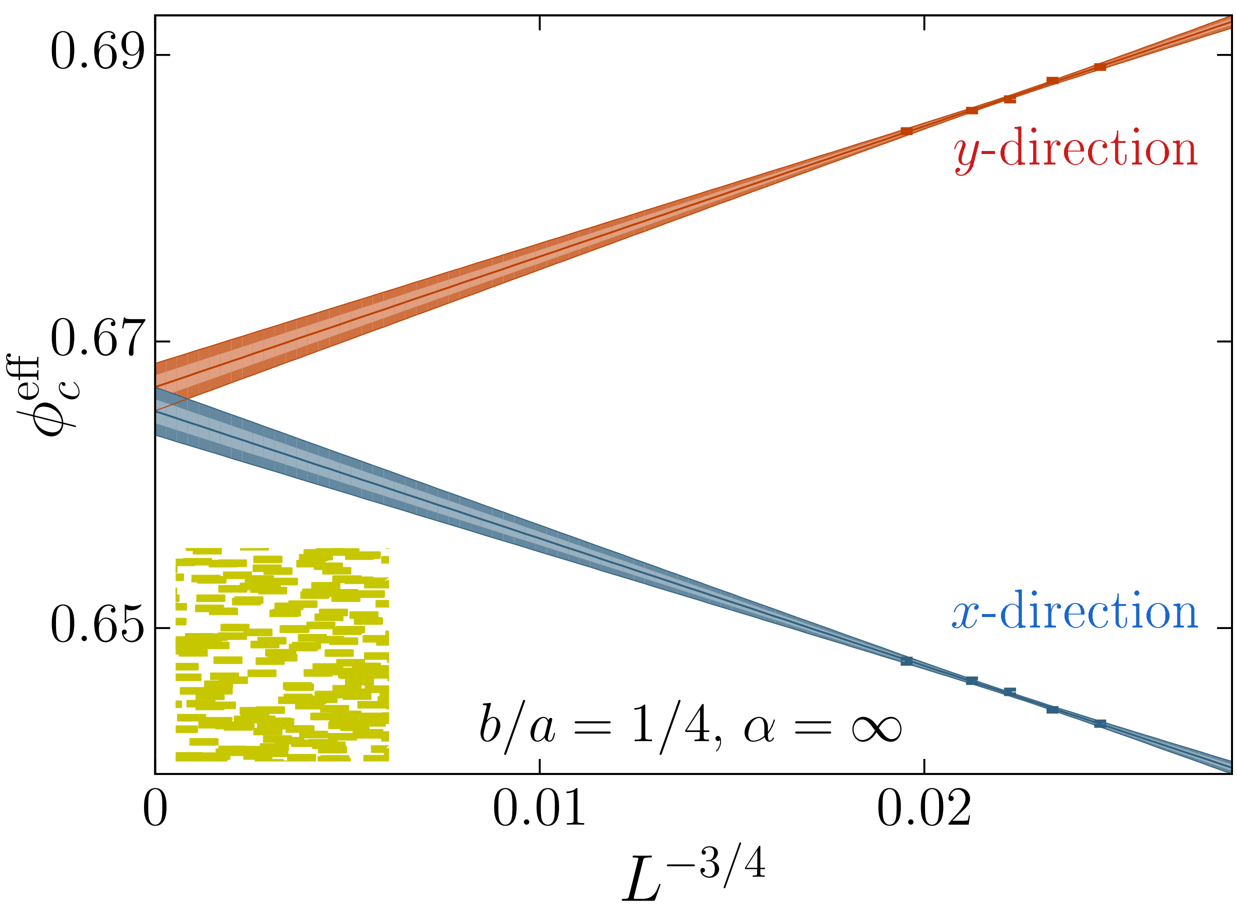

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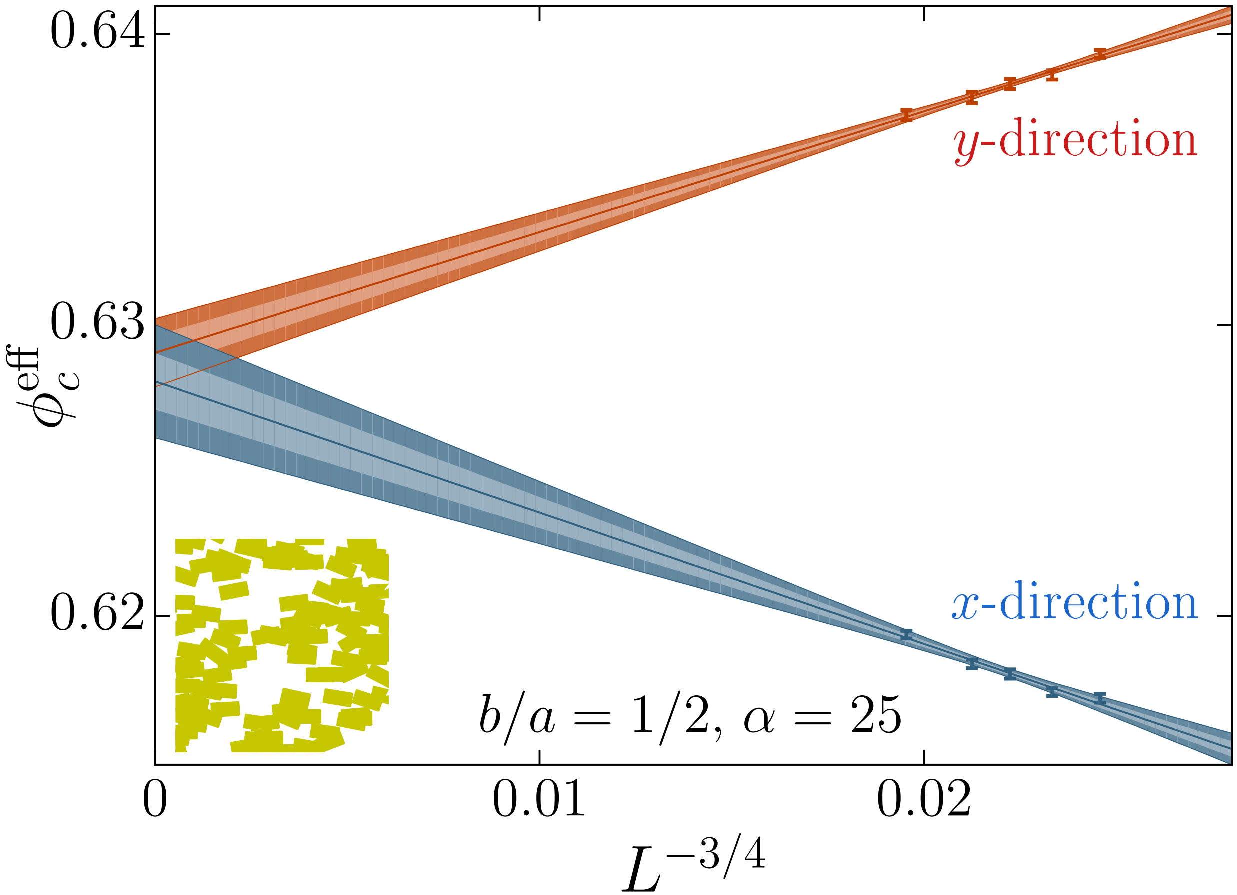

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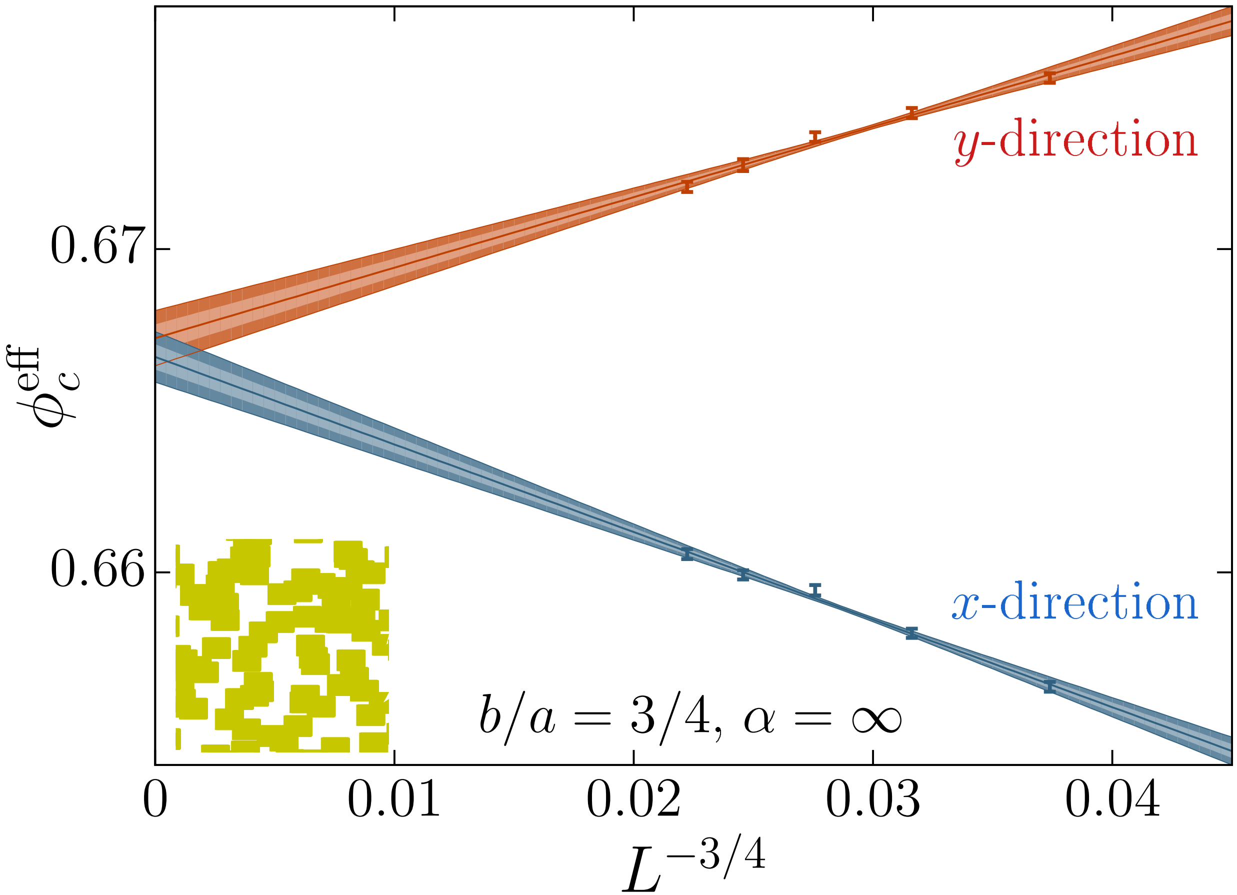

