On maximal hard-core thinnings of stationary particle processes

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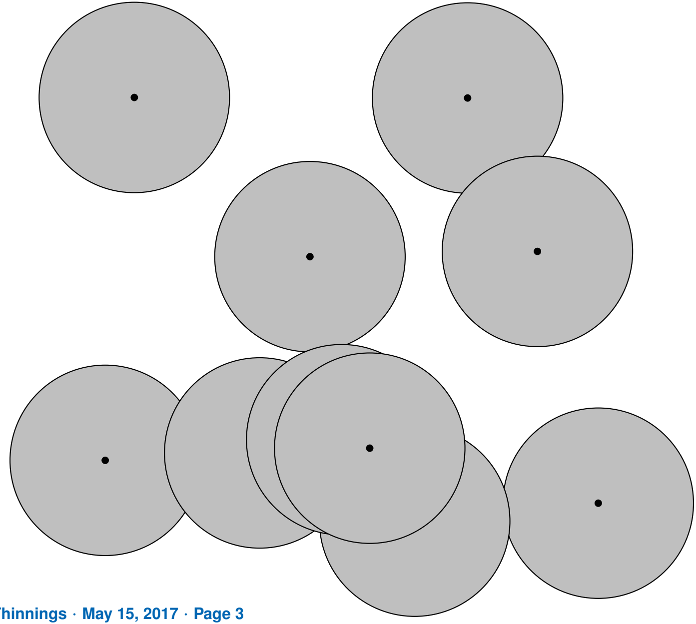
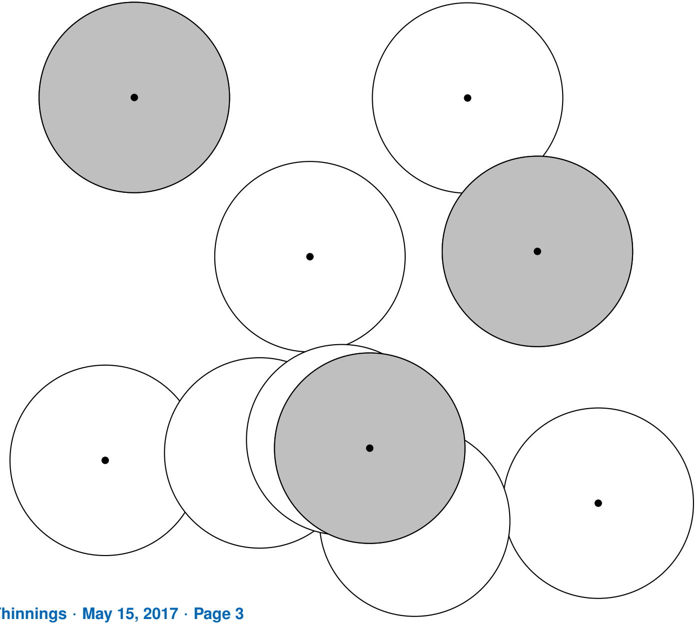
15/05/17, SGSIA 19, CIRM Luminy