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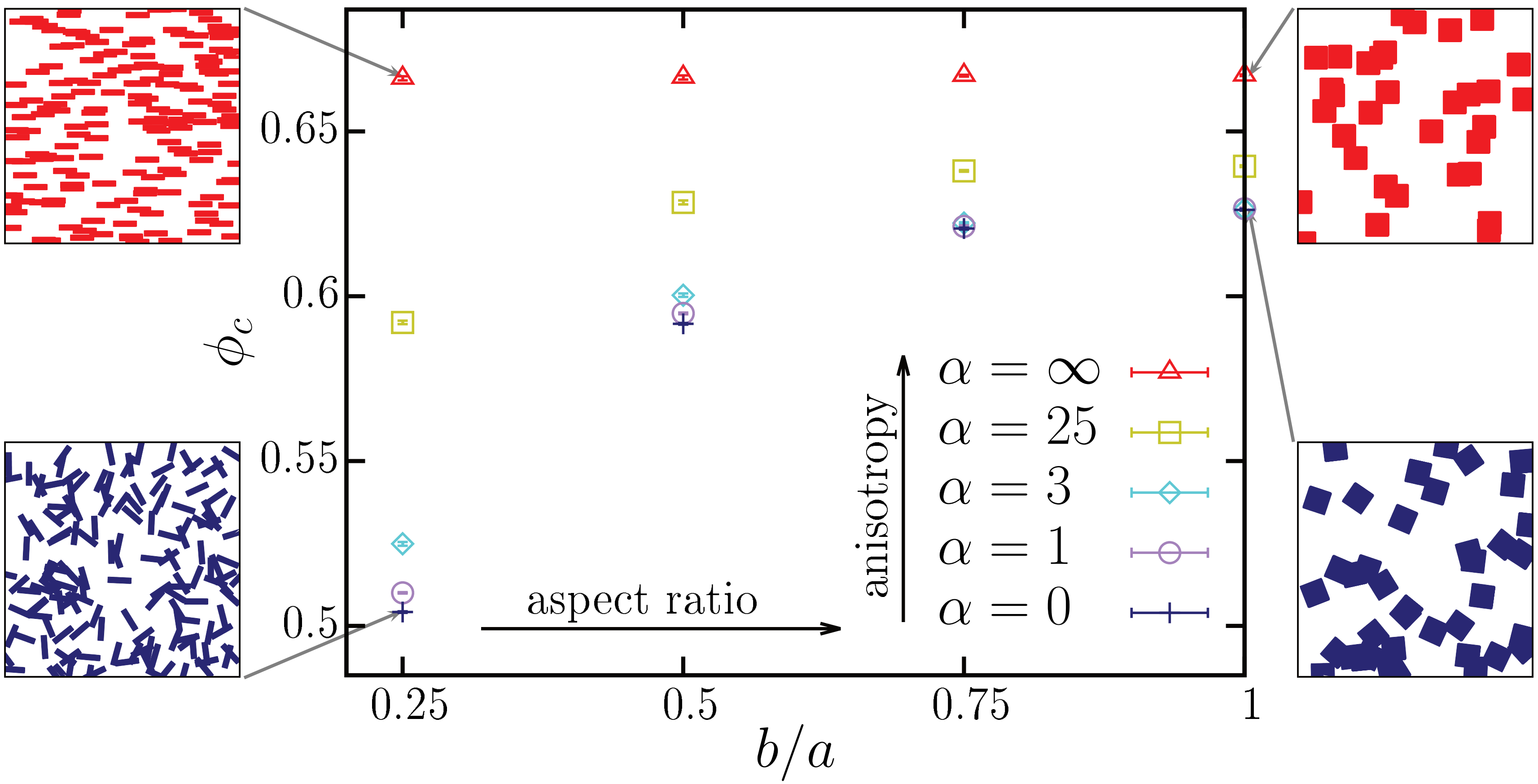
In infinite system: simultaneous percolation in x- and y-direction

Isotropic percolation threshold: $\phi_c^x \equiv \phi_c^y$

see also Balberg, Binenbaum, Phys. Rev. B (1983)

Threshold value depends on anisotropy of the system

Percolation probability depends on the grain distribution



Anisotropy and threshold estimation

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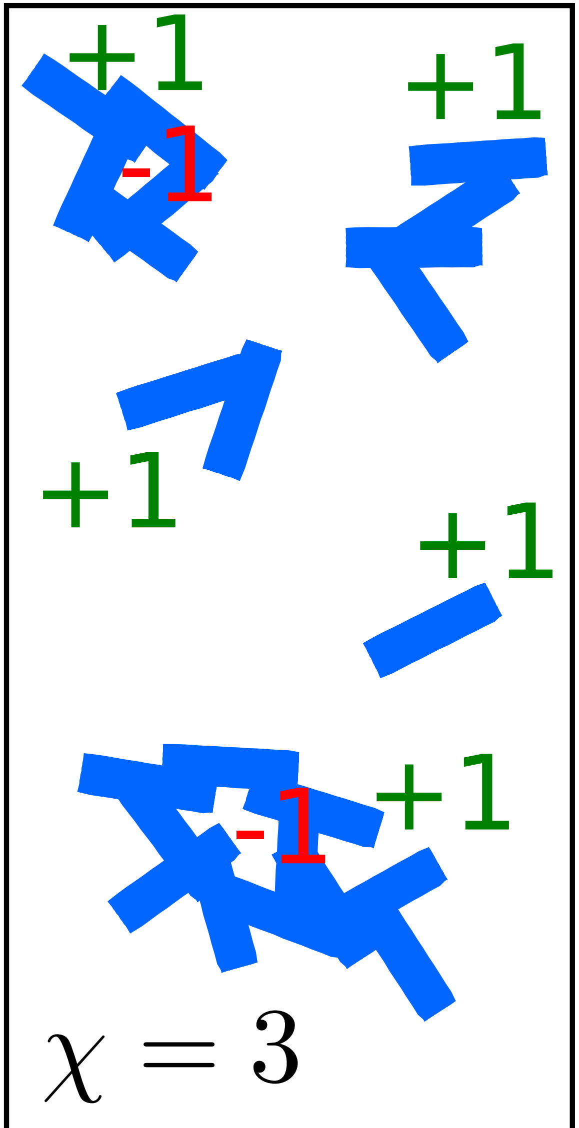
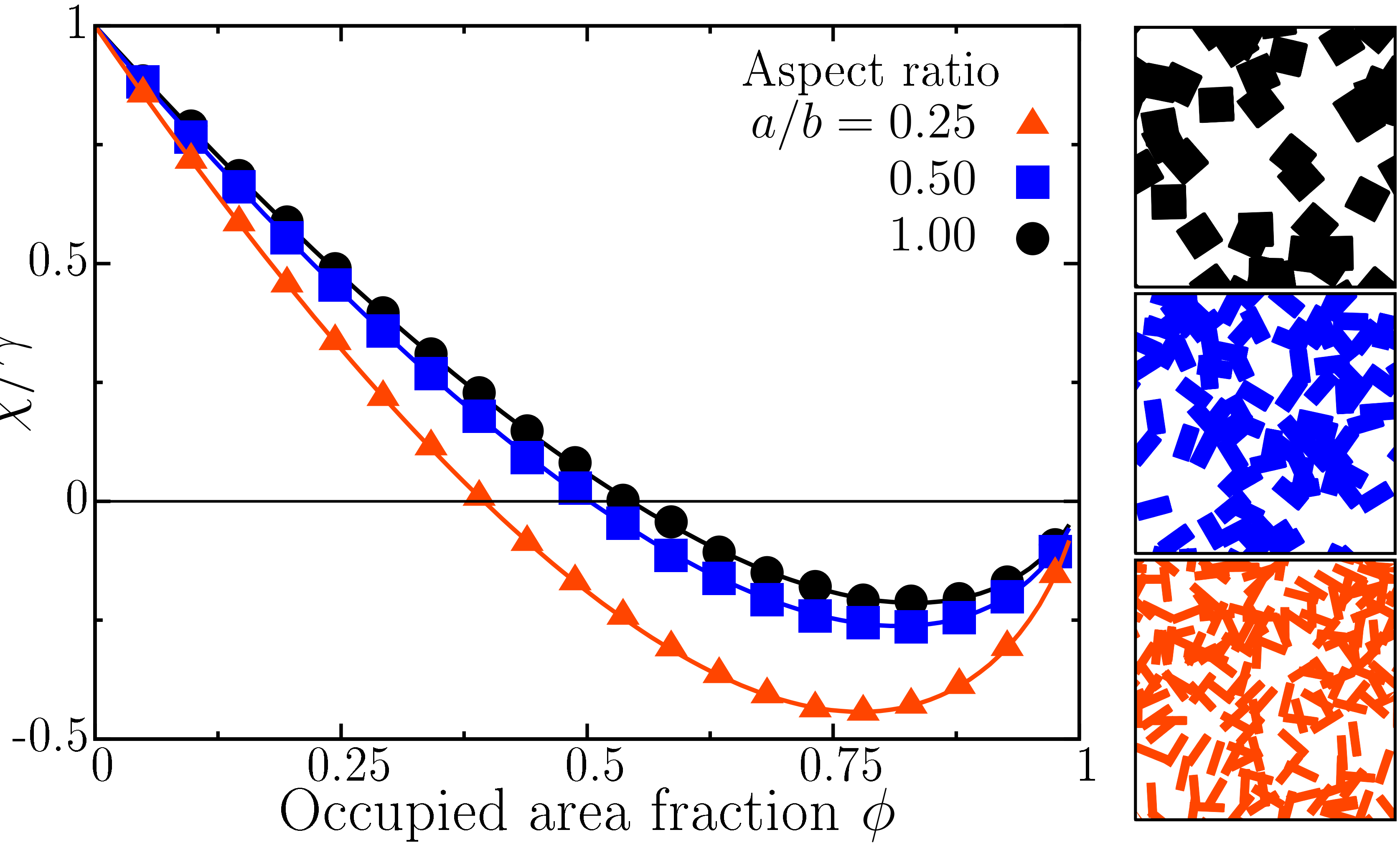
   1st and 2nd moments of Minkowski functionals

Euler characteristic χ



- Topological invariant
- For compact sets:
  χ = # connected components $-$ # holes
- Density in the limit of infinite system size $|O|$:
  $\overline{\chi} := \lim_{|O|\to\infty} \frac{\mathbb{E}[\chi]}{|O|}$
- $\overline{\chi} > 0$: predominant role of single clusters
  $\overline{\chi} < 0$: network-like structure
- Density of the Euler characteristic as a function of the intensity γ changes its sign.
- Mecke, Wagner (1991) suggested the zero of the Euler characteristic as an approximation of the percolation threshold.

Density of the Euler characteristic



Hörrmann, Hug, K., Mecke, *Adv. in Appl. Math.* (2014)
Weil, *Math. Z.* (1990)

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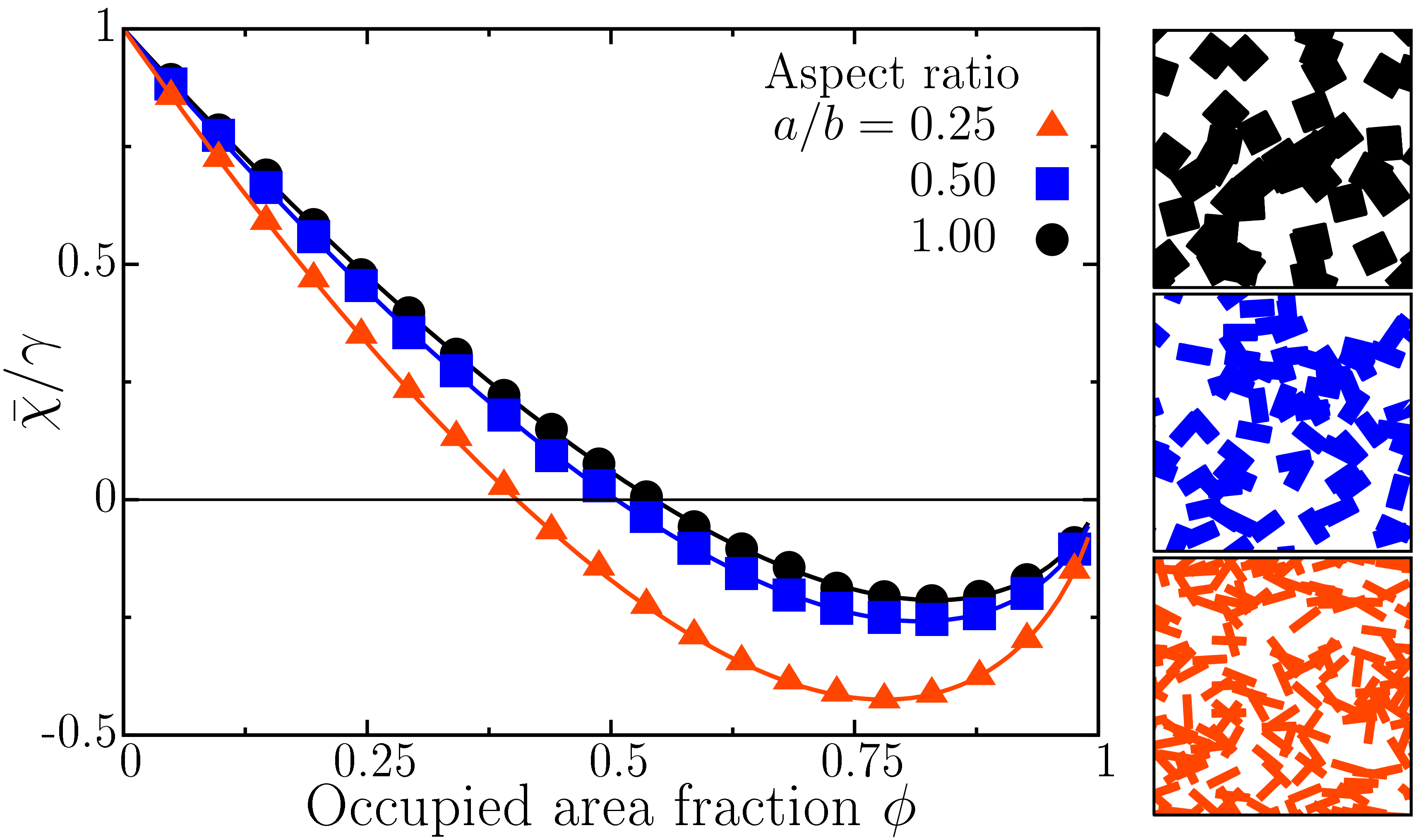

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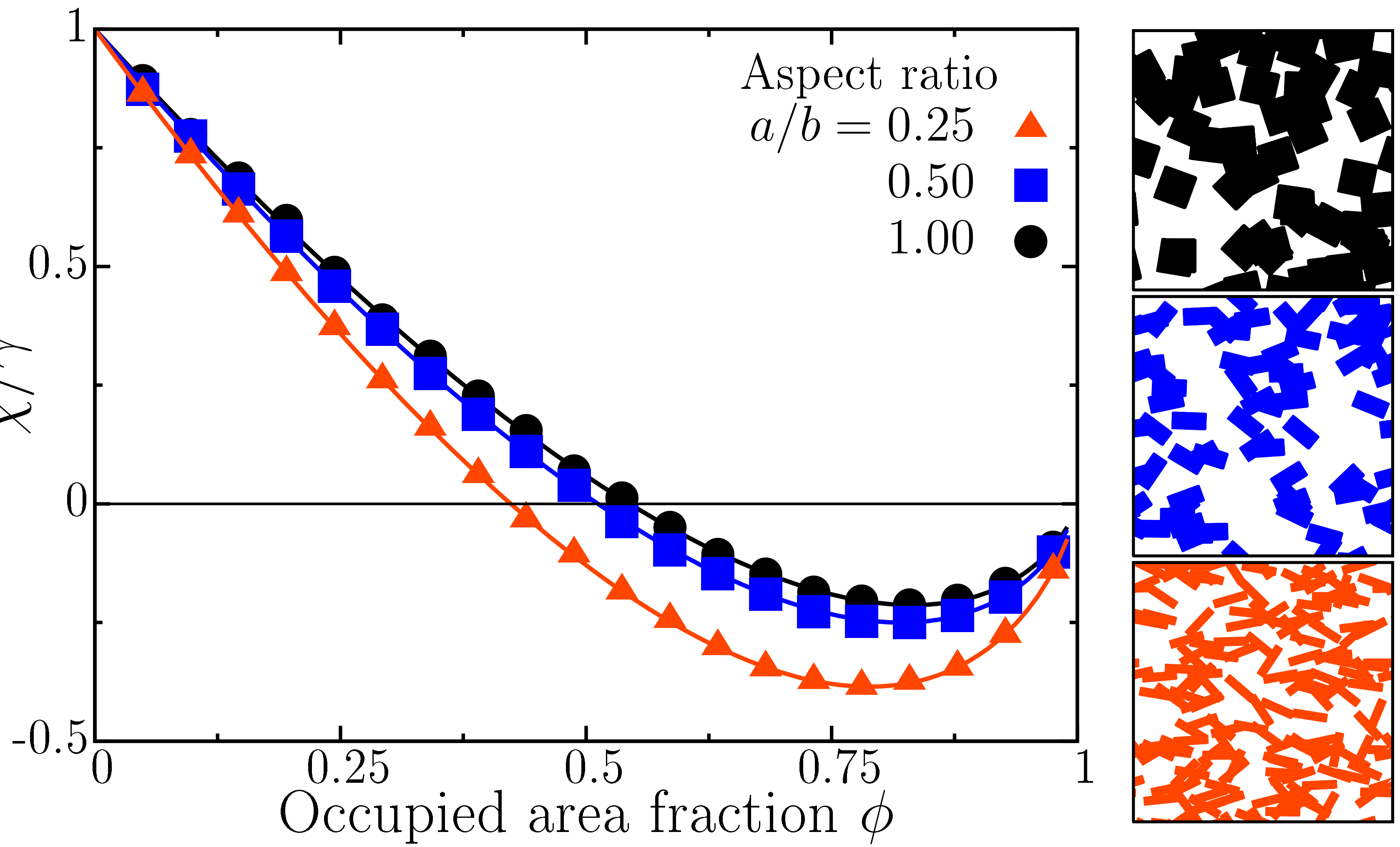