Riddle

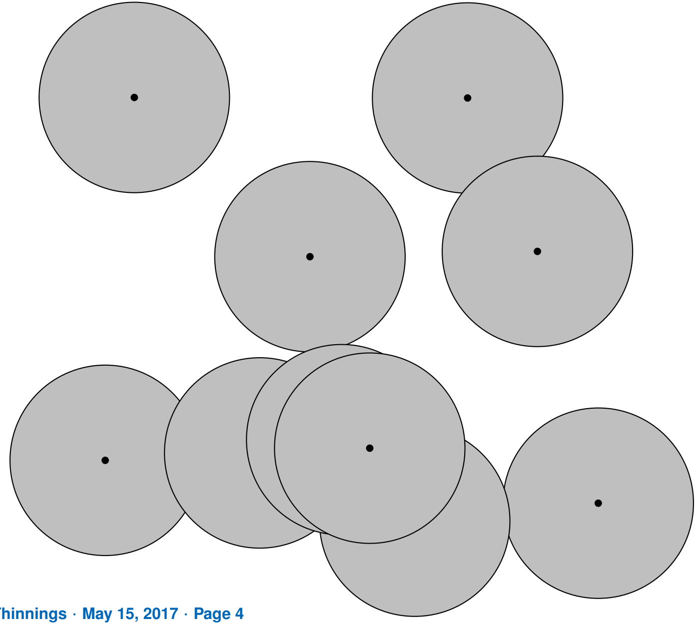
Riddle

Matérn hard-core thinning





Random sequential adsorption



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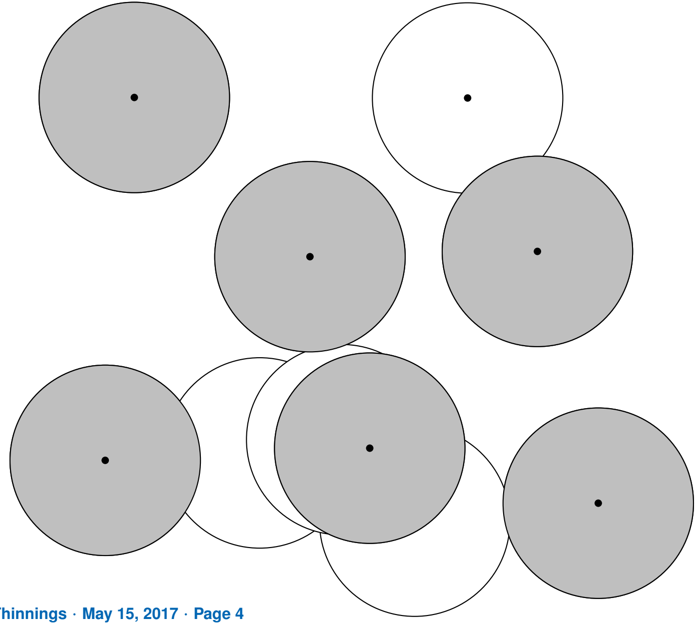

Infinite-volume limits of optimal thinnings?

1 **Definition & Existence**

2 Uniqueness in barely supercritical regime

3 Non-uniqueness at high intensities?

4 Questions

- ***Goal***
  - ***intrinsic characterization*** of infinite-volume maximal hard-core thinnings

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  - ***intrinsic characterization*** of infinite-volume maximal hard-core thinnings
- Two approaches
  - ***locally maximal.*** no improvement through exchange of finitely many particles
  - ***intensity-maximal.*** hard-core thinnings achieving maximum possible intensity
- intensity- and locally maximal thinnings ***equivalent*** under weak moment assumption

Basic set-up

- $\Phi = \{K_i\}_{i\geq 1}$ ***stationary process of particles*** in $\mathcal{K} =$ convex compact bodies
  - e.g. ***Boolean model:*** $\Phi = \{B_{R_i}(X_i)\}_{i\geq 1}$, where
  - $\{X_i\}_{i\geq 1} =$ Poisson point process in $\mathbb{R}^d$ with intensity $\gamma > 0$
  - $\{R_i\}_{i\geq 1} =$ iid family of radii

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  - $\{R_i\}_{i\geq 1}$ = iid family of radii
- important characteristics
  - ***intensity*** $\gamma \mathrel{\widehat{=}}$ mean number of particles per unit volume

$$\gamma = \mathbb{E}\#\{K_i \in \Phi : c(K_i) \in [0,1]^d\},$$

    where $c(K)$ = center of mass of K
  - ***typical grain distribution*** $\mathbb{Q}$

$$\mathbb{E}_{\mathbb{Q}} f(K) = \frac{1}{\gamma}\mathbb{E}\sum_{c(K_i)\in[0,1]^d} f(K_i - c(K_i))$$

    for any measurable function $f : \mathcal{K} \to [0,\infty)$

- investigation of thinnings that are ***stationary*** and ***hard-core***
- ***stationarity***
  - $\mathcal{T}(\mathcal{L}(\Phi)) =$ family of all distributions of ***stationary*** $\{0,1\}$-marked particle processes whose underlying unmarked particle process is distributed according to $\mathcal{L}(\Phi)$
- ***hard-core property***
  - $\mathcal{T}_{\mathsf{hc}}(\mathcal{L}(\Phi)) =$ distributions of ***hard-core*** particle systems that are stationary thinnings of the particle process Φ, i.e.,
  - if $\mathcal{L}(\Psi) \in \mathcal{T}_{\mathsf{hc}}(\mathcal{L}(\Phi))$, then, a.s., $K_i \cap K_j = \emptyset$ for all $K_i, K_j \in \Psi$, $K_i \neq K_j$
- maximization of ***volume fraction***
  - that is, for $\mathcal{L}(\Psi) \in \mathcal{T}_{\mathsf{hc}}(\mathcal{L}(\Phi))$ put

$$\gamma_v(\mathcal{L}(\Psi)) = \mathbb{E} \sum_{\substack{K_i \in \Psi \\ c(K_i) \in [0,1]^d}} \nu_d(K_i),$$

where $\nu_d =$ Lebesgue measure in $\mathbb{R}^d$

Intensity-maximal vs. locally maximal thinnings

- two approaches for ***intrinsic definition*** of infinite-volume maximal thinnings
- ***intensity-maximal***
- ***locally maximal thinnings***