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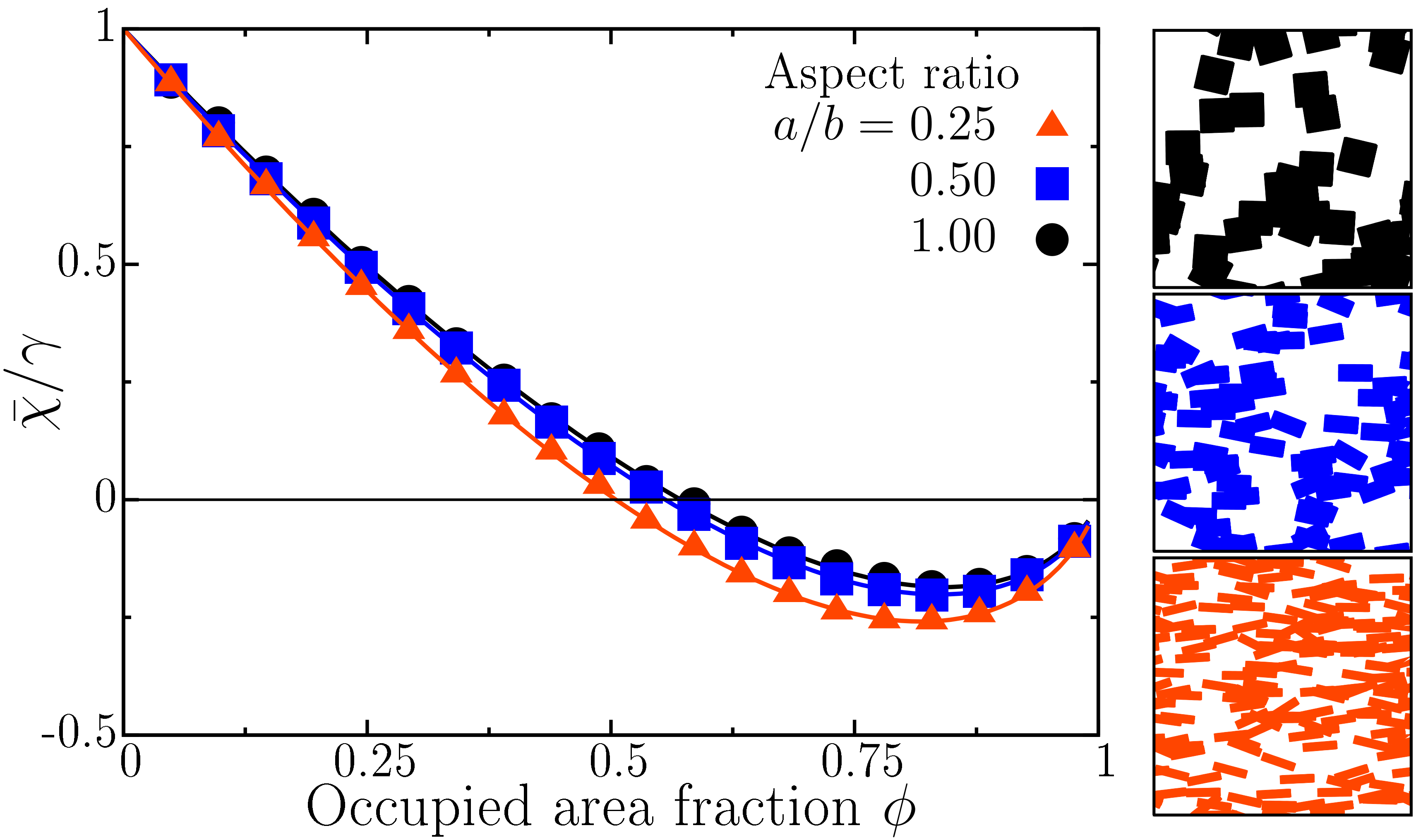

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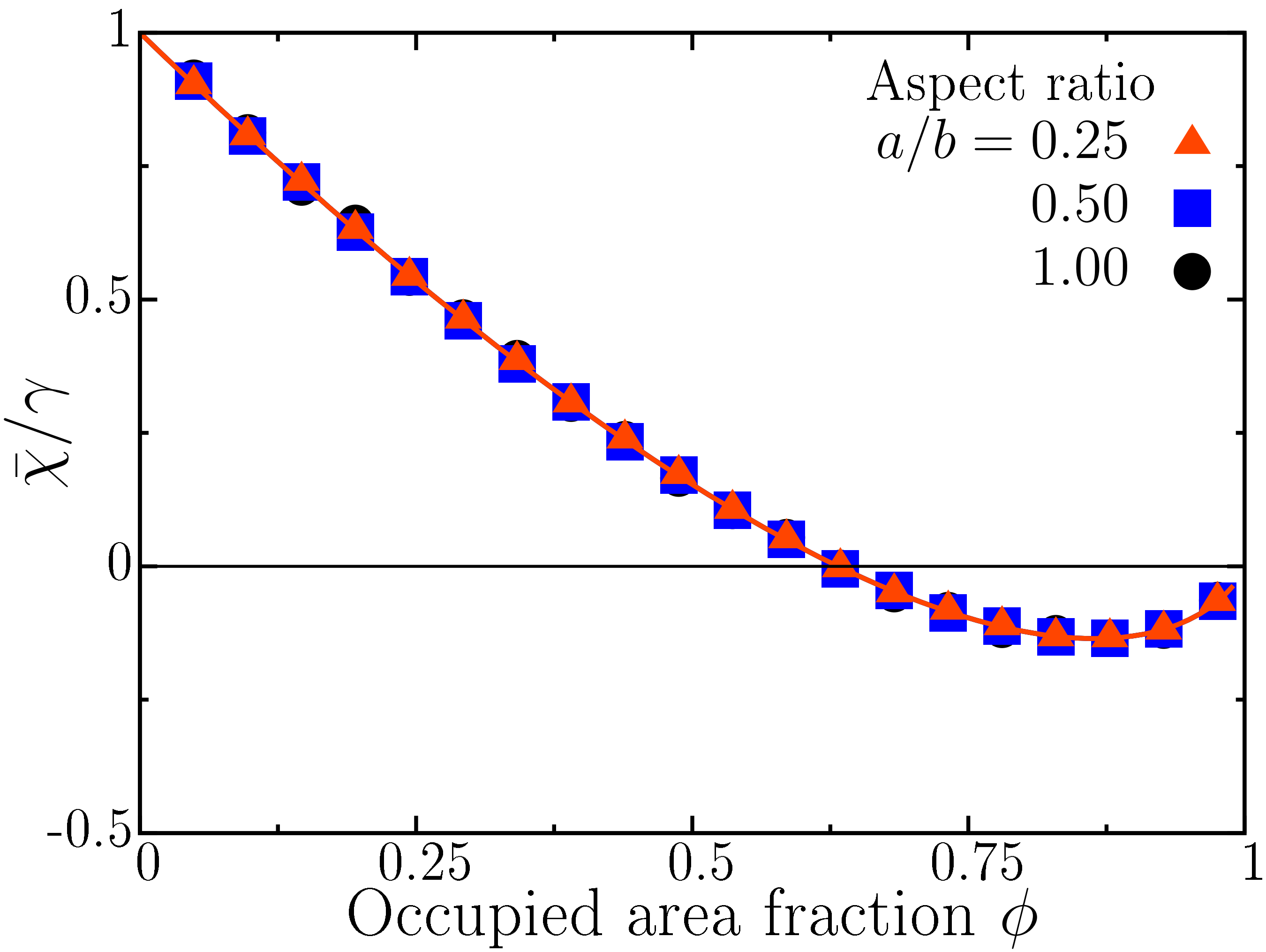

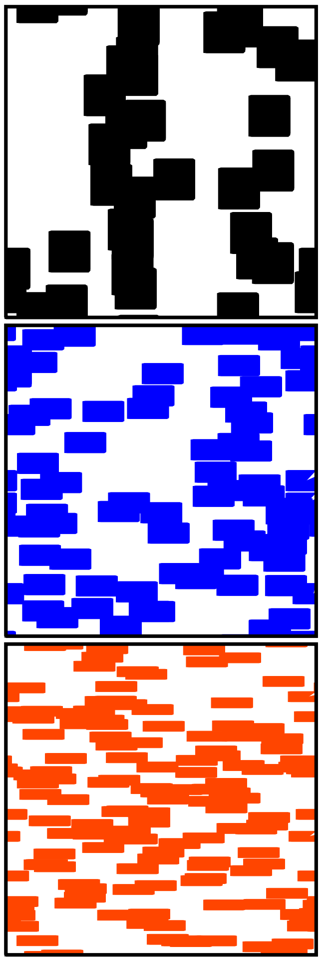

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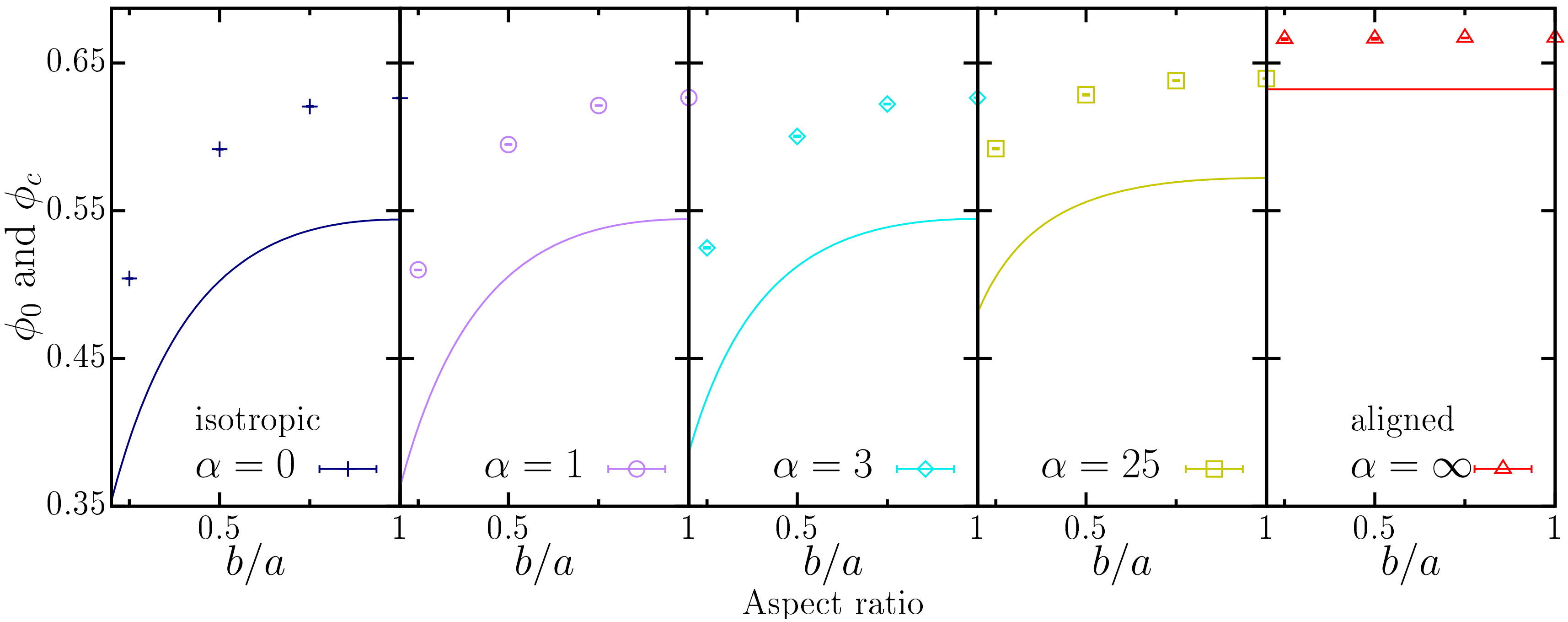
Zero of density of Euler characteristic

Explicit function of intensity γ or occupied area fraction ϕ (Weil 1990):

$$\overline{\chi}(\gamma) = \gamma\left(1 - \frac{\gamma}{2}\langle V^0_{1,1}\rangle\right)\exp(-\gamma A),$$

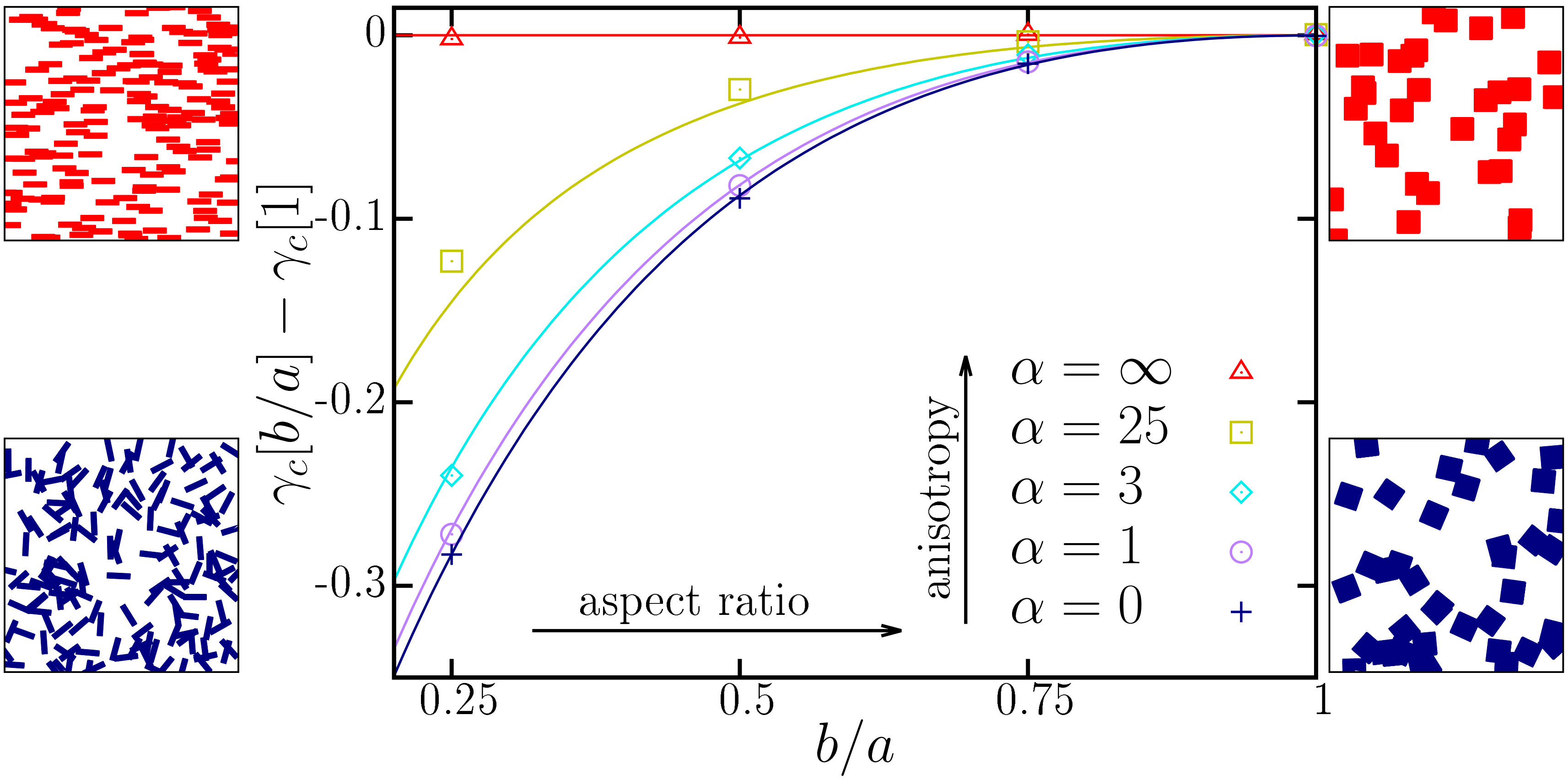
where $\langle V^0_{1,1}\rangle$ is the average of the mixed intrinsic volume of two grains and A the expected area of the typical grain.

$$\chi(\gamma_0) = 0 \Leftrightarrow \gamma_0 = \frac{2}{\langle V^0_{1,1}\rangle} \Rightarrow \phi_0 := 1 - \mathrm{e}^{-\gamma_0 A}$$



Area frac. ϕ_0 with $\chi(\phi_0) = 0$ is a putative bound on percolation threshold ϕ_c.

Qualitative behavior captured by Euler characteristic



Using the critical intensity of a Boolean model with squares allows for precise estimates for similar grain shapes.