- two approaches for ***intrinsic definition*** of infinite-volume maximal thinnings
- ***intensity-maximal***
  - $\mathcal{L}(\Psi) \in \mathcal{T}_{\mathsf{hc}}(\mathcal{L}(\Phi))$ is ***intensity-maximal*** if it achieves the maximal volume fraction among stationary hard-core thinnings, i.e.,

$$\gamma_v(\mathcal{L}(\Psi)) = \gamma_{\mathsf{max}},$$

    where $\gamma_{\mathsf{max}} = \sup_{\mathcal{L}(\Psi') \in \mathcal{T}_{\mathsf{hc}}(\mathcal{L}(\Phi))} \gamma_v(\mathcal{L}(\Psi'))$.
- ***locally maximal thinnings***

- two approaches for ***intrinsic definition*** of infinite-volume maximal thinnings
- ***intensity-maximal***
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- ***locally maximal thinnings***
  - volume fraction cannot be increased by exchanging ***finite number*** of particles
  - more precisely, let $\psi \subset \varphi$ be hard-core thinning of φ. ψ is ***locally maximal*** if

$$\sum_{K \in \psi' \setminus \psi} \nu_d(K) < \sum_{K \in \psi \setminus \psi'} \nu_d(K)$$

holds for any other hard-core thinning $\psi' \subset \varphi$ such that $\psi \Delta \psi' = (\psi \setminus \psi') \cup (\psi' \setminus \psi)$ is finite

Theorem

1. *if* $\gamma_v(\mathcal{L}(\Phi)) < \infty$, *then* ***intensity-maximal thinnings exist****. Moreover,* ***every intensity-maximal thinning is locally maximal****.*
2. *if additionally* $\int \nu_d(K \oplus [-1/2, 1/2]^d)\nu_d(K)\mathbb{Q}(\mathrm{d}K) < \infty$, *then* ***every locally maximal thinning is intensity-maximal****.*

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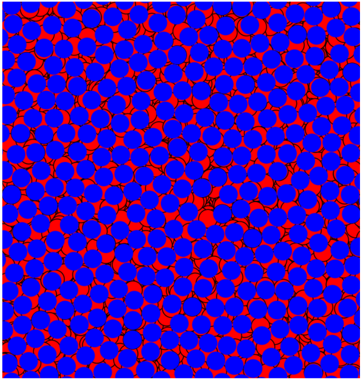
- proof sketch
  - ***existence.*** abstract compactness argument
  - ***intensity-maximal*** $\Rightarrow$ ***locally maximal.*** local modification argument
  - ***locally maximal*** $\Rightarrow$ ***intensity-maximal.*** approximability of maximal intensity through finite configurations



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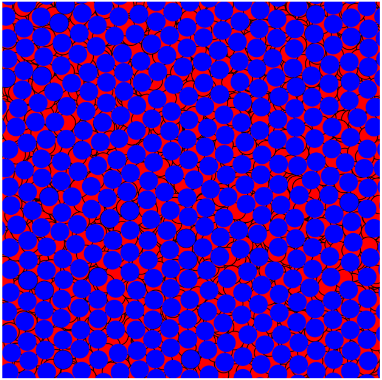

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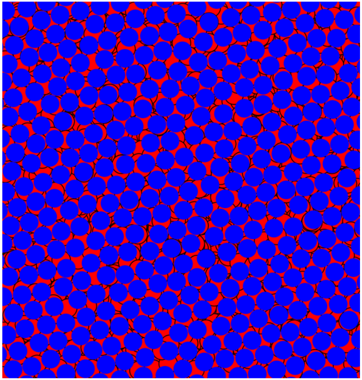

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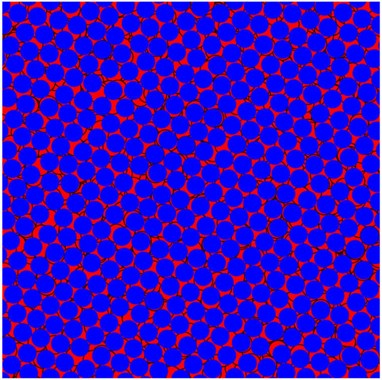

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Gibbsian world

- ***existence*** through abstract argument ✓
- ***uniqueness*** in regime of ***subcritical percolation*** ✓
- ***uniqueness*** in regime of ***barely supercritical percolation*** **?**
- ***non-uniqueness*** at high intensities **conjectured**

Thinning world

- ***existence*** through abstract argument ✓
- ***uniqueness*** in regime of ***subcritical percolation*** ✓
- ***uniqueness*** in regime of ***barely supercritical percolation*** ✓
- ***non-uniqueness*** at high intensities **unclear**