Asymptotic variances and convariances

Minkowski functionals:

$$W_0 \propto \text{Area}$$
$$W_1 \propto \text{Perimeter}$$
$$W_2 \propto \text{Euler characterisic}$$

Covariances of Minkowski functionals W_μ rescaled by system size $|O|$:

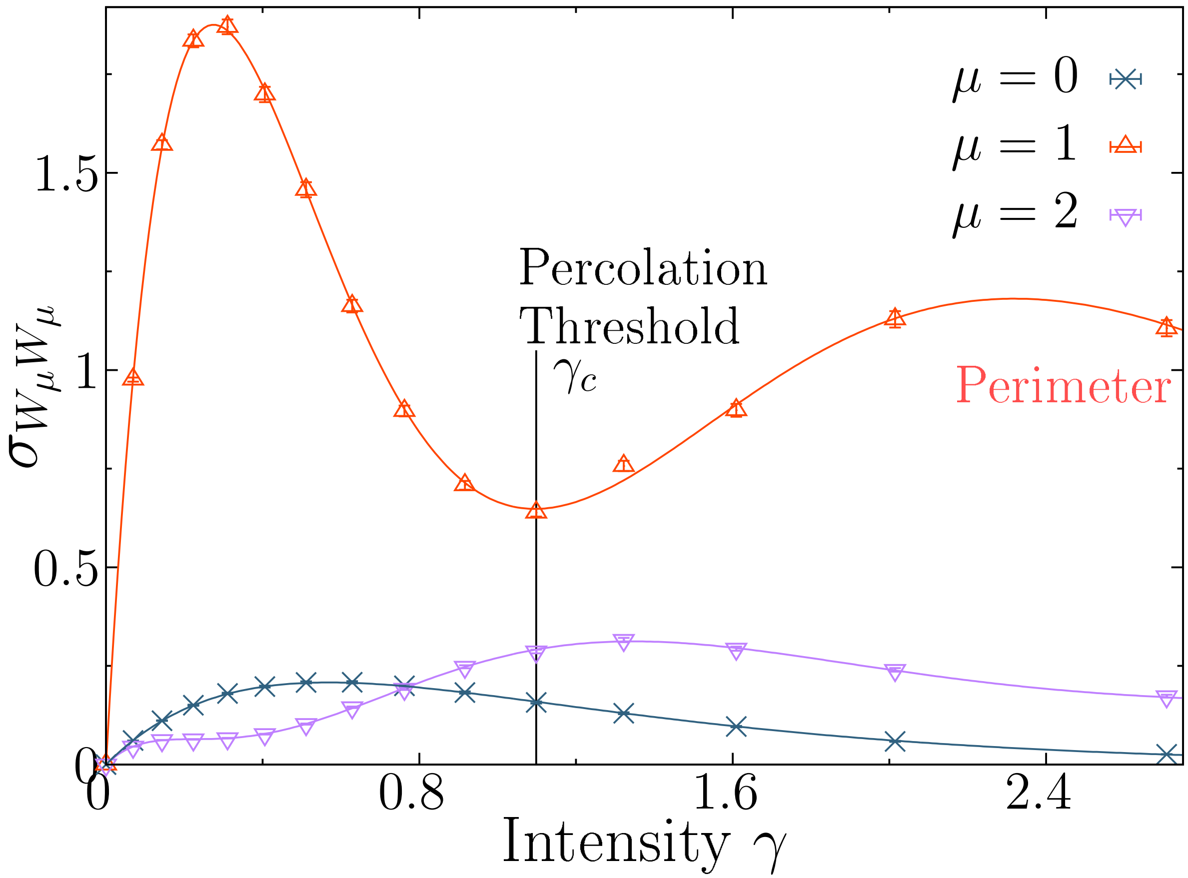
$$\sigma_{W_\mu W_\nu} := \lim_{|O|\to\infty} \frac{\mathrm{Cov}(W_\mu, W_\nu)}{|O|}$$

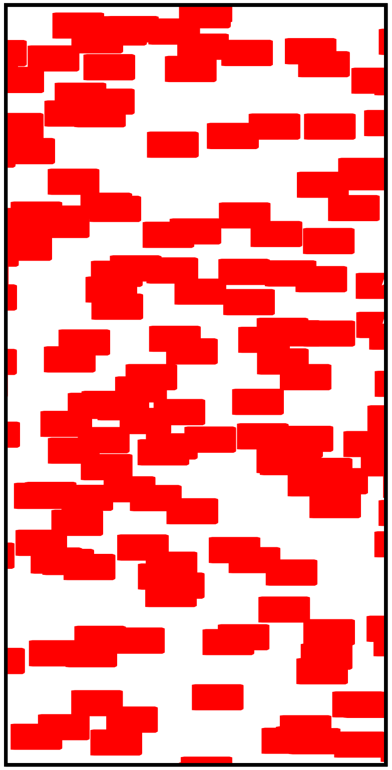
Explicit integral formulas for volume, surface area, and planar systems

Hug, Last, Schulte, *Ann. Appl. Probab.* (2016)
Hug, K., Last, Schulte, arXiv:1601.06718

Asymptotic variances: threshold approximations?

$$\sigma_{W_\mu W_\mu} := \lim_{|O|\to\infty} \frac{\mathrm{Var}(W_\mu)}{|O|}$$





Threshold from Torquato, Jiao *Phys. Rev. E* (2013)

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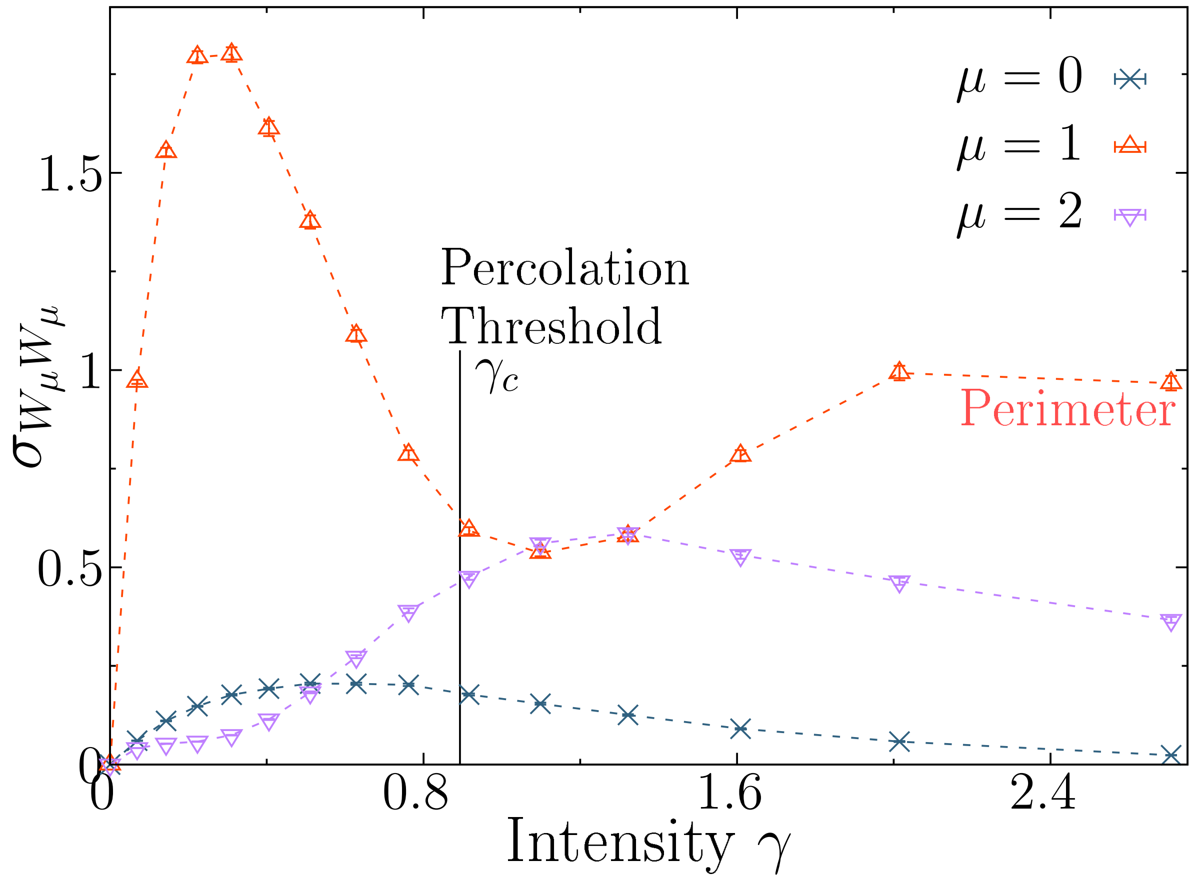