Theorem

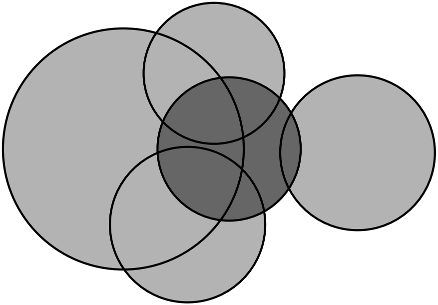
Assume that

- *the union of particles in Φ almost surely does not percolate, and*
- *with probability 1, any two finite collections of distinct particles have different volume.*

Then, locally maximal hard-core thinning of Φ are ***unique****.*

- proof by ***explicit construction***
- choose maximal hard-core thinning in each finite cluster and then take the union

- uniqueness holds ***slightly above criticality*** due to ***dispensable balls***
- $K \in \Phi$ is ***dispensable*** if there exists $K' \in \Phi$ such that
  1. $K \cap K' \neq \emptyset$,
  2. $\nu_d(K) < \nu_d(K')$,
  3. if $K'' \in \Phi$ is such that $K'' \cap K' \neq \emptyset$, then $K'' \cap K \neq \emptyset$.



- intensity of relevant balls ***strictly smaller*** than initial intensity

- use ***essential enhancements*** to make idea rigorous
- similar continuous set-up as in (Franceschetti, Penrose & Rosoman, 2011)
- still modifying essential-enhancement arguments requires fair amount of diligence

"In 1991 Aizenman and Grimmett claimed that any "essential enhancement" of site or bond percolation on a lattice lowers the critical probability, an important result with many implications, such as strict inequalities between critical probabilities on suitable pairs of lattices. Their proof has two parts, one probabilistic and one combinatorial. In this paper ***we point out that a key combinatorial lemma****, for which they provide only a figure as proof,* ***is false****."*

Essential enhancements revisited; Balister, Bollobás & Riordan, 2014

Theorem

- *Assume that Φ is a Boolean model of balls with intensity $\gamma > 0$ and random radii uniformly distributed on $[1, 2]$*

$\Rightarrow$ $\exists \gamma_{\mathrm{u}} > \gamma_{\mathrm{c}}$ *such that* ***locally maximal thinnings are unique whenever*** $\gamma < \gamma_{\mathrm{u}}$

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- construct ***two-type percolation process***
- balls in Φ are ***red*** independently with probability $1 - p$, $p \in (0, 1)$
- dispensable balls are ***green*** independently with probability $1 - q$, $q \in (0, 1)$

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- construct ***two-type percolation process***
- balls in Φ are ***red*** independently with probability $1-p$, $p \in (0,1)$
- dispensable balls are ***green*** independently with probability $1-q$, $q \in (0,1)$
- define ***percolation probability*** $\theta(p,q) = \lim_{n\to\infty} \theta_n(p,q)$, where

$$\theta_n(p,q) = \mathbb{P}(o \leftrightsquigarrow \partial B_n \text{ via uncolored overlapping balls})$$

- essential enhancement $\Rightarrow$

$$\theta(\gamma^{-1}\gamma_{\mathrm{c}} + \varepsilon, 0) \leq \theta(\gamma^{-1}\gamma_{\mathrm{c}} - \varepsilon, 1)$$

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Thinning world

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Partial Result

*Locally maximal thinnings are **unique at arbitrarily high intensities** if grain sizes are measured through **highly fluctuating functional***

Partial Result

Locally maximal thinnings are ***unique at arbitrarily high intensities*** *if grain sizes are measured through* ***highly fluctuating functional***

- $\Phi =$ ***Boolean model*** with random radii distributed uniformly on the interval $[0, 1]$.
- for $a \geq 1$ put

$$\nu_a(B_r(x)) = e^{ar}$$

Theorem

If $a \geq 1$ is sufficiently large, then there is ***uniqueness*** *of locally ν_a-maximal thinnings*

- ***idea.*** Similarity with RSA + disagreement percolation

Summary & Questions

- intensity maximal thinnings and locally maximal thinnings as ***equivalent intrinsic characterization*** of infinite-volume limit of maximal thinnings in bounded windows
- existence via ***abstract compactness argument***
- uniqueness in ***sub-critical and barely supercritical*** percolation regime
- uniqueness for ***highly fluctuating grain sizes***

- intensity maximal thinnings and locally maximal thinnings as ***equivalent intrinsic characterization*** of infinite-volume limit of maximal thinnings in bounded windows
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- uniqueness for ***highly fluctuating grain sizes***

- non-uniqueness at ***high intensities***? In certain dimensions?
- ***symmetry-breaking***?
- ***constructive algorithm*** for maximal thinnings?

Thank you!

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