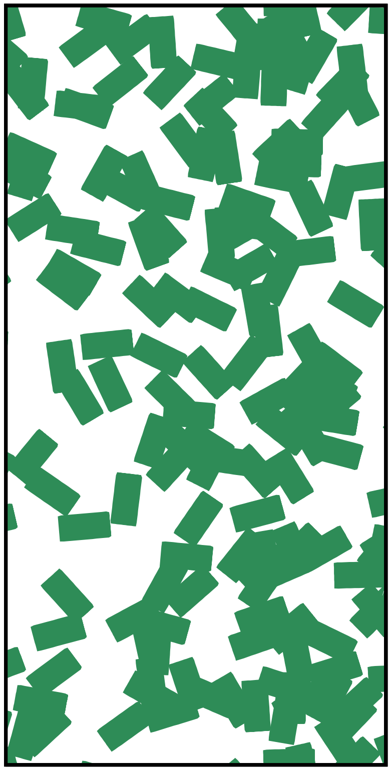

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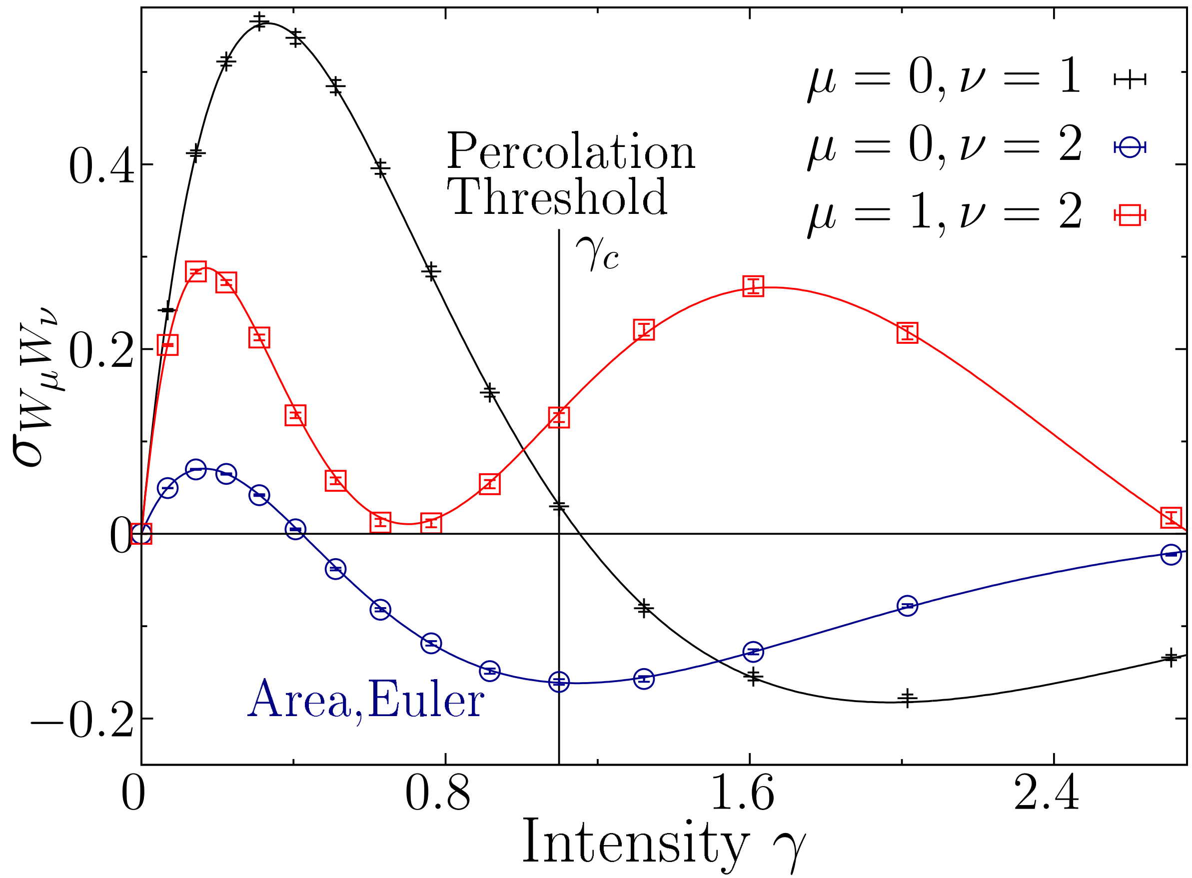

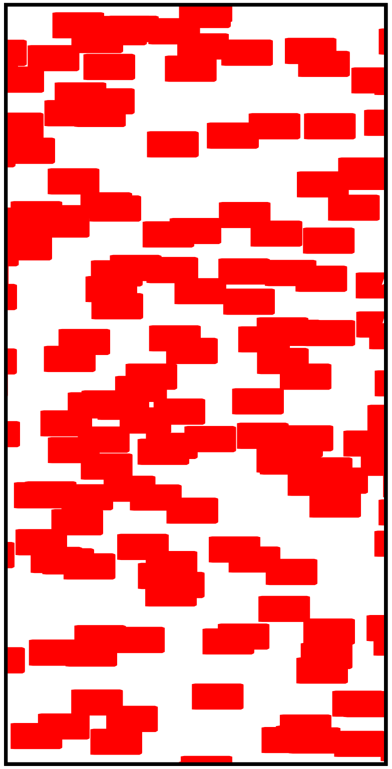

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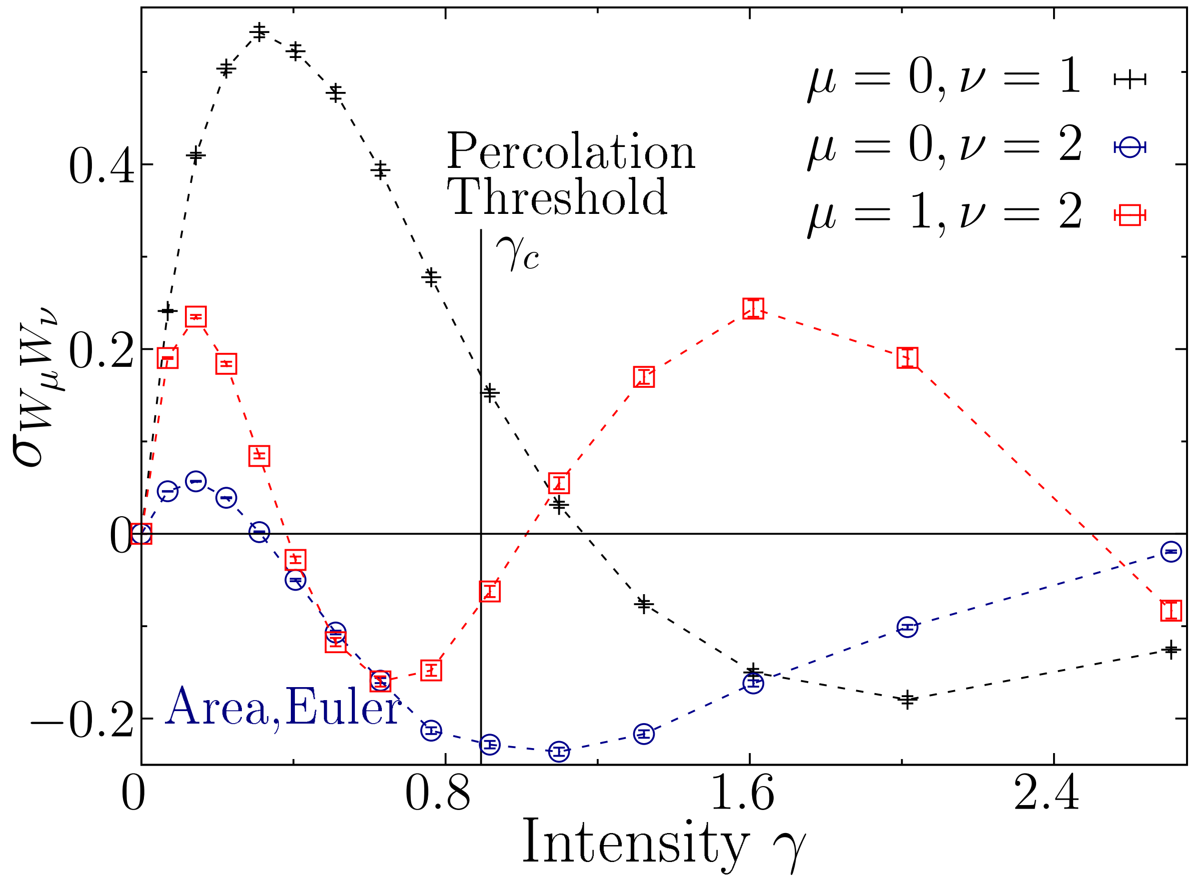

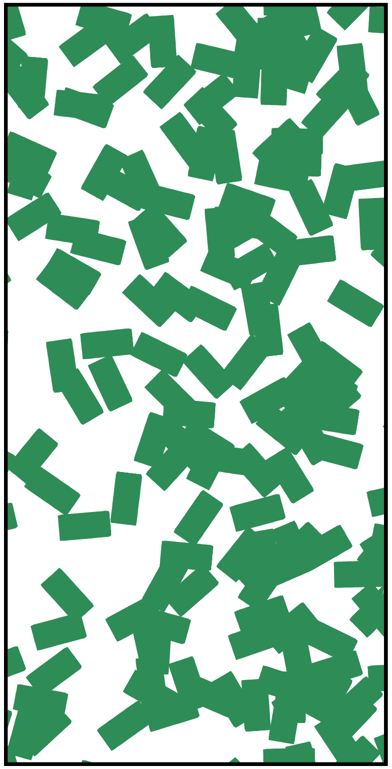

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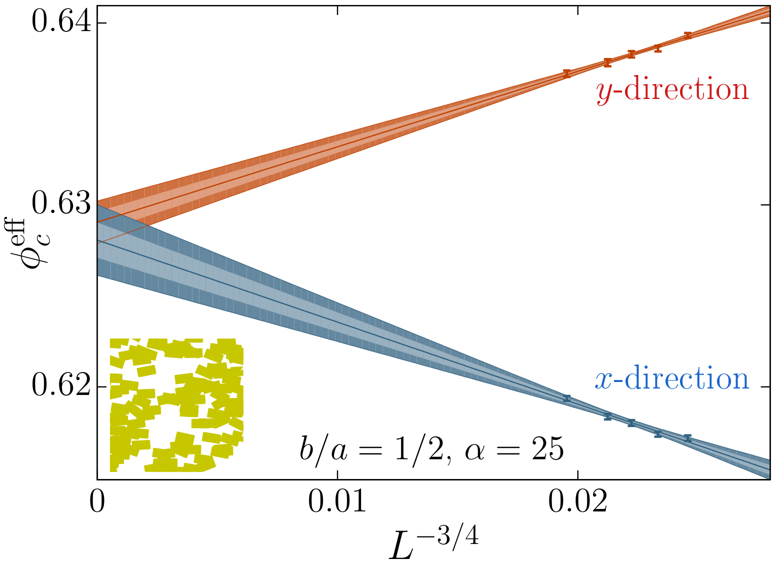
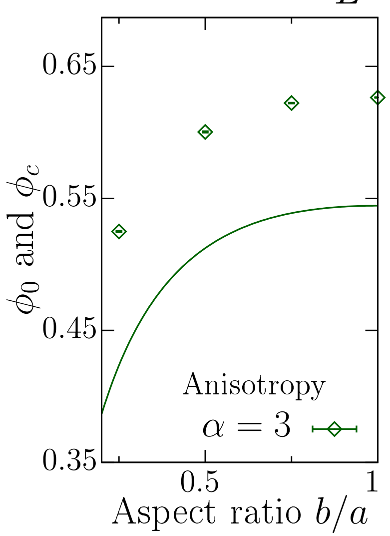
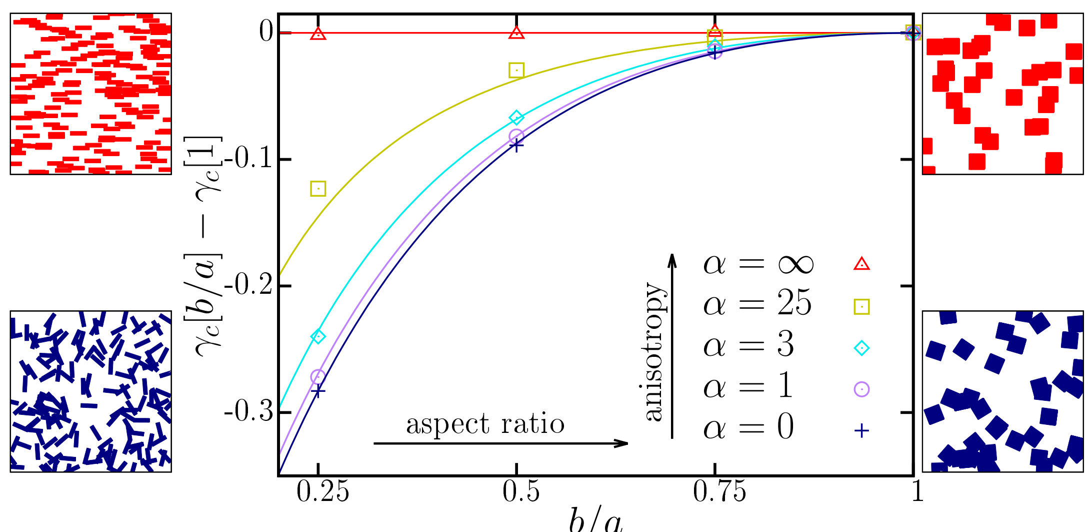


- Anisotropy in finite percolation
- Isotropic threshold because of uniqueness of percolation cluster
- Explicit threshold estimation via 1st and 2nd moments of Minkowski functionals





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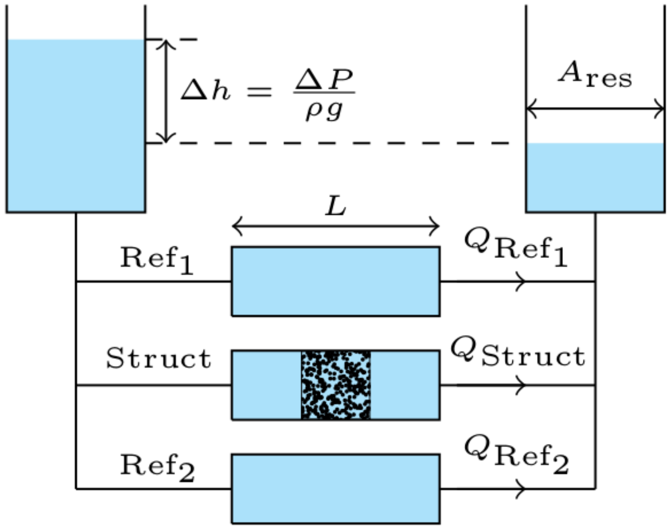
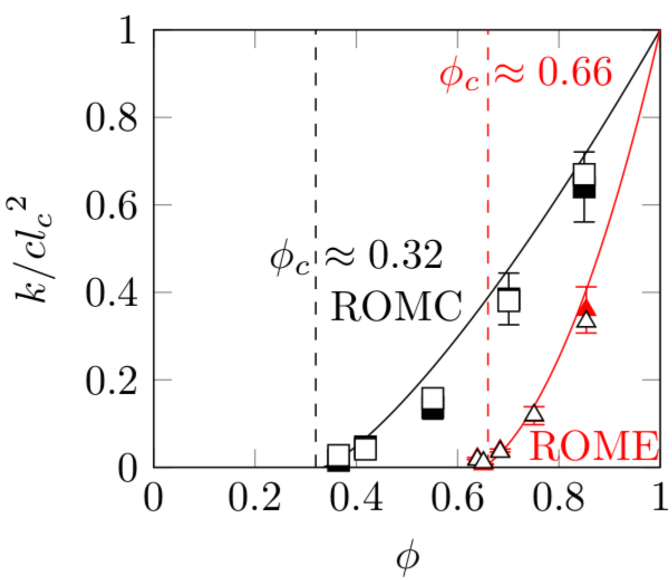
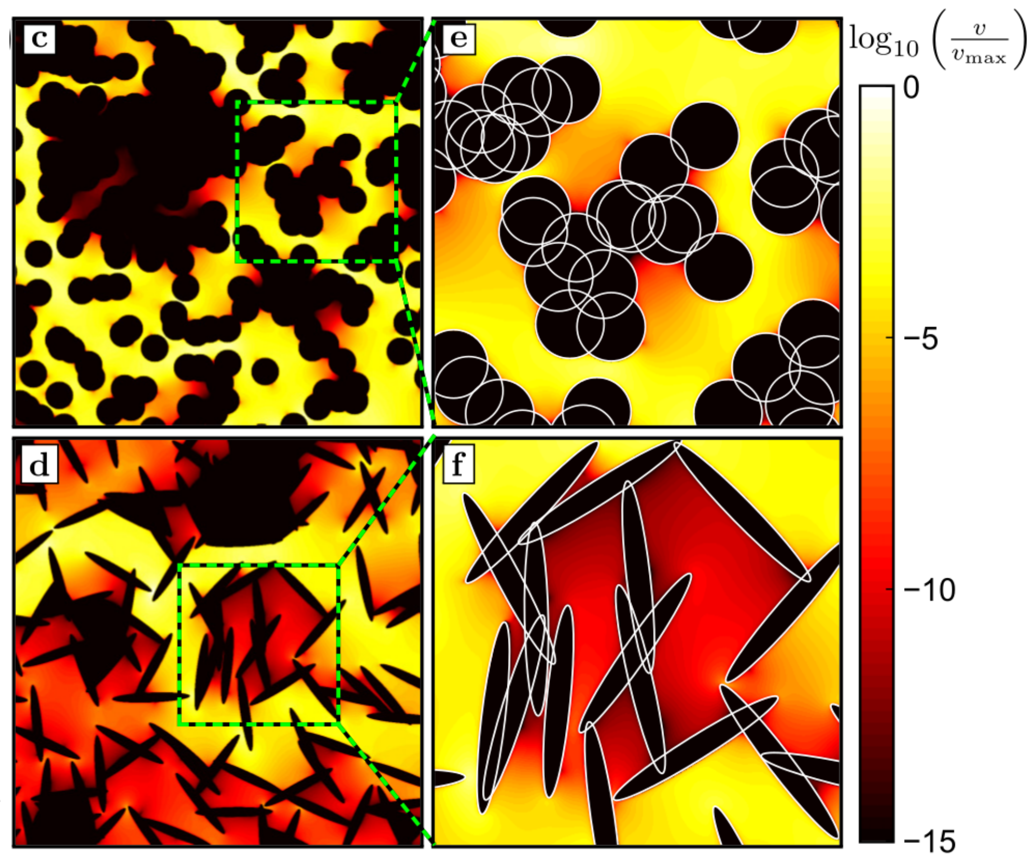
Morphology and Transport



Predict transport properties in heterogeneous media using Minkowski functionals





Scholz, Wirner, K., Hirneise, Schröder-Turk, Mecke, and Bechinger, *Phys. Rev. E* (2015)

Back Up

Orientation distribution